Math 615, Winter 2007 **Problem Set #3 Solutions**

1. When one localizes at $(*)$ $P \in \mathrm{Supp}\,(M/IM)$, a regular sequence in I on M maps to a regular sequence in IR_P on M_P by flatness: one has $M_P/IM_P \neq 0$ by $(*)$. So the depth cannot decrease. Pick a maximal regular sequence $\underline{f} = f_1, \dots, f_d$ in I, so that $\mathrm{depth}_I M/(\underline{f})M = 0$. Then I kills $u \in M/(\underline{f})M - \{0\}$, and so $I \subseteq P \in \mathrm{Ass}\,\big(M/(\underline{f})M\big)$. This still holds after localization at P, so $\mathrm{depth}_I M_P/(\underline{f})M_P = 0$ and $\mathrm{depth}_{IR_P} M_P = d$.

To prove that the depth cannot exceed the number of generators of I, nor $\dim\,(M)$, we may use the first part: localize at a minimal prime of M/IM (the depth cannot decrease and the number of generators and the dimension cannot increase) and replace R by $R/\mathrm{Ann}_I M$ (the number of generators cannot increase, while $\mathrm{depth}_I M$ and $\dim\,(M)$ stay the same). Now (R, m) is local with $\mathrm{Rad}\,(I) = m$, and $\dim\,(M) = \dim\,(R) \leq$ the number of generators of I. Since $\mathrm{depth}_I M \leq \mathrm{depth}_m M$, it suffices to show $d = \mathrm{depth}_m M \leq \dim\,(R) = \dim\,(M)$. Use induction on $\dim\,(R)$. If $\dim\,(R) = 0$, a power of m is 0 and kills M. Then $\mathrm{depth}_I M = \dim\,(R) = 0$. If $\dim\,(R) > 0$ and $d = 0$ we are done. If $d > 0$ choose $x \in m$ avoiding the minimal primes of R and the primes of $\mathrm{Ass}\,(M)$. The result now follows from the induction hypothesis applied to R/xR ($\dim\,(R/xR) = \dim\,(R) - 1$) and M/xM (depth $M/xM = d - 1$). For the final statement note that if one has a regular sequence on M, say $\underline{f} = f_1, \dots, f_d$, in $\mathrm{Rad}\,(I)$, for some n, $f_1^n, \dots f_d^n \in I$ *and* is a regular sequence on M. This is not so easy to prove in general. However, we can first reduce to the local case, where regular sequences are permutable, and it is easy to see that replacing the last term by a power does not affect whether one has a regular sequence. By permutability, this is true for any term. (Alternate: if $\underline{f}$ is maximal regular in I and $I \subseteq P \in \mathrm{Ass}\,\big(M/(\underline{f})M\big)$, then $\mathrm{Rad}\,(I) \subseteq P$ as well $\Rightarrow \underline{f}$ maximal regular in $\mathrm{Rad}\,(I)$.) $\square$

2. (a) If $\mathrm{depth}_I(M'') = 0$ choose $u \in M$ whose image in M'' is killed by I. Then choose any nonzerodivisor $f \in I$ on M. Then f is a nonzerodivisor on $M' \subseteq M$, and $fu \in M'$, since I kills the image of u in $M'' \cong M/M'$. However, $fu \notin fM'$ or else $u \in M'$. But $I(fu) = f(Iu) \subseteq fM'$, so that the image of fu in M'/fM' is a nonzero element killed by I. It follows that f is a maximal regular sequence M'. We complete the proof by induction on $d = \mathrm{depth}_I M''$. If $d > 0$ then all three depths are positive. We can choose $f \in I$ avoiding all three sets of associated primes. Then $0 \to fM' \to fM \to fM'' \to 0$ is isomorphic to the original exact sequence, and so is exact, and the sequence of cokernels $(*)\quad 0 \to M'/fM' \to M/fM \to M''/fM'' \to 0$ is exact. The result is immediate from the induction hypothesis applied to $(*)$. $\square$

(b) We have $\mathrm{depth}_m R = n$. By part (a), each time we take a graded module of syzygies M_1 of M, if $\mathrm{depth}_m M < n$ then $\mathrm{depth}_m M_1 = \mathrm{depth}_m M + 1$. The result is immediate from the given characterization of free modules. $\square$

3. (a) Suppose one has an infinite strictly decreasing sequence of finite sets. The largest elements are non-increasing and so must be eventually stable. Call the stable value a_1. Recursively, suppose that for $i \leq k$ the i th largest elements are eventually stable with stable value a_i. From the point where all are stable on, the $k+1$ st largest elements are non-increasing and so eventually stable as well, say with stable value a_{k+1}. The recursion yields an infinite strictlly decreasing sequence $a_1 > \cdots > a_k > \cdots$, a contradiction. $\square$

(b) At each step in the process, the set of monomials in the current remainder decreases in the sense of part (a), and so the process must terminate. □

4. (a) Let $f = r/s$, $r \in R$, $s \in R - \{0\}$, be invariant. Let $t = \prod_{g \in G - \{e\}} g(s)$, so that st is the product of elements in an orbit and is in R^G. Then $f = rt/st$, and since f, st are fixed by G, so is $rt = f(st)$. Hence $rt \in R^G$ and $f \in \mathrm{frac}\,(R^G)$. □

(b) If K is infinite, $R^G = K$: a polynomial is fixed by G iff all the monomials in it are fixed by G, and G fixes no monomial except 1. Hence, $\mathrm{frac}\,(R^G) = K$. But $y/x \in \mathcal{F}^G$.

