Categories and functors, the Zariski topology, and the functor Spec

We do not want to dwell too much on set-theoretic issues but they arise naturally here. We shall allow a class of all sets. Typically, classes are very large and are not allowed to be elements. The objects of a category are allowed to be a class, but morphisms between two objects are required to be a set.

A category \( \mathcal{C} \) consists of a class \( \text{Ob} (\mathcal{C}) \) called the objects of \( \mathcal{C} \) and, for each pair of objects \( X, Y \in \text{Ob} (\mathcal{C}) \) a set \( \text{Mor} (X, Y) \) called the morphisms from \( X \) to \( Y \) with the following additional structure: for any three given objects \( X, Y \) and \( Z \) there is a map

\[
\text{Mor} (X, Y) \times \text{Mor} (Y, Z) \rightarrow \text{Mor} (X, Z)
\]

called composition such that three axioms given below hold. One writes \( f : X \rightarrow Y \) or \( X \xrightarrow{f} Y \) to mean that \( f \in \text{Mor} (X, Y) \). If \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) then the composition is denoted \( g \circ f \) or \( gf \). The axioms are as follows:

\begin{enumerate}
\item[(0)] \( \text{Mor} (X, Y) \) and \( \text{Mor} (X', Y') \) are disjoint unless \( X = X' \) and \( Y = Y' \).
\item[(1)] For every object \( X \) there is an element denoted \( 1_X \) or \( \text{id}_X \) in \( \text{Mor} (X, X) \) such that if \( g : W \rightarrow X \) then \( 1_X \circ g = g \) while if \( h : X \rightarrow Y \) then \( h \circ 1_X = h \).
\item[(2)] If \( f : W \rightarrow X \), \( g : X \rightarrow Y \), and \( h : Y \rightarrow Z \) then \( h \circ (g \circ f) = (h \circ g) \circ f \) (associativity of composition).
\end{enumerate}

The morphism \( 1_X \) is called the identity morphism on \( X \) and one can show that it is unique. If \( f : X \rightarrow Y \) then \( X \) is called the domain of \( f \) and \( Y \) is called the codomain, target, or range of \( f \), but it is preferable to avoid the term “range” because it is used for the set of values that a function actually takes on. A morphism \( f : X \rightarrow Y \) is called an isomorphism if there is a morphism \( g : Y \rightarrow X \) such that \( gf = 1_X \) and \( fg = 1_Y \). If it exists, \( g \) is unique and is an isomorphism from \( Y \rightarrow X \). If there is an isomorphism from \( X \rightarrow Y \) then \( X \) and \( Y \) are called isomorphic.

**Examples.** (a) Let the class of objects be the class of all sets, let the morphisms from a set \( X \) to a set \( Y \) be the functions from \( X \) to \( Y \), and let composition be ordinary composition of functions. In this category of sets and functions, two sets are isomorphic if and only if they have the same cardinality.

In the next few examples the objects have underlying sets and composition coincides with composition of functions.

(b) Rings and ring homomorphisms form a category.

(c) Commutative rings with identity and ring homomorphisms that preserve the identity form a category.

(d) For a fixed ring \( R \), \( R \)-modules and \( R \)-linear homomorphisms form a category.

Examples (c) and (d) give the environments in which we’ll be “living” during this course.

(e) Groups and group homomorphisms are another example of a category.
We pause to review some basics about topological spaces before continuing with our examples.

A topology on a set \( X \) is a family of sets, called the open sets of the topology satisfying the following three axioms:

0) The empty set and \( X \) itself are open.
1) A finite intersection of open sets is open.
2) An arbitrary union of open sets is open.

A set is called closed if its complement is open. A topological space is a set \( X \) together with a topology. Such a space may be described equally well by specifying what the closed sets are. They must satisfy:

0) The empty set and \( X \) itself are closed.
1) A finite union of closed sets is closed.
2) An arbitrary intersection of closed sets is closed.

A subset \( Y \) of a topological space \( X \) becomes a topological space in its own right: one gets the topology by intersecting the open sets of \( X \) with \( Y \). (The closed sets of \( Y \) are likewise gotten by intersecting the closed sets of \( X \) with \( Y \).) The resulting topology on \( Y \) is called the inherited topology, and \( Y \) with this topology is called a (topological) subspace of \( X \).

