1. Prove that a direct limit of formally unramified (respectively, formally étale) $R$-algebras is formally unramified (respectively, formally étale).

2. Show that a direct limit of Henselian quasilocal rings in which the homomorphisms are local is Henselian.

3. Let $K$ be a field of characteristic 0, and construct an increasing sequence of $K$-algebras $R_n$ as follows. Let $X = X_1$ be an indeterminate, and let $R_1 = K[X_1]$. Recursively, suppose that $R_n = K[X_n]$ is a polynomial ring in one variable over $K$, $n \geq 1$, and let $R_{n+1} = R_n[X_{n+1}]/(X_{n+1}^2 - X_n)$. (One may think of $X_n$ as $X_1/2^{n-1}$.) Let $R = \bigcup_{n=1}^{\infty} R_n$, a directed union of polynomial rings in one variable. Determine whether $R$ is formally smooth over $K$. Prove your answer.

4. Let $K_1, \ldots, K_n, \ldots$ be fields, and let $R = \prod_{n=1}^{\infty} K_n$ be their product.
   (a) Show that every element of $R$ is the product of a unit and an indempotent, and show that $R$ is a zero-dimensional ring, i.e., that every prime ideal is maximal.
   (b) Suppose all of the $K_n$ are $K$-algebras, where $K$ is a field, and assume as well either that (1) infinitely many $K_n$ are infinite fields or (2) there is an infinite set of $K_n$ of finite cardinality such that the cardinals of these $K_n$ are not bounded. (1) is immediate if $K$ is infinite.) Show that $R$ contains an element that is transcendental over $K$.
   (c) Suppose also that $K$ has characteristic 0 (e.g., this holds if $K = K_n = \mathbb{Q}$ for all $n$). Show that $R$ is not formally unramified over $K$.

5. Let $(R, m, K)$ be a Henselian ring. Let $Z_1, \ldots, Z_n$ be indeterminates over $R$: the subscripts should be read modulo $n$. Let $u_1, \ldots, u_n \in m$ and let $r_1, \ldots, r_n \in R$. Let $h_1, \ldots, h_n$ be integers all of which are $\geq 2$.

$$u_1 Z_1^{h_1} + Z_2 = r_1$$
$$\ldots$$
$$u_i Z_i^{h_i} + Z_{i+1} = r_i$$
$$\ldots$$
$$u_n Z_n^{h_n} + Z_1 = r_n$$

Show that the $n$ simultaneous equations have a solution in $R$. (One approach is to make use of a suitable pointed étale extension.)

6. Let $A = K[[x, y]]$ be the formal power series ring in two variables over a field $K$. Let $P = xA$, which is a prime ideal. Prove that the local ring $R_P$ is not Henselian by showing that there is a monic polynomial $F = F(Z)$ in one indeterminate over $R$ such that when $F$ is considered modulo $PR_P$, it has simple roots, but such that $F$ has no root in $R$.

EXTRA CREDIT 5. Show that there exists an algebra $S$ over a Noetherian ring $R$ of prime characteristic $p > 0$ such that $S$ is module-finite and free as an $R$-module and $\Omega_{S/R}$ is free as an $S$-module, but $S$ is not smooth over $R$. (This shows that in characteristic $p$, one needs a further condition, e.g., on the rank of $\Omega_{S/R}$, in order to conclude that $S$ is smooth over $R$.)