Let $R = K[x_1, \ldots, x_n]$ and let $\mu_1, \ldots, \mu_k$ be a sequence of monomials in $R$. Let $M$ be a monomial submodule of the finitely generated free module $F$. Then $\mu_1, \ldots, \mu_k$ is a regular sequence on $F/M$ if and only if no variable that occurs in $\mu_i$ occurs in another $\mu_j$, nor in any of the minimal monomial generators of $M$.

Proof. Since $M = I_1e_1 + \cdots + I_se_s$ where the $I_j$ are monomial ideals, we reduce at once to the case where $M = I$ is a monomial ideal: call the minimal monomial generators $\nu_1, \ldots, \nu_h$. We use induction on $k$. If $k = 1$, note that if $\mu_1$ shares a variable $x_i$ with $\nu_i$ then $\nu_i :R \mu_1$ is generated by a monomial that divides $\nu_i$ and has a smaller exponent on $x_i$ then $\nu_i$ does. This element is not in $I$, by the minimality of $\nu_i$, but is in $I : \mu_1$. Hence the condition that $\mu_1$ not involve a variable occurring in any $\nu_i$ is necessary. On the other hand, if that is true then $\nu_i :R \mu = \nu_i R$ for every $i$, $1 \leq i \leq h$, and since colon distributes over sum we have that

$$I :R \mu_1 = \left( \sum_{i=1}^h \nu_i R \right) :R \mu_1 = \sum_{i=1}^h (\nu_i R :R \mu_1) = \sum_{i=1}^h \nu_i R = I,$$

as required. Moreover it is clear that $\nu_1, \ldots, \nu_h, \mu_1$ are minimal generators for $I + \mu_1 R$. The inductive step is then an application of the case where $k = 1$. \qed

Compatible orders and a sufficient condition for regularity of a sequence

Given a polynomial ring $K[x_1, \ldots, x_n]$ over a field $K$ and a monomial order $>$ on a finitely generated $R$-free module $F$ with ordered free basis $e_1, \ldots, e_s$, recall that for every $t$, $1 \leq t \leq s$, there is a monomial order $>_t$ on $R$ defined by the condition $\mu > \mu'$ precisely if $\mu e_t > \mu' e_t$. Moreover, if $g \in R - \{0\}$ and $f \in F - \{0\}$ are such that $\text{in}(f)$ involves $e_t$, then

$$(\dagger) \quad \text{in}(gf) = \text{in}_{>_t}(g) \text{in}(f).$$

See the second page of the Lecture Notes of January 19. We shall say that a monomial order $>_R$ on $R$ is compatible with a given monomial order $>_F$ on $F$ if all of the orders $>_t$
are the same, and agree with \( >_R \). It follows at once that if \( >_R \) and \( > \) on \( F \) are compatible, then for all \( g \in R - \{0\} \) and \( f \in F - \{0\} \),

\[
(††) \quad \text{in}(fg) = \text{in}_{>_R}(g\text{in}(f)).
\]

In fact, condition \((††)\) is easily seen to be equivalent to compatibility. In working with compatible monomial orders, we typically use the same symbol \( > \) for both.

If two of the \( >_t \) are distinct, which can happen, there is no compatible order on \( R \). If there is a compatible order on \( R \), it is unique. The standard method of extending a monomial order on \( R \) to a monomial order on \( F \) (i.e., \( \mu e_i > \mu' e_j \) if \( \mu > \mu' \) or \( \mu = \mu' \) and \( i < j \)) always produces a monomial order on \( F \) with which the original monomial order is compatible. In particular, revlex on \( F \) is compatible with revlex on \( R \). In the sequel, when \( F \) is graded so that its generators do not necessarily all have degree 0, we give a slightly different way of extending revlex to \( F \)—but it is still compatible with revlex on \( R \).

We next observe the following sufficient (but not necessary) condition for elements of \( R \) to be a regular sequence on \( F/M \). Notice that we are not assuming that \( M \) is graded, nor that \( > \) is revlex.

**Theorem.** Let \( R = K[x_1, \ldots, x_n], f_1, \ldots, f_k \in R \) and let \( M \) be any submodule of a finitely generated free \( R \)-module \( F \). Suppose that we have compatible monomial orders on \( R \) and \( F \). If \( \text{in}(f_1), \ldots, \text{in}(f_k) \) form a regular sequence on \( \text{in}(M) \), then \( f_1, \ldots, f_k \) is a regular sequence on \( M \) and, for \( 1 \leq i \leq k \), \( \text{in}(M + (f_1, \ldots, f_i)) = \text{in}(M) + (\text{in}(f_1), \ldots, \text{in}(f_i)) \).