A topological space is called \( T_0 \) if for any two distinct points there is an open set that contains one of them and not the other. It is called \( T_1 \) if every point is closed. It is called \( T_2 \) or Hausdorff if for any two distinct points \( x \) and \( y \) there are disjoint open sets \( U \) and \( V \) such that \( x \in U \) and \( y \in V \).

A family of open subsets of a topological space \( X \) (following the usual imprecise practice, we mention the underlying set without mentioning the topology) is called an open cover if its union is all of \( X \). A subset of such a family whose union is all of \( X \) is called a subcover. A topological space is called quasi-compact if every open cover has a subcover containing only finitely many open sets, i.e., a finite subcover.

A family of sets is said to have the finite intersection property if every finite subfamily has non-empty intersection. Being quasi-compact is equivalent to the condition that every family of closed sets with the finite intersection property has non-empty intersection. (This is only interesting when the family is infinite.) A quasi-compact Hausdorff space is called compact. We assume familiarity with the usual topology on \( \mathbb{R}^n \), in which a set is closed if and only if for every convergent sequence of points in the set, the limit point of the sequence is also in the set. Alternatively, a set \( U \) is open if and only if for any point \( x \) in the set, there exists \( a > 0 \) in \( \mathbb{R} \) such that all points of \( \mathbb{R}^n \) within distance of \( a \) of \( x \) are in \( U \).

The compact subspaces of \( \mathbb{R}^n \) are precisely the closed, bounded sets.

A topological space is called connected if it is not the union of two non-empty disjoint open subsets (which will then both be closed as well). The connected subsets of the real line are identical with the intervals: these are the subsets with the property that if they
contain $a$ and $b$, they contain all real numbers in between $a$ and $b$. They include the empty set, individual points, open intervals, half-open intervals, closed intervals, and the whole line.

A function $f$ from a topological space $X$ to a topological space $Y$ is called continuous if for every open set $V$ of $Y$, $f^{-1}V = \{x \in X : f(x) \in V\}$ is open. It is an equivalent condition to require that the inverse image of every closed set be closed.

We are now ready to continue with our discussion of examples of categories.

(f) Topological spaces and continuous maps give a category. In this category, isomorphism is called homeomorphism.

We now consider some examples in which composition is not necessarily composition of functions.

(g) A partially ordered set (or poset) consists of a set $P$ together with a relation $\leq$ such that for all $x, y, z \in P$, (1) if $x \leq y$ and $y \leq x$ then $x = y$ and (2) if $x \leq y$ and $y \leq z$ then $x \leq z$. Given a partially ordered set, we can construct a category in which the objects are the elements of the partially ordered set. We artificially define there to be one morphism from $x$ to $y$ when $x \leq y$, and no morphisms otherwise. In this category, isomorphic objects are equal. Note that there is a unique way to define composition: if we have a morphism $f$ from $x$ to $y$ and one $g$ from $y$ to $z$, then $x \leq y$ and $y \leq z$. Therefore, $x \leq z$, and there is a unique morphism from $x$ to $z$, which we define to be the composition $gf$. Conversely, a category in which (1) the objects form a set, (2) there is at most one morphism between any two objects, and (3) isomorphic objects are equal is essentially the same thing as a partially ordered set. One defines a partial ordering on the objects by $x \leq y$ if and only if there is a morphism from $x$ to $y$.

(h) A category with just one object in which every morphism is an isomorphism is essentially the same thing as a group. The morphisms of the object to itself are the elements of the group.

Given any category $C$ we can construct an opposite category $C^{\text{op}}$. It has the same objects as $C$, but for any two objects $X$ and $Y$ in $\text{Ob}(C)$, $\text{Mor}_{C^{\text{op}}}(X, Y) = \text{Mor}_C(Y, X)$. There turns out to be an obvious way of defining composition using the composition in $C$: if $f \in \text{Mor}_{C^{\text{op}}}(X, Y)$ and $g \in \text{Mor}_{C^{\text{op}}}(Y, Z)$ we have that $f : Y \to X$ in $C$ and $g : Z \to Y$, in $C$, so that $f \circ g$ in $C$ is a morphism $Z \to X$ in $C$, i.e., a morphism $X \to Z$ in $C^{\text{op}}$, and thus $g \circ_{C^{\text{op}}} f$ is $f \circ_C g$.

