Math 615, Winter 2007 **Problem Set #2 Solutions**

1. For any monomial order on R, when $f,\, g \in R - \{0\}$, $\mathrm{in}(fg) = \mathrm{in}(f)\mathrm{in}(g) \Rightarrow \mathrm{in}(f^k) = \big(\mathrm{in}(f)\big)^k$. If $f \in \mathrm{Rad}\,(I)$ then $f^k \in I$ for some k, and $\big(\mathrm{in}(f)\big)^k = \mathrm{in}(f^k) \in \mathrm{in}(I)$, which is radical, and so $\mathrm{in}(f) \in \mathrm{in}(I)$. Hence $\mathrm{in}(I) \subseteq \mathrm{in}\big(\mathrm{Rad}\,(I)\big) \subseteq \mathrm{in}(I) \Rightarrow \mathrm{in}(I) = \mathrm{in}\big(\mathrm{Rad}\,(I)\big)$ and so $\mathrm{Rad}\,(I) = I$. $\square$ The application to 6. of Problem Set #1 is immediate, since the initial ideal is generated by the square-free monomials $x_i x_{n+j}$ for $i < j$. $\square$

2. $I_\Sigma \subseteq P \in \mathrm{Spec}\,(R) \Rightarrow P$ meets each non-face $\tau \subseteq \{x_1, \ldots, x_n\}$: $\prod_{x_i \in \tau} x_i \in I_\Sigma \subseteq P$. Conversely, if $\tau \subseteq \{x_1, \ldots, x_n\}$ meets every non-face, τ generates a prime $\supseteq I_\Sigma$. Hence, minimal primes of I_Σ correspond to and are generated by minimal sets $\tau \subseteq \{x_1, \ldots, x_n\}$ that meet every non-face of Σ. If $\tau \subseteq \{x_1, \ldots, x_n\}$ generates a prime, $\sigma = \{x_1, \ldots, x_n\} - \tau$ must be a face: the product of the x_i in it is $\neq 0$ even in the quotient. Also, if $\sigma \in \Sigma$, then $\sigma' = \{x_1, \ldots, x_n\} - \sigma$ must meet every non-face τ: $\tau \cap \sigma' = \emptyset \Rightarrow \tau \subseteq \sigma$. It follows at once that the minimal primes of I_Σ correspond bijectively with the maximal faces of Σ. $\square$

3. (a) Let $\mathfrak{P}_R(z) = \sum_{i=0}^\infty a_i z_i$ and $\mathfrak{P}_S(z) = \sum_{j=0}^\infty b_j z^j$ By the definition of multiplication for power series, the coefficient of z^n in the product is $\sum_{i+j=n}^n a_i b_j$, where $i,\, j \in \mathbb{N}$, and since $\dim_K(R_i \otimes_K S_j) = a_i b_j$, the result follows. Since $K[x_1, \ldots, x_n]$ is the iterated tensor product of the rings $R_i = K[x_i]$, and $\mathfrak{P}_{R_i}(z) = \sum_{t=0}^\infty (z^{m_i})^t = 1/(1 - z^{m_i})$, (b) follows. $\square$

4. (a) Consider $0 \to M \xrightarrow{x\cdot} M$. If M is flat (respectively, faithully flat), it remains exact (respectively, its exactness is not affected) after applying $S \otimes_R \_$. Then (b) follows: by (a), each x_{i+1} is a nonzerodivisor on $M_i = M/(x_1, \ldots, x_i)M$ iff it is a nonzerodivisor on $W_i = S \otimes_R M_i \cong (S \otimes_R M)/(x_1, \ldots, x_i)(S \otimes_R M)$; moreover, $M_n \neq 0 \iff W_n \neq 0$ by *faithful* flatness. $\square$ (c) is clear, since $L \otimes_K \_$ preserves module-finiteness (the same generators work), and module-finite (or even integral) extensions preserve dimension. (d) is immediate from (c), the fact that $F_1, \ldots, F_n$ is a homogeneous system of parameters iff R is module-finite over the polynomial ring $K[F_1, \ldots, F_n]$, the faithful flatness of L over K, the definition of the Cohen-Macaulay property, and part (b).

5. The quotient of $R/(x_1, x_2) \cap (x_3, x_4) = R/(x_1x_3, x_1x_4, x_2x_3, x_2x_4)$ by each minimal prime has dimension 2, and so $\dim\,(R/I) = 2$. Mod $(x_1 - x_3,\, x_2 - x_4)$, we obtain $K[x_1, x_2]/(x_1^2, x_1x_2, , x_2^2)$, which is 0-dimensional. Hence, $x_1 - x_3,\, x_2 - x_4$ is a homgeneous system of parameters. Mod $(x_1 - x_3)$ R/I becomes $K[x_1,\, x_2,\, x_4]/(x_1^2, x_1x_4, x_1x_2, x_2x_4)$, The image of $x_2 - x_4$ is a zerodivisor, since it kills the image of x_1, which is nonzero because it has too small a degree to be in $(x_1^2,\, x_1x_2,\, x_2^2)$. $\square$

6. Since intersection distributes over sum for monomial ideals, $IT \cap JT$ is the sum of the ideals $\mu T \cap \nu T$ where μ and ν are monomial generators of I and J, respectively. Since μ and ν involve disjoint sets of variables, every $\mu T \cap \nu T = \mu\nu T$. Hence, $IT \cap JT = IJT$. We have a short exact sequence $(*)$ $\;0 \to T/\big((IT) \cap (JT)\big) \to T/IT \bigoplus T/JT \to T/(I+J)T \to 0$, where the maps preserve degree. The leftmost nonzero term is T/IJT. Now $T/IT \cong (R/I) \otimes_K S$ and so, by Problem 3., $\mathfrak{P}(T/IT) = \dfrac{F(z)}{(1-z)^r}$. Similarly, $\mathfrak{P}(T/JT) = \dfrac{G(z)}{(1-z)^n}$. Also, $T/(I+J)T \cong (R/I) \otimes (S/J)$ and so $\mathfrak{P}_{T/(I+J)T}(z) = F(z)G(z)$. The alternating sum of the Hilbert-Poincaré series of the terms in the exact sequence $(*)$ is zero. $\square$