Remark. It is worth noting that Cauchy sequences in an $I$-adic topology are much easier to study, in some ways, than Cauchy sequences of, say, real numbers. In an $I$-adic topology, for $\{r_n\}_n$ to be a Cauchy sequence it suffices that $r_n - r_{n+1} \to 0$ as $n \to \infty$, i.e., that for any specified $N \in \mathbb{N}$, the differences $r_n - r_{n+1}$ are eventually in $I^N$. The reason is that if this is true for all $n \geq n_0$, we also have that

$$r_{n'} - r_n = r_{n'} - r_{n' - 1} + \cdots + r_{n+1} - r_n \in I^n$$

for all $n' \geq n \geq n_0$. In consequence, a necessary and sufficient condition for an infinite series $\sum_{n=0}^{\infty} r_n$ to converge in the $I$-adic topology is that $r_n \to 0$ as $n \to \infty$, which, of course, is false over $\mathbb{R}$: the series $\sum_{n=1}^{\infty} 1/n$ does not converge, and the corresponding sequence of partial sums $\{r_n\}_n$ does not converge, even though $r_{n+1} - r_n = 1/(n+1) \to 0$ as $n \to \infty$.

Our next result on coefficient fields uses a completely different argument:

**Theorem.** Let $(R, m, K)$ be a complete local ring of positive prime characteristic $p$. Suppose that $K$ is perfect. Let $R^{p^n} = \{r^{p^n} : r \in R\}$ for every $n \in \mathbb{N}$. Then $K_0 = \bigcap_{n=0}^{\infty} R^{p^n}$ is a coefficient field for $R$, and it is the only coefficient field for $R$.

**Proof.** Consider any coefficient field $L$ for $R$, assuming for the moment that one exists. Then $L \cong K$, and so $L$ is perfect. Then

$$L = L^p = \cdots = L^{p^n} = \cdots,$$

and so for all $n$,

$$L \subseteq L^{p^n} \subseteq R^{p^n}.$$

Therefore, $L \subseteq K_0$. If we know that $K_0$ is a field, it follows that $L = K_0$, proving uniqueness.

It therefore suffices to show that $K_0$ is a coefficient field for $K$. We first observe that $K_0$ meets $m$ only in 0. For if $u \in K_0 \cap m$, then $u$ is a $p^n$ th power for all $n$. But if $u = v^{p^n}$ then $v \in m$, so $u \in \bigcap_n m^{p^n} = \{0\}$.

Thus, every element of $K_0 - \{0\}$ is a unit of $R$. Now if $u = v^{p^n}$ and $u$ is a unit of $R$, then $1/u = (1/v)^{p^n}$. Therefore, the inverse of every nonzero element of $K_0$ is in $K_0$. Since $K_0$ is clearly a ring, it is a subfield of $R$.

Finally, we want to show that given $\theta \in K$ some element of $K_0$ maps to $\theta$. Let $r_n$ denote an element of $R$ that maps to $\theta^{1/p^n} \in K$. Then $r_n^{p^n}$ maps to $\theta$. We claim that $\{r_n^{p^n}\}_n$ is a
Cauchy sequence in $R$, and so has a limit $r$. To see this, note that $r_n$ and $r_{n+1}^p$ both map to $\theta^{1/p^n}$ in $K$, and so $r_n - r_{n+1}^p$ is in $m$. Taking $p^n$ powers, we find that

$$r_n^p - r_{n+1}^{p+1} \in m^{p^n}.$$ 

Therefore, the sequence is Cauchy, and has a limit $r \in R$. It is clear that $r$ maps to $\theta$. Therefore, it suffices to show that $r \in R^p$ for every $k$. But

$$r_k, r_{k+1}^p, \ldots, r_{k+h}^p$$

is a sequence of the same sort for the element $\theta^{1/p^k}$, and so is Cauchy and has a limit $s_k$ in $R$. But $s_k^p = r$ and so $r \in R^{p^k}$ for all $k$. □

Before pursuing the issue of the existence of coefficient fields further, we show that the existence of a coefficient field implies that the complete local ring is a homomorphic image of a power series ring in finitely many variables over a field, and is also a module-finite extension of such a ring.

