Math 615, Winter 2007 **Problem Set #1 Solutions**

1. For positive integers a, b, $x_1^a >_\gamma x_2^b \Leftrightarrow a\gamma > b \Leftrightarrow \gamma > b/a$ from which we have $\gamma = \sup\{b/a : x_1^a >_\gamma x_2^b\}$. It follows that if $>_\gamma$ and $>_{\gamma'}$ are the same order then $\gamma = \gamma'$. $\square$

2. (a) $\mu \neq 1 \Rightarrow W(\mu) > 0 \Rightarrow$ for all $k \gg 0$ $W(\mu^k) = kW(\mu) > W(\mu')$, while $x_1 >_{\mathrm{lex}} x_2^k$ for all k.

(b) Choose integers a, $b > 0$ such that $\gamma_2/\gamma_1 < a/b < 1$, so that $a\gamma_1 > b\gamma_2$ but $a < b$. Then $\deg(x_1^a) < \deg(x_2^b)$ but $x_1^a > x_2^b$, showing that $<$ is neither hlex nor revlex.

3. (a) $G_{1,2} = x_2g_1 - x_1g_2 = -x_1^2x_3 + x_2x_3^2$. Since neither term is divisible by $\mathrm{in}(g_1)$ nor $\mathrm{in}(g_2)$, this is already the remainder in a standard expression for division by g_1, g_2, and we let $g_3 = -x_1^2x_3 + x_2x_3^2$. Then $G_{1,3} = x_1x_3g_1 + x_2g_3 = x_1x_3^3 + x_2^2x_3^2 = x_3^2g_2 + 0$, while $G_{2,3}$ need not be checked since the initial terms are relatively prime. Hence, g_1, g_2, g_3 is a Gröbner basis.

(b) Since the initial terms are relatively prime, h_1, h_2, is already a Gröbner basis.

4. For $k = i$, j, let $g_k = c_k\mu_ke_t + \widetilde{g}_ke_t$, where $\widetilde{g}_k$ is a polynomial such that all terms of $\widetilde{g}_ke_t$ are smaller than $c_k\mu_ke_t$. Since all terms involve e_t we omit it from the notation. We have that $\mathrm{GCD}(\mu_i, \mu_j) = 1$. Then $G_{ij} = c_j\mu_jg_i - c_i\mu_ig_j = (g_j - \widetilde{g}_j)g_i - (g_i - \widetilde{g}_i)g_j = -\widetilde{g}_jg_i + \widetilde{g}_ig_j + 0$. This will be the required standard expression. If either or both of $\widetilde{g}_i$ or $\widetilde{g}_j$ is 0 this is clear. If not, it suffices to show that the initial terms of the two summands do not cancel, for then one is the initial term of G_{ij} and the other is no larger. But if they cancel then $\mathrm{in}(\widetilde{g}_j)\mu_i$ and $\mathrm{in}(\widetilde{g}_i)\mu_j$ are the same up to scalar multiplication, and since $\mathrm{GCD}(\mu_I, \mu_J) = 1$, we must have that $\mu_i|\mathrm{in}(\widetilde{g}_i)$, a contradiction, since $\mathrm{in}(\widetilde{g}_i) < \mu_i$. $\square$

5. The initial forms of the given elements are:
$x_1x_2,\ x_3x_4x_5,\ x_6x_7x_8x_9,\ \ldots,\ x_{\binom{k+1}{2}}x_{\binom{k+1}{2}+1}\cdots x_{\binom{k+1}{2}+k},\ \ldots,\ x_{\binom{n+1}{2}}x_{\binom{n+1}{2}+1}\cdots x_{\binom{n+1}{2}+n}.$
These are relatively prime in pairs: hence, one need not perform any tests in the Buchberger algorithm, and the given elements are a Gröbner basis for the ideal they generate. The displayed initial forms span $\mathrm{in}(I)$. No initial form divides any other, so this is a minimal Gröbner basis. No initial term divides any term of any other element of the Gröbner basis, the coefficients in the initial terms are all 1, and the initial terms are decreasing. Hence, the given elements are already a reduced Gröbner basis for the ideal.

6. For $1 \le i < j \le n$, let $D_{i,j}$ denote $x_ix_{n+j} - x_jx_{n+i}$, the 2×2 minor formed from the columns of x_i and x_j. We claim that $D_{1,2}, \ldots D_{1,n}, D_{2,3}, \ldots, D_{2,n}, \ldots, D_{n-1,n}$ is already a reduced Gröbner basis. Since the initial forms x_ix_{n+j} for $i < j$ are already decreasing with coefficient 1 and no term divides any other, we need only apply the (sharpened) Buchberger criterion. The only cases where the initial terms are *not* relatively prime and need to be checked are (1) $D_{i,j}$, $D_{i,k}$ and (2) $D_{i,k}$, $D_{j,k}$, where $i < j < k$ in both cases. In case (1), we have $x_{n+k}D_{ij} - x_{n+j}D_{ik} = -x_{n+k}x_jx_{n+i} + x_{n+j}x_kx_{n+i} = -x_{n+i}D_{j,k} + 0$, which is already a standard expression for division by the proposed Gröbner basis with remainder 0, since there is only one term. In case (2), similarly, we have that $x_jD_{i,k} - x_iD_{j,k} = -x_jx_kx_{n+i} + x_ix_kx_{n+j} = x_kD_{i,j} + 0$. $\square$