

1. From 6. of Problem Set #1 with  $n = 3$  and  $x_{ij} = x_{i+j-1}$  the  $\Delta_i$  are a Gröbner basis for hlex: and generating relations come from the checks in Buchberger's criterion: the ones coming from cases (1) and (2) of the Problem Set #1 Solutions are  $x_{i1}\Delta_1 - x_{i2}\Delta_2 + x_{i3}\Delta_3 = 0$  for  $i = 2, i = 1$ , resp., yield relations  $\rho_i = (x_{i1}, -x_{i2}, x_{i3})$  for  $i = 1, 2$ . The third check is not needed here, but we have not proved this. It gives  $x_{11}x_{22}\Delta_1 - x_{12}x_{23}\Delta_3 = x_{12}x_{21}\Delta_1 - x_{13}x_{22}\Delta_2$ ; hence the relation  $(\Delta_3, 0, -\Delta_1) = x_{22}\rho_1 - x_{11}\rho_2$ . Thus,  $\rho_1$  and  $\rho_2$  generate, confirming the resolution from the Notes for February 26.  $\square$

2. Let  $\underline{F} = F_1, \dots, F_n$  be a homogenous system of parameters for  $R$ .  $S$  is module-finite over  $R \Rightarrow S/(\underline{F})S$  module-finite over  $R/(\underline{F})R \Rightarrow \dim(S/(\underline{F})S) = 0 \Rightarrow F_1, \dots, F_n$  is a homogeneous system of parameters in  $S$ , and so a regular sequence on  $S$ . Since  $S = R \oplus W$  over  $R$  for some  $W$ ,  $\underline{F}$  regular on  $R \oplus W \Rightarrow$  regular on  $R$  by induction on  $n$ .

The Veronese subring contains the  $k$ th power of each generator, and so  $R$  is module-finite over  $R^{(k)}$ . This map is split:  $\bigoplus_{j \not\equiv 0 \pmod k} [R]_j$  is a complement. Apply the first part.

3. Let  $X' = X - x_{11}\mathbf{I}$  and  $Y' = Y - y_{11}\mathbf{I}$ . Then  $XY - YX = X'Y' - Y'X'$ , and  $K[X, Y] = K[X', Y'][x_{11}, y_{11}]$ . If we use  $x'_{22} = x_{22} - x_{11}$  and  $y'_{22} = y_{22} - y_{11}$ , we have that  $K[X, Y]/I_1(XY - YX) \cong (K[X', Y']/I_1(X'Y' - Y'X'))[x_{11}, y_{11}]$ . Here,  $X', Y'$  are matrices of indeterminates except that there are 0s in the upper left corners. But  $I_1(X'Y' - Y'X')$  is  $I_2(Z)$  where  $Z$  is the  $2 \times 3$  matrix with rows  $x_{12}, x_{21}, x'_{22}$  and  $y_{12}, y_{21}, y'_{22}$ .

4. Let  $f \in R^H$  and suppose  $f \in V \subseteq R$  where  $V$  is finite-dimensional as a  $K$ -vector space and  $G$ -stable. We have a regular map  $G \rightarrow V \times V$  via  $\gamma \mapsto (\gamma(v), v)$ . The inverse image of the diagonal  $\Delta \subseteq V \times V$  is closed, and contains  $H$ . Since  $H$  is dense, it contains  $G$ . The fact that  $G$  maps into  $\Delta$  implies that all of  $G$  fixes  $f$ .

Let  $\overline{H}$  be the closure of the unitary matrices  $H$ . Fix  $h \in H$ . Left multiplication by  $h$  is an automorphism of  $G$  as an algebraic set that stabilizes  $H$ . Hence, it stabilizes  $\overline{H}$ , and  $H\overline{H} \subseteq \overline{H}$ . Fix  $g \in \overline{H}$ . Then right multiplication by  $g^{-1}$  is an automorphism of  $G$  as an algebraic set and since  $\overline{H}g^{-1}$  is a closed set containing  $H$ , it contains  $\overline{H}$ . Hence,  $\overline{H}\overline{H} \subseteq \overline{H}$ . Likewise,  $g \mapsto g^{-1}$  is an automorphism of the algebraic set  $G$  that stabilizes  $H$ , and so it stabilizes  $\overline{H}$ . Hence,  $\overline{H}$  is a subgroup of  $G$ .

To show that  $\overline{H} = G$ , it suffices to show  $\overline{H}$  contains the diagonal and elementary matrices. Let  $D_1$  be the set of diagonal matrices  $D_1$  with an element of  $\mathbb{C} - \{0\}$  in the 1, 1 spot and 1 elsewhere on the diagonal. Since the elements of absolute value 1 are Zariski dense in  $\mathbb{C} - \{0\}$  (any infinite set is dense),  $\overline{H}$  contains  $D_1$ , and similarly, all diagonal matrices such that diagonal entries are 1 except in one spot. Since  $\overline{H}$  is a group,  $\overline{H}$  contains all diagonal matrices. Since an elementary matrix is the direct sum of an identity matrix with a  $2 \times 2$  matrix, whether the elementary matrices are in  $\overline{H}$  reduces to the case  $n = 2$ . Orthogonal matrices are unitary, and symmetric matrices are orthogonally conjugate to diagonal matrices. Thus,  $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in \overline{H}$  and  $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \overline{H}$ , so that  $\begin{pmatrix} 1 & 0 \\ 0 & c^{-1} \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \in \overline{H}$ ,  $c \neq 0$ , and this is a typical upper triangular elementary matrix. Lower triangular elementary matrices can be obtained similarly. (Alternatively one may use the fact that every invertible matrix is the product of a unitary matrix and a Hermitian matrix.)

