Math 615, Fall 2010

Problem Set #5

Due: Wednesday, April 21

1. Prove that if $R$ and $S$ are separated $K$-algebras, then $R \otimes_K S$ is separated.

2. Prove that if $(R, m, K)$ is a local ring that is an approximation ring then
   (a) if $R$ is a domain, so is $\hat{R}$ and (b) if $R$ is reduced, so is $\hat{R}$.

3. Let $T$ be a formally smooth $R$-algebra, $I \subseteq T$ an ideal, and let $S = T/I$. Let $\sim$ indicate images mod $I^2$, and $[\cdot]$ mod $I$ or mod $\Omega T_R$. The restriction of $d : T \rightarrow \Omega T_R$ to $I$ gives a map $I \rightarrow \Omega T_R$ and, hence, $I \rightarrow \Omega T_R/I\Omega T_R \cong S \otimes_T \Omega T_R$.
   (a) Show that the map described kills $I^2$, and so induces a map $\eta : I/I^2 \rightarrow \Omega T_R/I\Omega T_R$ such that if $f \in I$, $\overline{f} \mapsto [df]$. Prove also that $\eta$ is $S$-linear.

   We know that for any surjection $T \rightarrow T/I = S$ of $R$-algebras, $\Omega_S$ is the quotient of $S \otimes_T \Omega T_R$ by the $R$-span of the image of $\{df : f \in I\}$. Hence, there is an exact sequence $I/I^2 \xrightarrow{\sim} \Omega T_R/I\Omega T_R \rightarrow \Omega S_R \rightarrow 0$ of $S$-modules.

   (b) Show that giving a splitting, $\phi : T/I \rightarrow T/I^2$, as $R$-algebras, of the map $T/I^2 \rightarrow T/I$, is equivalent to giving, for all $t \in T$, an element $\delta(t) \in I/I^2$ such that $\phi([t]) = I - \delta(t)$, subject to the conditions that $\delta : T \rightarrow I/I^2$ be an $R$-derivation and that $\delta(df) = \overline{f}$ for all $f \in I$. Thus, $\delta$ corresponds to a $T$-linear map $\theta : \Omega T_R \rightarrow I/I^2$ such that $\delta = \theta \circ d$, and for all $f \in I$, $\theta([\eta f]) = \overline{f}$. Show also that if $f \in I$ and $t \in T$, $\theta(df t) = 0$, so that $\delta$ induces an $S$-linear map map $\Omega T_R/I\Omega T_R \rightarrow I/I^2$.

   (c) Conclude that $S$ is formally smooth over $R$ iff $\eta : I/I^2 \rightarrow \Omega T_R/I\Omega T_R$ splits as a map of $S$-modules (hence, also, iff $\eta$ is injective and $\Omega T_R/I\Omega T_R \rightarrow \Omega S_R$ splits over $S$).

4. Let $K$ be a field, and let $R = K[X_1, X_2, X_3, \ldots, X_n]$ be the polynomial ring in countably many variables over $K$, let $m = (X_n : n \geq 1)\hat{R}$, let $S$ by the $m$-adic completion of $R$, and let $f_n = \sum_{j=n}^{\infty} x_j^n$. Is $f_n \in m^n S$? Prove your answer.

5. Let $(A, m, K)$ be a complete local ring and let $R = A[[X_1, \ldots, X_n]]$ be a formal power series ring over $A$. Let $f_1, \ldots, f_n$ be elements of the maximal ideal of $R$ whose images in $R/mR \cong K[[X_1, \ldots, X_n]]$ form a system of parameters. Prove that $R/(f_1, \ldots, f_n)R$ is module-finite over $A$.

6. Let $D$ be a principal ideal domain such that $p_1 D, \ldots, p_n D \ldots$ is an enumeration of the distinct maximal ideals of $D$ (e.g., one may take $D = \mathbb{Z}$ or $\mathbb{Q}[x]$). Let $G$ be the free $D$-module with free basis $e_1, \ldots, e_n, \ldots$, and let $f : G \rightarrow G$ be such that $f(e_n) = e_n - p_n e_{n+1}$ for all $n$. Let $C = \text{Coker}(f)$, so that $(\ast) 0 \rightarrow G \xrightarrow{f} G \rightarrow C \rightarrow 0$ is exact. Let $L$ be the fraction field of $D$. Let $W$ be the set of square-free elements of $D$. Show that $C \cong \{a/b : a \in D, b \in W\} \subseteq L$. Show that $(\ast)$ is locally split but not split.

EXTRA CREDIT 8. Let $D$ be as in #6, and let $T = K[X_n : n \geq 1]$ be a polynomial ring in a countably infinite set of variables over $D$. Let $I = (X_n - p_n X_{n+1} : n \geq 1)\hat{T}$, and $S = T/I$. Prove that $S_Q$ is formally smooth over $D$ for all $Q \in \text{Spec}(S)$: in fact $S_P$ is smooth over $D_P$ for all $P \in \text{Spec}(D)$. Prove that $S$ is not formally smooth over $D$.  