We first prove the following result, which bears some resemblance to Nakayama’s Lemma, but is rather different, since $M$ is not assumed to be finitely generated.

**Proposition.** Let $R$ be separated and complete in the $I$-adic topology, where $I$ is a finitely generated ideal of $R$, and let $M$ be an $I$-adically separated $R$-module. Let $u_1, \ldots, u_h \in M$ have images that span $M/IM$ over $R/I$. Then $u_1, \ldots, u_h$ span $M$ over $R$.

**Proof.** Since $M = Ru_1 + \cdots + Ru_h + IM$, we find that for all $n$,

$$I^n M = I^n u_1 + \cdots + I^n u_h + I^{n+1} M.$$ 

Let $u \in M$ be given. Then $u$ can be written in the form $r_{01} u_1 + \cdots + r_{0h} u_h + \Delta_1$ where $\Delta_1 \in IM$. Therefore $\Delta_1 = r_{11} u_1 + \cdots r_{1h} u_h + \Delta_2$ where the $r_{ij} \in IM$ and $\Delta_2 \in I^2 M$. Then

$$u = (r_{01} + r_{11})u_1 + \cdots + (r_{0h} + r_{1h})u_h + \Delta_2,$$

where $\Delta_2 \in I^2 M$. By a straightforward induction on $n$ we obtain, for every $n$, that

$$u = (r_{01} + r_{11} + \cdots + r_{0n}) u_1 + \cdots + (r_{0h} + r_{1h} + \cdots + r_{nh}) u_n + \Delta_{n+1}$$

where every $r_{jk} \in I$ for $1 \leq k \leq h$ and $j \geq 0$ and $\Delta_{n+1} \in I^{n+1} M$. In the recursive step, the formula $(*)$ is applied to the element $\Delta_{n+1} \in I^{n+1} M$.

For every $k$, $\sum_{j=0}^\infty r_{jk}$ represents an element $s_k$ of the complete ring $R$. We claim that

$$u = s_1 u_1 + \cdots + s_h u_h.$$
The point is that if we subtract
\[ \sigma_n = (r_{01} + r_{11} + \cdots + r_{n1})u_1 + \cdots + (r_{0h} + r_{1h} + \cdots + r_{nh})u_h \]
from \( u \) we get \( \Delta_{n+1} \in I^{n+1}M \), and if we subtract \( \sigma_n \) from \( s_1u_1 + \cdots + s_hu_h \) we also get an element of \( I^{n+1}M \), which we shall justify in greater detail below. Therefore,
\[
\left( s_1u_1 + \cdots + s_hu_h \right) \in \bigcap_n I^{n+1}M = 0,
\]
since \( M \) is \( I \)-adically separated.

It remains to see why \( s_1u_1 + \cdots + s_hu_h - \sigma_n \) is in \( I^{n+1}M \). This difference can be rewritten as \( s'_1u_1 + \cdots + s'_hu_h \) where \( s'_k = r_{n+1,k} + r_{n+2,k} + \cdots \). Hence, we simply need to justify the assertion that if \( r_{jk} \in I^j \) for \( j \geq n+1 \) then
\[
r_{n+1,k} + r_{n+2,k} + \cdots + r_{n+t,k} + \cdots \in I^{n+1},
\]
which needs a short argument. Since \( I \) is finitely generated, we know that \( I^{n+1} \) is finitely generated by the monomials of degree \( n + 1 \) in the generators of \( I \), say, \( g_1, \ldots, g_d \). Then
\[
r_{n+1+t,k} = \sum_{\nu=1}^d q_{\nu}g_{\nu} \quad {\text{with every}} \quad q_{\nu} \in I^t \quad \text{and} \quad \sum_{t=0}^\infty r_{n+1+t,k} = \sum_{\nu=1}^d (\sum_{t=0}^\infty q_{\nu})g_{\nu}. \]