(c) $R^G = R \cap \mathcal{F}^G$, both of which are intergrally closed, and so R^G is as well. □

5. It is easy to see that an elementary row or column operation does not affect the ideal generated by the $t \times t$ minors of the matrix. With $x = x_{mn}$ invertible, we may subtract x_{in}/x times the bottom row from the ith row, $1 \le i \le m-1$. Then x_{ij} changes to $y_{ij} = x_{ij} - x_{in}x_{nj}/x$, $1 \le i \le m-1$, $1 \le j \le n-1$, while the bottom row is unchanged and the last column becomes 0 except for the bottom entry x. We may then subtract x_{mj}/x times the last column from the jth column, $1 \le j \le n-1$, and multiply the last row by $1/x$. The matrix has become $\begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix}$ in block form, where $Y = (y_{ij})$ is $(m-1) \times (n-1)$. The $t \times t$ minors that involve 1 correspond bijectively to the $t-1$ size minors of Y, and each, up to sign, equals one of them. Those that do not involve 1 are obviously in $I_{t-1}(Y)$: expand by minors with respect one row or column. Hence, $I_{t-1}(Y)K[X]_x = I_t(X)K[X]_x$. The y_{ij} together with the variables x_{mj}, $1 \le j \le n$, x_{in}, $1 \le i \le n-1$ are algebraically independent over K and together with $1/x = 1/x_{mn}$ generate $K[X]_x$. So $K[X]_x$ is S_x where S is the polynomial ring in the x_{nj} and x_{in} over $K[Y]$. The claim follows. □

6. We show the 2×2 minors are a Gröbner basis using the Buchberger criterion. Thus, the initial ideal is generated by products of pairs of elements on the “back” diagonals: the *back diagonal* consists of the two elements not on the main diagonal. Since the ideal is graded and x_{mn} is not on a back diagonal, it is not a zerodivisor on the initial ideal, and so is not a zerodivisor on $I_2(X)$. With notation as in 5., one wants to show that $I_1(Y)$ is prime, which is clear: it is generated by a subset of the variables. It remains to apply the Buchberger criterion. If the two minors involve 4 columns or 4 rows, the back diagonals do not overlap, and so the initial terms are relatively prime. If the minors lie in a 2×3 (resp., 3×2) submatrix, the check is the same as in 6. of Problem Set #1: if x is the other variable in the column (resp., row) of the common element of the back diagonals, one gets $\pm x$ times the third 2×2 minor. Let $\Delta_{ab,ij} = x_{ai}x_{bj} - x_{aj}x_{bi}$. We may assume the 2 minors lie in rows indexed $a < b < c$ and columns indexed $i < j < k$, all needed, and that their back diagonals meet in one element. There are five cases not already covered: the pairs of back diagonals (each written as a product) are: (1) $x_{ci}x_{bj}$, $x_{ci}x_{ak}$, (2) $x_{ci}x_{aj}$, $x_{ci}x_{bk}$, (3) $x_{bj}x_{ci}$, $x_{bj}x_{ak}$, (4) $x_{ak}x_{bj}, x_{ak}x_{ci}$, and (5) $x_{ak}x_{bi}, x_{ak}x_{cj}$. The 5 checks, all similar, are:

(1) $x_{ak}\Delta_{bc,ij} - x_{bj}\Delta_{ac,ik} = -x_{bi}\Delta_{ac,jk} - x_{ck}\Delta_{ab,ij}$.

(2) $x_{bk}\Delta_{ac,ij} - x_{aj}\Delta_{bc,ik} = -x_{ai}\Delta_{bc,jk} + x_{ck}\Delta_{ab,ij}$.

(3) $x_{ak}\Delta_{bc,ij} - x_{ci}\Delta_{ab,jk} = x_{aj}\Delta_{bc,ik} - x_{bi}\Delta_{ac,jk}$.

(4) $x_{ci}\Delta_{ab,jk} - x_{bj}\Delta_{ac,ik} = -x_{aj}\Delta_{bc,ik} - x_{ck}\Delta_{ab,ij}$.

(5) $x_{cj}\Delta_{ab,ik} - x_{bi}\Delta_{ac,jk} = -x_{ai}\Delta_{bc,jk} + x_{ck}\Delta_{ab,ij}$.

The initial terms of righthand products don’t cancel, so these are standard epxressions. □