**Proof.** We use induction on \( k \), and we consequently can reduce at once to the case where \( k = 1 \). We write \( f \) for \( f_1 \), and we must show that if \( \text{in}(f) \) is not a zerodivisor on \( F/\text{in}(M) \) then \( f \) is not a zerodivisor on \( F/M \) and \( \text{in}(M + fM) = \text{in}(M) + \text{in}(f)F) \).

If \((1)\) fails we have \( fu \in v \in M \) with \( u \notin M \), and we can choose such an example with \( \text{in}(u) \) minimum, since the monomial order on \( F \) is a well-ordering. By the compatibility of orders, \( \text{in}(fu) = \text{in}(f)\text{in}(u) = \text{in}(v) \in \text{in}(M) \), and since \( \text{in}(f) \) is not a zerodivisor on \( \text{in}(M) \), we have that \( \text{in}(u) \in \text{in}(M) \), so that we can choose \( u' \in M \) with \( \text{in}(u) = \text{in}(u') \). Then \( fu \) and \( fu' \) are both in \( M \), and so \( f(u - u') \in M \). But the initial terms of \( u \) and \( u' \) cancel, so that \( u = u' \) or \( \text{in}(u - u') < \text{in}(u) \). The latter contradicts the minimality of the choice of \( u \), and the former shows that \( u \in M \).

To prove \((2)\), note that \( \text{in}(M) + \text{in}(f)F \subseteq \text{in}(M + fF) \) is obvious, and so we need only prove the opposite inclusion. If it fails, we can choose \( u + fv \in M + fF \) where \( u \in M \), \( v \in F \), such that \( \text{in}(u + fv) \notin \text{in}(M) + \text{in}(f)F \), and, again, we can make this choice so that \( \text{in}(v) \) is minimum (note that \( v \) cannot be 0). We consider two cases.

First case: \( \text{in}(fv) \in \text{in}(M) \). Then \( \text{in}(f)\text{in}(v) \in \text{in}(M) \) and, since \( \text{in}(f) \) is not a zerodivisor on \( \text{in}(M) \), we have that \( \text{in}(v) \in \text{in}(M) \) and we can choose \( v' \in M \) such that \( \text{in}(v) = \text{in}(v') \). Then \( u + fv = (u + fv') + f(v - v') \) still has initial form not in \( M + fV \), and we have \( u + fv' \in M \) while \( v - v' \) has smaller initial form than \( v \), a contradiction.

Second case: \( \text{in}(fv) \notin \text{in}(M) \). In this case, \( \text{in}(fv) \) and \( \text{in}(u) \in \text{in}(M) \) cannot cancel, and so one of them must be \( \text{in}(u + fv) \). But then either \( \text{in}(u + fv) = \text{in}(u) \in \text{in}(M) \) or \( \text{in}(u + fv) = \text{in}(fv) = \text{in}(f)\text{in}(v) \in \text{in}(f)F \), as required. □
Special properties of reverse lexicographic order and a converse result

Throughout this section, $R = K[x_1, \ldots, x_n]$ is a polynomial ring over $K$ considered with reverse lexicographic order, $F$ is a finitely generated graded free $R$-module with ordered free homogeneous basis $e_1, \ldots, e_s$, also with reverse lexicographic order, which we define as follows. In the graded case we still want revlex to define total degree. Therefore, we define $\mu e_i >_{\text{revlex}} \mu' e_j$ to mean either that (1) $\deg(\mu e_i) > \deg(\mu' e_j)$ or (2) $\deg(\mu e_i) = \deg(\mu' e_j)$ and $\mu < \mu'$ in lexicographic order for the variables ordered so that

$$x_n > x_{n-1} > \cdots > x_2 > x_1,$$

or (3) $\deg(\mu e_i) = \deg(\mu' e_j)$, $\mu = \mu'$, and $i < j$.