By a (covariant) functor from a category $C$ to a category $D$ we mean a function $F$ that assigns to every object $X$ in $C$ an object $F(X)$ in $D$ and to every morphism $f : X \to Y$ in $C$ a morphism $F(f) : F(X) \to F(Y)$ in $D$ such that

(1) For all $X \in \text{Ob}(C)$, $F(1_X) = 1_{F(X)}$ and
(2) For all $f : X \to Y$ and $g : Y \to Z$ in $C$, $F(g \circ f) = F(g) \circ F(f)$.

A contravariant functor from $C$ to $D$ is a covariant functor to $C^{\text{op}}$ to $D^{\text{op}}$. This means that when $f : X \to Y$ in $C$, $F(f) : F(Y) \to F(X)$ in $D$, and $F(g \circ f) = F(f) \circ F(g)$ whenever $g \circ f$ is defined in $C$. 
Here are some examples.

(a) Given any category $\mathcal{C}$, there is an identity functor $1_{\mathcal{C}}$ on $\mathcal{C}$: it sends the object $X$ to the object $X$ and the morphism $f$ to the morphism $f$. This is a covariant functor.

(b) There is a functor from the category of groups and group homomorphisms to the category of abelian groups and homomorphisms that sends the group $G$ to $G/G'$, where $G'$ is the commutator subgroup of $G$: $G'$ is generated by the set of all commutators $\{ghg^{-1}h^{-1} : g, h \in G\}$: it is a normal subgroup of $G$. The group $G/G'$ is abelian. Note also that any homomorphism from $G$ to an abelian group must kill all commutators, and factors through $G/G'$, which is called the abelianization of $G$.

Given $\phi : G \to H$, $\phi$ automatically takes commutators to commutators. Therefore, it maps $G'$ into $H'$ and so induces a homomorphism $G/G' \to H/H'$. This explains how this functor behaves on homomorphisms. It is covariant.

(c) Note that the composition of two functors is a functor. If both are covariant or both are contravariant the composition is covariant. If one is covariant and the other is contravariant, the composition is contravariant.

(d) There is a contravariant functor $F$ from the category of topological spaces to the category of rings that maps $X$ to the ring of continuous $\mathbb{R}$-valued functions on $X$. Given a continuous map $f : X \to Y$, the ring homomorphism $F(Y) \to F(X)$ is induced by composition: if $h : Y \to \mathbb{R}$ is any continuous function on $Y$, then $h \circ f$ is a continuous function on $X$.

(e) Given a category such as groups and group homomorphisms in which the objects have underlying sets and the morphisms are given by certain functions on those sets, we can give a covariant functor to the category of sets: it assigns to each object its underlying set, and to each morphism the corresponding function. Functors of this sort are called forgetful functors. The category of rings and ring homomorphisms and the category of topological spaces and continuous maps both have forgetful functors as well.

We next want to give a contravariant functor from commutative rings to topological spaces.

We first want to review some terminological conventions. All rings, unless otherwise specified, are commutative with multiplicative identity $1$. We use $1_R$ for the identity in the ring $R$ if greater precision is needed. We recall that $1 = 0$ is allowed, but this forces every element of the ring to be 0. Up to unique isomorphism, there is a unique ring with one element, which we denote 0.

By a domain or integral domain we mean a commutative ring such that $1 \neq 0$ and such that if $ab = 0$ then either $a = 0$ or $b = 0$. It is an arbitrary convention to exclude the ring in which every element is zero, but this turns out to be convenient. By a field we mean a ring in which $1 \neq 0$ and in which every nonzero element has an inverse under multiplication. A field $K$ has only two ideals: $\{0\}$ and $K$. A field is an integral domain, although the converse is not true in general.
An ideal \( P \) in \( R \) is called prime if \( R/P \) is an integral domain. This means that \( P \) is prime in \( R \) if and only if \( 1 \notin P \) and for all \( a, b \in R \), if \( ab \in P \) then either \( a \in P \) or \( b \in P \).

An ideal \( m \in R \) is called maximal if, equivalently, either \( R/m \) is a field or \( m \) is maximal among all proper ideals of \( R \). A maximal ideal is prime.