We also note:

**Proposition.** Let \( f : R \to S \) be a ring homomorphism. Suppose that \( S \) is \( J \)-adically complete and separated for an ideal \( J \subseteq S \) and that \( I \subseteq R \) maps into \( J \). Then there is a unique induced homomorphism \( \hat{R}^I \to S \) that is continuous (i.e., preserves limits of Cauchy sequences in the appropriate ideal-adic topology).

**Proof.** \( \hat{R}^I \) is the ring of \( I \)-adic Cauchy sequences mod the ideal of sequences that converge to 0. The continuity condition forces the element represented by \( \{r_n\}_n \) to map to
\[
\lim_{n \to \infty} f(r_n)
\]
(Cauchy sequences map to Cauchy sequences: if \( r_m - r_n \in I^N \), then \( f(r_m) - f(r_n) \in J^N \), since \( f(I) \subseteq J \).) It is trivial to check that this is a ring homomorphism that kills the ideal of Cauchy sequences that converge to 0, which gives the required map \( \hat{R}^I \to S \). \( \square \)

A homomorphism of quasilocal rings \( h : (A, \mu, \kappa) \to (R, m, K) \) is called a local homomorphism if \( h(\mu) \subseteq m \). If \( A \) is a local domain, not a field, the inclusion of \( A \) in its fraction field is not local. If \( A \) is a local domain, any quotient map arising from killing a proper ideal is local. A local homomorphism induces a homomorphism of residue class fields \( \kappa = A/\mu \to R/m = K \).
Proposition. Let $A$ be a Noetherian ring that is complete and separated with respect to an ideal $\mu$, which may be 0, let $(R, m, K)$ be a complete local ring, and let $h : A \to R$ be a homomorphism, so that $R$ is an $A$-algebra and $\mu$ maps into $m$. Thus, if $(A, \mu)$ is local, we are requiring that $A \to R$ be local. Suppose that $f_1, \ldots, f_n \in m$ together with $\mu R$ generate an $m$-primary ideal. Then:

(a) There is a unique continuous homomorphism $h : A[[X_1, \ldots, X_n]] \to R$ extending the $A$-algebra map $A[X_1, \ldots, X_n]$ taking $X_i$ to $f_i$ for all $i$.

(b) If $K$ is module-finite over the image of $A$, then $R$ is module-finite over the image of $A[[X_1, \ldots, X_n]]$ under the map discussed in part (a).

(c) If the composite map $A \to R \to K$ is surjective, and $\mu + (f_1, \ldots, f_n)R = m$, then the map $h$ described in (a) is surjective.

Proof. (a) This is immediate from the preceding Proposition, since $(X_1, \ldots, X_n)$ maps into $m$.

(b) $A[[X_1, \ldots, X_n]]$ is complete and separated with respect to the the $\mathfrak{A}$-adic topology, where $\mathfrak{A} = (\mu, X_1, \ldots, X_n)A[[X_1, \ldots, X_n]]$. Given a Cauchy sequence of power series $\{f_k\}_k$, it is easy to see that the sequence of coefficients of a fixed monomial $X_1^{n_1} \cdots X_n^{n_n} = X^n$ is a Cauchy sequence in $A$ in the $\mu$-adic topology, and so has a limit $a_\nu \in A$. The only possible limit for the Cauchy sequence $\{f_k\}_k$ is the power series

$$\sum_{\nu \in \mathbb{N}^n} a_\nu X_\nu,$$

and it is easy to verify that this is the limit.