5. (a) If  $f^2, f^3 \in R$  then  $f^n \in R$  for all  $n \geq 2$  in  $\mathbb{N}$ . Hence,  $f^p \in R$ , whence under  $F : R \rightarrow R = S$ ,  $F(f^3) \in F(f^2)S$ . Since  $F$  is split, ideals are contracted, and we have that  $f^3 \in f^2R$  so that  $f \in R$ .

If  $R$  is a polynomial ring over a perfect field  $K$ , a polynomial is a  $p$ th power iff all exponents are divisible by  $p$ .  $F(R) \subseteq R$  has a complement the  $K$ -span of all monomials such that at least one exponent occurring is not divisible by  $p$ . Call the corresponding splitting  $\phi$ . The same description, applied to monomials that are nonzero mod  $I$ , gives a splitting of  $F(R/I) \rightarrow R/I$  when  $I$  is generated by square-free monomials. In part (c),  $G$  acts on monomials, and a  $K$ -basis for  $R^G$  consists of sums of orbits. One monomial in an orbit has all exponents divisible by  $p$  if and only if all the monomials occurring do. It follows that  $\phi$  maps  $R^G$  into  $F(R^G)$  and this implies that it splits  $F(R^G) \subseteq R^G$ .

6. Let  $I_t = (\Delta_1^t, \Delta_2^t, \Delta_3^t)S$  and  $I = I_1$ . We shall sketch a proof that  $(\Delta_1\Delta_2\Delta_3)^{p-1} \in I_pS$ , which is clearly not true in  $R$ , where the  $\Delta_j$  are indeterminates. We leave filling in the details of the sketch as a bonus exercise. First note that we have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \xrightarrow{d_3} & S^3 & \xrightarrow{d_2} & S^3 & \xrightarrow{d_1} & S & \longrightarrow & 0 \\ & & \phi_3 \uparrow & & \phi_2 \uparrow & & \phi_1 \uparrow & & 1_S \uparrow & & \\ 0 & \longrightarrow & S & \xrightarrow{D_3} & S^3 & \xrightarrow{D_2} & S^3 & \xrightarrow{D_1} & S & \longrightarrow & 0 \end{array}$$

where the matrix of  $d_1$  is  $(\Delta_1 \ \Delta_2 \ \Delta_3)$ , the matrix of  $d_2$  is  $\begin{pmatrix} 0 & \Delta_3 & -\Delta_2 \\ -\Delta_3 & 0 & \Delta_1 \\ \Delta_2 & -\Delta_1 & 0 \end{pmatrix}$ ,  $d_3$

is the transpose of  $d_1$ , and each  $D_j = F(d_j)$ . Also,  $\phi_1 = \text{diag}(\Delta_1^{p-1}, \Delta_2^{p-1}, \Delta_3^{p-1})$ ,  $\phi_2 = \text{diag}(\Delta_2^{p-1}\Delta_3^{p-1}, \Delta_1^{p-1}\Delta_3^{p-1}, \Delta_1^{p-1}\Delta_2^{p-1})$ , and  $\phi_3 = (\Delta_1^{p-1}\Delta_2^{p-1}\Delta_3^{p-1})$ . The top row resolves the ideal generated by the  $\Delta_j$  over  $R$ , but is not acyclic over  $S$ , although one can check that  $\text{Im}(d_3) = \text{Ker}(d_2)$ . One can give a different map of complexes using a middle row:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \xrightarrow{d_3} & S^3 & \xrightarrow{d_2} & S^3 & \xrightarrow{d_1} & S & \longrightarrow & 0 \\ & & 0 \uparrow & & H \uparrow & & \phi_1 \uparrow & & 1 \uparrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & S^2 & \xrightarrow{C} & S^3 & \xrightarrow{D_1} & S & \longrightarrow & 0 \\ & & 0 \uparrow & & G \uparrow & & 1 \uparrow & & 1 \uparrow & & \\ 0 & \longrightarrow & S & \xrightarrow{D_3} & S^3 & \xrightarrow{D_2} & S^3 & \xrightarrow{D_1} & S & \longrightarrow & 0 \end{array}$$

The middle row is acyclic: it is the result of applying  $F$  to the resolution of  $I$  developed in problem 1, and resolves  $I_p = I^{[p]}$ . The map  $G$  is easy to construct using that the middle row is exact at  $S^3$ . The existence of  $H$  is subtle and critical. In the top row, let  $B = \text{Im}(d_2) \subseteq \text{Ker}(d_3) = Z$ . One shows that  $(IS^3 \cap Z) \subseteq B$  (this part of the remaining problem). Then one uses that  $\phi_1 \circ C$  has image in  $(IS^3) \cap Z$  to construct  $H$ . The maps  $H \circ G$  and  $\phi_2$  both make the middle square commute (the other arrows are the same in the two diagrams). It follows that  $d_2 \circ (\phi_2 - HG) = 0$ . Hence  $g = \phi_2 - HG$  maps  $S^3$  into  $d_3(S) \cong S$ . But then  $\phi_3 - 0 = g \circ D_3$ : considering the definitions of  $\phi_3$  and  $D_3$ , this gives the required result.  $\square$