Every proper ideal is contained in a maximal ideal. To see this, we first recall Zorn’s lemma, which we shall not prove. It is equivalent to the axiom of choice in set theory. A subset of a partially ordered set is called a chain if it is linearly ordered, i.e., if any two of its elements are comparable.

**Zorn’s lemma.** Let \( P \) be a non-empty partially ordered set in which every chain has an upper bound. Then \( P \) has a maximal element.

**Corollary.** Let \( I \) be a proper ideal of the commutative ring \( R \). Then \( I \) is contained in a maximal ideal \( m \).

**Proof.** We apply Zorn’s lemma to the partially ordered set of proper ideals containing \( I \). Given any chain containing \( I \), its union is a proper ideal containing \( I \) and is an upper bound for the chain. Thus there are maximal elements in the set of proper ideals containing \( I \), and these will be maximal ideals. □

We are now ready to introduce our functor, \( \text{Spec} \), from commutative rings to topological spaces. If \( R \) is a ring, let \( \text{Spec} (R) \) denote the set of all prime ideals of \( R \). Note that \( \text{Spec} (R) \) is empty if and only if \( R \) is the 0 ring. We place a topology, the Zariski topology, on \( \text{Spec} (R) \) as follows. For any subset \( I \) of \( R \), let \( V(I) \) denote the set \( \{ P \in \text{Spec} (R) : I \subseteq P \} \). If the set \( I \) is replaced by the ideal it generates, \( V(I) \) is unaffected. The Zariski topology has the subsets of \( \text{Spec} (R) \) of the form \( V(I) \) as its closed sets. Note that \( V(0) = \text{Spec} (R) \), that \( V(R) = \emptyset \), and that for any family of ideals \( \{ I_\lambda \}_{\lambda \in \Lambda} \),

\[
\bigcap_{\lambda \in \Lambda} V(I_\lambda) = V\left( \sum_{\lambda \in \Lambda} I_\lambda \right).
\]

It remains only to show that the union of two closed sets (and, hence, any finite number) is closed, and this will follow if we can show that for any two ideals \( I, J \), \( V(I) \cup V(J) = V(I \cap J) = V(\sum I \lambda) \). It is clear that the leftmost term is smallest. Suppose that a prime \( P \) contains \( IJ \) but not \( I \), so that \( u \in I \) but \( u \notin P \). For every \( v \in J \), \( uv \in P \), and since \( u \notin P \), we have \( v \in P \). Thus, if \( P \) does not contain \( I \), it contains \( J \). It follows that \( V(\sum I \lambda) \subseteq V(I) \cup V(J) \), and the result follows.

The Zariski topology is \( T_0 \). If \( P \) and \( Q \) are distinct primes, one of them contains an element not in the other. Suppose, say, that \( u \in P \) and \( u \notin Q \). The closed set \( V(u) \) contains \( P \) but not \( Q \).

It is easy to show that the closure of the one point set \( \{ P \} \), where \( P \) is prime, is the set \( V(P) \). The closure has the form \( V(I) \), and is the smallest set of this form such that \( P \in V(I) \), i.e., such that \( I \subseteq P \). As \( I \) gets smaller, \( V(I) \) gets larger. It is therefore immediate that the smallest closed set containing \( P \) is \( V(P) \).

It follows that \( \{ P \} \) is closed if and only if \( P \) is maximal. In general, \( \text{Spec} (R) \) is not \( T_1 \).
Spec becomes a contravariant functor from the category of commutative rings with identity to the category of topological spaces if, given a ring homomorphism \( f : R \to S \), we define \( \text{Spec}(f) \) by having it send \( Q \in \text{Spec}(S) \) to \( f^{-1}(Q) = \{ r \in R : f(r) \in Q \} \). There is an induced ring homomorphism \( R/f^{-1}(Q) \to S/Q \) which is injective. Since \( S/Q \) is an integral domain, so is its subring \( R/f^{-1}(Q) \). (We are also using tacitly that the inverse image of a proper ideal is proper, which is a consequence of our convention that \( f(1_R) = 1_S \).) \( f^{-1}(Q) \) is sometimes denoted \( Q^c \) and called the contraction of \( Q \) to \( R \). This is a highly ambiguous notation.

