1. Let $x$ and $y$ be relatively prime elements in a UFD $R$ such that $I = (x, y)R$ is a proper ideal. You may assume that the surjection $R^2 \to I$ that sends $(r, s) \mapsto sx - ry$ has kernel $Rv$ spanned by $v = (x, y) \in R^2$, so that $I \cong R^2/Rv$. Represent $I \otimes_R I$ as the cokernel of a map of finitely generated free modules. Prove that the map $I \otimes I \to I^2$ has a nonzero kernel spanned by $\theta = x \otimes y - y \otimes x \in I \otimes_R I$, and that the annihilator of $\theta$ in $R$ is $I$.

2. Let $X = (x_{ij})$ be an $n \times n$ matrix of indeterminates over an algebraically closed field $K$. Show that the algebraic set $Z_n$ in $A^{n^2} K$ defined by the vanishing of the entries of the matrix $X^n$ can also be defined by the vanishing of $n$ polynomials in the $x_{ij}$. Show that $Z_n$ is irreducible, i.e., is a variety. (For the latter, it may be useful to show that $Z_n$ is the image of $GL(n, K) \times Y$ under a regular map, where $Y$ consists of the upper triangular matrices with 0 entries on the main diagonal.) What is the dimension of the variety $Z_n$?

3. Let $M$ be a finitely generated module over a Noetherian ring $R$. Show that the set $\{P \in \text{Spec}(R) : M_P$ is free over $R_P\}$ is Zariski open in $\text{Spec}(R)$.

4. Let $K_1 \subset \cdots \subset K_n \subset \cdots$ be an infinite chain of proper field extensions. Let $x$ be a power series indeterminate over all the $K_n$. Let $R = \bigcup_{n=1}^{\infty} K_n[[x]]$. Show that $R$ is a local (Noetherian) domain in which every nonzero proper ideal is generated by a power of $x$.

5. Let $f$ and $g$ be elements of a ring $R$ such that $f + g = 1$. Let $M$ be an $R$-module. Suppose that $u, v \in M$ are such that $gu - fv = 0$. Show that there is a unique element $m \in M$ such that the image of $m$ in $M_f$ is $u/f$ and the image of $m$ in $M_g$ is $v/g$.

6. Let $R$ be any ring, and let $Y \subseteq X =: \text{Spec}(R)$ be a subset that is a finite union of sets of the form $U_i \cap Z_i$, where $U_i$ is quasicompact and open in $X$ and $Z_i$ is closed in $X$. Show that there is a finitely generated $R$-algebra $S$ such that the image of the map $\text{Spec}(S) \to \text{Spec}(R)$ is $Y$. [Suggestion: if $R \to S_1$ and $R \to S_2$ are homomorphisms, compare the image of $\text{Spec}(S_1 \times S_2)$ with the images of $\text{Spec}(S_1)$ and $\text{Spec}(S_2)$.]

**Extra Credit 8.** Let $L$ be a field extension of $K$ that is not finitely generated as a field extension. Prove that $L \otimes_K L$ is not a Noetherian ring.

**Extra Credit 9.** Let $K_1 \subset \cdots \subset K_n \subset \cdots$ be as in 3. and let $x_1, \ldots, x_d$ be power series indeterminates over all the of the fields $K_n$. Let $R_d = \bigcup_{n=1}^{\infty} K_n[[x_1, \ldots, x_d]]$. Is $R_d$ Noetherian? Prove your answer.