Let $M$ be a graded submodule of $F$. We already noted at the end of the Lecture of January 31 that $x_{k+1}, \ldots, x_n$ is a regular sequence on $F/M$ if and only if $x_{k+1}, \ldots, x_n$ is a regular sequence on $F/\text{in}(M)$, which we know is equivalent to the condition that no minimal monomial generator of $\text{in}(M)$ involves any of the variables $x_{k+1}, \ldots, x_n$. The preceding Theorem already shows that the condition is sufficient. We next want to prove that it is necessary as well. The following very easy result is a key fact about revlex that we shall use repeatedly.

**Lemma.** Let notation be as above and let $u \in F - \{0\}$ be a homogeneous element. Then for every positive integer $h$, $x_n^h$ divides $u$ if and only if $x_n^h$ divides $\text{in}(u)$.

**Proof.** “Only if” is obvious. The “if” part is immediate from the definition: since all terms have the same degree, any term not divisible by $x_n^h$ is strictly larger than any term divisible by $x_n^h$. □

**Proposition.** Let notation be as above, with $M \subseteq F$ graded, and let $g_1, \ldots, g_r$ be a Gröbner basis for $M$ consisting of homogeneous elements. Let $k$ be a positive integer.

(a) $\text{in}(M + x_n^h F) = \text{in}(M) + x_n^h F$, and $g_1, \ldots, g_r, x_n^h e_1, \ldots, x_n^h e_s$ is a Gröbner basis for $M + x_n^h F$.

(b) $\text{in}(M :_F x_n^h) = \text{in}(M) :_F x_n^h$. Moreover, if for $1 \leq j \leq r$, $t_j$ denotes the greatest integer in the interval $[0, h]$ such that $x^{t_j} | g_j$ and $h_j = g_j / x_n^{t_j}$, then $h_1, \ldots, h_r$ is a Gröbner basis for $M :_F x_n^h$.

**Proof.** (a) Clearly, $\text{in}(M) + x_n^h F \subseteq \text{in}(M + x_n^h F)$. Now consider $\text{in}(u + x_n^h f)$ where $u \in U$ and $f \in F$. In revlex, the homogeneous component of an element of highest degree has the same initial form as the element, and so we may assume that $u + x_n^h f$ is homogeneous. If the initial term is divisible by $x_n^h$ the result is proved. If not, it must be a term of $u$, and $x_n$ must occur with a strictly smaller exponent than $h$. All other terms of $u$ must be smaller: either they are not divisible by $x_n^h$ and persist in $u + x_n^h f$, or they are divisible by $x_n^h$, which forces them to be smaller than $u$ in revlex, by the definition of revlex. The statement
about the Gröbner basis is immediate, since the specified elements are in \( M + x_n^h F \) and their initial terms span \( \text{in}(M) + x_n^h F \).

(b) We have that a monomial \( \nu \in \text{in}(M : F x_n^h) \) iff and \( x_n^h \nu \in \text{in}(M) \) iff \( x_n^h \nu = \text{in}(w) \) with \( w \in M \) homogeneous. But \( x_n^h \) divides \( w \) if and only \( x_n^h \) divides \( \text{in}(w) \), by the Lemma above, and the result is immediate. We then have that \( \text{in}(M) \) is the span of the \( \text{in}(g_j)R : F x_n^h \), and these are the same as the \( \text{in}(g_j/x_j^h)R \). Again, we are using that a power of \( x_n \) divides \( g_j \) if and only if it divides \( \text{in}(g_j) \). □

We can now prove:

**Theorem.** Let notation be as above, with \( M \subseteq F \) graded, and use revlex order on \( F \) and \( R \). Then \( x_{k+1}, \ldots, x_n \) is a regular sequence on \( F/M \) if and only if it is a regular sequence on \( F/\text{in}(M) \).

**Proof.** Since regular sequences are permutable in the graded case, we may show instead the same result for \( x_n, \ldots, x_{k+1} \). We already know the “if” part. Now suppose that \( x_n \) is not a zerodivisor on \( F/M \). Then \( M : F x_n = M \), and so

\[
\text{in}(M) = \text{in}(M : F x_n) = \text{in}(M) : F x_n = \text{in}(M).
\]

The proof is now completed by induction: when we work mod \( x_n \), \( R \) is replaced by \( R/x_n R = K[x_1, \ldots, x_{n-1}] \), \( F \) by \( F/x_n F \), and \( M \) by \( M/x_n M \rightarrow F/x_n F \), since \( x_n \) is not a zerodivisor on \( M/x_n M \). The hypothesis is preserved because of the preceding Proposition. □