We want to talk about when two functors are isomorphic and to do that, we need to have a notion of morphism between two functors. Let \( F,G \) be functors from \( C \to D \) with the same variance. For simplicity, we shall assume that they are both covariant. The case where they are both contravariant is handled automatically by thinking instead of the case of covariant functors from \( C \) to \( D^{op} \). A natural transformation from \( F \) to \( G \) assigns to every object \( X \in \text{Ob}(C) \) a morphism \( T_X : F(X) \to G(X) \) in such a way that for all morphisms \( f : X \to Y \) in \( C \), there is a commutative diagram:

\[
\begin{array}{ccc}
F(X) & \xrightarrow{F(f)} & F(Y) \\
\downarrow T_X & & \downarrow T_Y \\
G(X) & \xrightarrow{G(f)} & G(Y)
\end{array}
\]

The commutativity of the diagram simply means that \( T_Y \circ F(f) = G(f) \circ T_X \).

This may seem like a complicated notion at first glance, but it is actually very “natural,” if you will forgive the expression.

This example may clarify. If \( V \) is a vector space write \( V^* \) for the space of linear functionals on \( V \), i.e., for \( \text{Hom}_K(V, K) \), the \( K \)-vector space of \( K \)-linear maps from \( V \to K \). Then \( * \) is a contravariant functor from \( K \)-vector spaces and \( K \)-linear maps to itself. (If \( \theta : V \to W \) is linear, \( \theta^* : W^* \to V^* \) is induced by composition: if \( g \in W^* \), so that \( g : W \to K \), then \( \theta^*(g) = g \circ \theta \).)

The composition of * with itself gives a covariant functor \( ** \): the double dual functor. We claim that there is a natural transformation \( T \) from the identity functor to \( ** \). To give \( T \) is the same as giving a map \( T_V : V \to V^{**} \) for every vector space \( V \). To specify \( T_V(v) \) for \( v \in V \), we need to give a map from \( V^* \) to \( K \). If \( g \in V^* \), the value of \( T_V(v) \) on \( g \) is simply \( g(v) \). To check that this is a natural transformation, one needs to check that for every \( K \)-linear map \( f : V \to W \), the diagram

\[
\begin{array}{ccc}
V & \xrightarrow{f} & W \\
\downarrow T_V & & \downarrow T_W \\
V^{**} & \xrightarrow{f^{**}} & W^{**}
\end{array}
\]
commutes. This is straightforward. Note that the map $V \to V^{**}$ is not necessarily an isomorphism. It is always injective, and is an isomorphism when $V$ is finite-dimensional over $K$.

Here is another example of a natural transformation: in this case, the functors are contravariant. Let $F$ and $G$ be the functors from topological spaces to rings such that $F(X)$ (respectively, $G(X)$) is the ring of continuous real-valued (respectively, complex-valued) functions on $X$. (The values on continuous maps are both induced by composition.) The inclusions $F(X) \subseteq G(X)$ give a natural transformation from $F$ to $G$.

Let $C$ be the category of pairs $(X,x)$ where $X$ is a non-empty topological space and $x \in X$, i.e., of topological spaces with basepoint. A morphism from $(X,x)$ to $(Y,y)$ is a continuous function from $X$ to $Y$ such that $f(x) = y$. For every $X$ there is a group homomorphism from $T_X : \pi_1(X,x) \to H_1(X,\mathbb{Z})$ where the former is the fundamental group and the latter is singular homology with integer coefficients. (Let $S^1$ be a circle and fix a generator $\theta$ of $H_1(S^1,\mathbb{Z}) \cong \mathbb{Z}$. Every element of $\pi_1(X,x)$ is represented by (the homotopy class of) a continuous map $f : S^1 \to X$. $T_X([f]) = f_\ast(\theta) \in H_1(X,\mathbb{Z})$.) These $T_X$ give a natural transformation from $\pi_1$ to the functor $H_1(\_,\mathbb{Z})$, both regarded as functors from $C$ to the category of groups. There are also natural transformations $H_1(\_,\mathbb{Z}) \to H_1(\_,\mathbb{Q}) \to H_1(\_,\mathbb{R}) \to H_1(\_,\mathbb{C})$.

In giving definitions for natural transformations, we will stick with the case of covariant functors: the contravariant case may be handled by replacing $D$ by $D^{op}$.

Given functors $F, G, H$ from $C \to D$, a natural transformation $S : F \to G$, and a natural transformation $T : G \to H$, we may define a natural transformation $T \circ S$ from $F$ to $H$ by the rule $(T \circ S)_X = T_X \circ S_X$.