The expansion of the ideal $\mathfrak{A}$ of $A[[X_1, \ldots, X_n]]$ to $R$ is $\mu R + (f_1, \ldots, f_n)R$, which contains a power of $m$, say $m^N$. Thus, $R/\mathfrak{A}R$ is a quotient of $R/m^N$ and has finite length: the latter has a filtration whose factors are the finite-dimensional $K$-vector spaces $m^i/m^{i+1}$, $0 \leq i \leq N - 1$. Since $K$ is module-finite over the image of $A$, it follows that $R/\mathfrak{A}R$ is module finite over over $A[[X_1, \ldots, X_n]]/\mathfrak{A} = A/\mu$. Choose elements of $R$ whose images in $R/\mathfrak{A}R$ span it over $A/\mu$. By the Proposition stated on p. 2, these elements span $R$ as an $A[[X_1, \ldots, X_n]]$-module. We are using that $R$ is $\mathfrak{A}$-adically separated, but this follows because $\mathfrak{A}R \subseteq m$, and $R$ is $m$-adically separated.

(c) We repeat the argument of the proof of part (b), noting that now $R/\mathfrak{A}R \cong K \cong A/\mu$, so that $1 \in R$ generates $R$ as an $A[[X_1, \ldots, X_n]]$ module. But this says that $R$ is a cyclic $A[[X_1, \ldots, X_n]]$-module spanned by 1, which is equivalent to the assertion that $A[[X_1, \ldots, X_n]] \to R$ is surjective. \(\square\)

We have now done all the real work needed to prove the following:

Theorem. Let $(R, m, K)$ be a complete local ring with coefficient field $K_0 \subseteq K$, so that $K_0 \subseteq R \to R/m = K$ is an isomorphism. Let $f_1, \ldots, f_n$ be elements of $m$ generating
an ideal primary to m. Let $K_0[[X_1, \ldots, X_n]] \rightarrow R$ be constructed as in the preceding Proposition, with $X_i$ mapping to $f_i$ and with $A = K_0$. Then:

(a) $R$ is module-finite over $K_0[[X_1, \ldots, X_n]]$.

(b) Suppose that $f_1, \ldots, f_n$ generate $m$. Then the homomorphism $K_0[[x_1, \ldots, x_n]] \rightarrow R$ is surjective. (By Nakayama’s lemma, the least value of $n$ that may be used is the dimension as a $K$-vector space of $m/m^2$.)

(c) If $d = \dim (R)$ and $f_1, \ldots, f_d$ is a system of parameters for $R$, the homomorphism

$$K_0[[x_1, \ldots, x_d]] \rightarrow R$$

is injective, and so $R$ is a module-finite extension of a formal power series subring.

Proof. (a) and (b) are immediate from the preceding Proposition. For part (c), let $\mathfrak{A}$ denote the kernel of the map $K_0[[x_1, \ldots, x_d]] \rightarrow R$. Since $R$ is a module-finite extension of the ring $K_0[[x_1, \ldots, x_d]]/\mathfrak{A}$, $d = \dim (R) = \dim (K_0[[x_1, \ldots, x_d]]/\mathfrak{A})$. But we know that $\dim (K_0[[x_1, \ldots, x_d]]) = d$. Killing a nonzero prime in a local domain must lower the dimension. Therefore, we must have that $\mathfrak{A} = (0)$. □

Thus, when $R$ has a coefficient field $K_0$ and $f_1, \ldots, f_d$ are a system of parameters, we may consider a formal power series

$$\sum_{\nu \in \mathbb{N}^d} c_\nu f_\nu^\nu,$$

where $\nu = (\nu_1, \ldots, \nu_d)$ is a multi-index, the $c_\nu \in K_0$, and $f_\nu^\nu$ denotes $f_1^{\nu_1} \cdots f_d^{\nu_d}$. Because $R$ is complete, this expression represents an element of $R$. Part (c) of the preceding Theorem implies that this element is not 0 unless all of the coefficients $c_\nu$ vanish. This fact is sometimes referred to as the analytic independence of a system of parameters. The elements of a system of parameters behave like formal indeterminates over a coefficient field. Formal indeterminates are also referred to as analytic indeterminates.