There is an identity natural transformation, $1_F$, from the functor $F : C \to D$ to itself: $1_{F,X} : F(X) \to F(X)$ is $1_{F(X)}$. It behaves as an identity should under composition. Given two functors $F$ and $G$ from $C \to D$, we can now define them to be isomorphic if there are natural transformations $T : F \to G$ and $T' : G \to F$ such that $T' \circ T = 1_F$ and $T \circ T' = 1_G$. In fact, $T$ is an isomorphism of functors if and only if all the morphisms $T_X$ are isomorphisms, and in that case the unique way to define $T' \circ T$ is by the rule $T'_X = (T_X)^{-1}$.

Once we have a notion of isomorphism of functors we can define two categories $C$ and $D$ to be equivalent if there are functors $F : C \to D$ and $G : D \to C$ such that $G \circ F$ is isomorphic to the identity functor on $C$ and $F \circ G$ is isomorphic to the identity functor on $D$. If $C$ is equivalent to $D$ it is said to be antiequivalent to $D$. Roughly speaking, equivalence is like isomorphism, but there may not be the same number of objects in an isomorphism class in one of the two equivalent categories as there are in the other. For example, suppose that we have a category $D$ and another $C$ in which there is exactly one object of $D$ from each isomorphism class of objects in $D$. Also suppose that the morphisms from one object in $C$ to another are the same as when they are considered as objects of $D$, and likewise for composition. Then one can show, with a suitably strong form of the axiom of choice, that $C$ and $D$ are equivalent categories.

Another application of the notion of isomorphism of functors is the definition of a representable functor. This is a point of view that unifies numerous constructions, both in
set-theoretic ones. In the category associated with a partially ordered set, the product of spaces, the product topology works: the open sets are unions of Cartesian products of open rings, groups, abelian groups, \( R \). 

Identity maps also follows from the defining property of the product. \( \delta \). Similarly, we can define a contravariant functor \( h^Z \) to sets by \( h^Z(X) = \text{Mor} (Z, X) \) while \( h^Z(f) : \text{Mor} (Z, Y) \) sends \( g \) to \( g \circ f \). A contravariant functor is representable in \( C \) if it is isomorphic to \( h^Z \) for some \( Z \in \text{Ob}(C) \). We say that \( Z \) represents \( G \).

Examples. (a) Let \( C \) be the category of abelian groups and group homomorphisms. Let \( G \) be any group. We can define a functor \( F \) from abelian groups to sets by letting \( F(A) = \text{Hom}(G, A) \), the set of group homomorphisms from \( G \) to \( A \). Can we represent \( F \) in the category of abelian groups? Yes! Let \( G/\mathbb{Z} \), the abelianization of \( G \). Then every homomorphism \( G \to A \) factors uniquely \( G \to G/\mathbb{Z} \to A \), giving a bijection of \( F(A) \) with \( \text{Hom}(G/\mathbb{Z}, A) \). This yields an isomorphism of \( F \cong h_{G/\mathbb{Z}} \).

(b) Let \( R \) be a ring and and \( I \) be an ideal. Define a functor from the category of commutative rings with identity to the category of sets by letting \( F(S) = \text{Hom}(R/I, S) \) be the set of all ring homomorphisms \( f : R \to S \) such that \( f \) kills \( I \). Every homomorphism \( R \to S \) such that \( f \) kills \( I \) factors uniquely \( R \to R/I \to S \), from which it follows that the functor \( F \) is representable and is \( \cong h_{R/I} \).

(c) In this example we want to define products in an arbitrary category. Our motivation is the way the Cartesian product \( Z = X \times Y \) behaves in the category of sets. It has product projections \( \pi_X : Z \to X \) and \( \pi_Y : Z \to Y \) sending \((x, y)\) to \(x\) and \(y\) to \(y\). To give a function from \( W \to X \times Y \) is equivalent to giving a pair of functions, one \( \alpha : W \to X \) and another \( \beta : W \to Y \). The function \( f : W \to X \times Y \) then sends \( w \) to \((\alpha(w), \beta(w))\). The functions \( \alpha \) and \( \beta \) may be recovered from \( f \) as \( \pi_X \circ f \) and \( \pi_Y \circ f \), respectively.

Now let \( C \) be any category. Let \( X, Y \in \text{Ob}(C) \). An object \( Z \) together with morphisms \( \pi_X : Z \to X \) and \( \pi_Y : Z \to Y \) (called the product projections on \( X \) an \( Y \), respectively) is called a product for \( X \) and \( Y \) in \( C \) if for all objects \( W \) in \( C \) the function \( \text{Mor}(W, Z) \to \text{Mor}(W, X) \times \text{Mor}(W, Y) \) sending \( f \) to \((\pi_X \circ f, \pi_Y \circ f)\) is a bijection. This means that the functor \( \text{Mor}(W, \cdot) \) is representable in \( C \). Given another product \( Z' \), \( \pi'_X \) and \( \pi'_Y \), there are unique mutually inverse isomorphisms \( \gamma : Z \to Z' \) and \( \delta : Z' \to Z \) that are compatible with the product projections, i.e., such that \( \pi_X = \gamma \circ \pi'_X \) and \( \pi_Y = \gamma \circ \pi'_Y \) (the existence and uniqueness of \( \gamma \) are guaranteed by the defining property of the product) and similarly for \( \delta \). The fact that the compositions are the appropriate identity maps also follows from the defining property of the product.

Products exist in many categories, but they may fail to exist. In the categories of sets, rings, groups, abelian groups, \( R \)-modules over a given ring \( R \), and topological spaces, the product turns out to be the Cartesian product with the usual additional structure (in the algebraic examples, operations are performed coordinate-wise; in the case of topological spaces, the product topology works: the open sets are unions of Cartesian products of open sets from the two spaces). In all of these examples, the product projections are the usual set-theoretic ones. In the category associated with a partially ordered set, the product of
two elements $x$ and $y$ is the greatest lower bound of $x$ and $y$, if it exists. The point is that $w$ has (necessarily unique) morphisms to both $x$ and $y$ iff $w \leq x$ and $w \leq y$ iff $w$ is a lower bound for both $x$ and $y$. For $z$ to be a product, we must have that $z$ is a lower bound for $x$, $y$ such that every lower bound for $x$, $y$ has a morphism to $z$. This says that $z$ is a greatest lower bound for $x$, $y$ in the partially ordered set. It is easy to give examples of partially ordered sets where not all products exist: e.g., a partially ordered set that consists of two mutually incomparable elements (there is no lower bound for the two), or one in which there are four elements $a$, $b$, $x$, $y$ such that $a$ and $b$ are incomparable, $x$ and $y$ are incomparable, while both $a$ and $b$ are strictly less than both $x$ and $y$. Here, $a$ and $b$ are both lower bounds for the $x$, $y$, but neither is a greatest lower bound.

The product of two objects in $\mathcal{C}^{\text{op}}$ is called their coproduct in $\mathcal{C}$. Translating, the coproduct of $X$ and $Y$ in $\mathcal{C}$, if it exists, is given by an object $Z$ and two morphisms $\iota_X : X \to Z$, $\iota_Y : Y \to Z$ such that for every object $W$, the map $\text{Mor} (Z, W) \to \text{Mor} (X, W) \times \text{Mor} (Y, W)$ sending $f$ to $(f \circ \iota_X, f \circ \iota_Y)$ is bijective. This means that the functor sending $W$ to $\text{Mor} (X, W) \times \text{Mor} (Y, W)$ is representable in $\mathcal{C}$. Coproducts have the same sort of uniqueness that products do: they are products (in $\mathcal{C}^{\text{op}}$).

In the category of sets, coproduct corresponds to disjoint union: one takes the union of disjoint sets $X'$ and $Y'$ set-isomorphic to $X$ and $Y$ respectively. The function $\iota_X$ is an isomorphism of $X$ with $X'$ composed with the inclusion of $X'$ in $X' \cup Y'$, and similarly for $\iota_Y$. To give a function from the disjoint union of two sets to $W$ is the same as to give two functions to $W$, one from each set.

In the category of $R$-modules over a commutative ring $R$, coproduct corresponds to direct sum. In the category of commutative $A$-algebras, the coproduct of $R$ and $S$ turns out to be $R \otimes_A S$. In the category associated with a partially ordered set, the coproduct of two elements corresponds to the least upper bound of the two elements.