1. The number of copies of $E_R(R/P)$ in the direct sum decomposition does not change when we localize at $P$, since $E_{R_P}(N_P) \cong E_R(N)_P$. It follows that we may assume that $(R, P)$ is local. Then for any $R$-module $M$, $\text{Hom}_R(K, M) \cong \text{Ann}_M M$. Quite generally, if $N \subseteq M$ is essential, then $\text{Ann}_N I \subseteq \text{Ann}_M I$ for any ideal $I$ of $R$: every nonzero element of $\text{Ann}_M I$ has a nonzero multiple in $N$, and this multiple is clearly killed by $I$ and so in $\text{Ann}_N I$. Hence, $\text{Hom}_R(K, N) = \text{Ann}_N P \subseteq \text{Ann}_{E_R(N)} P = \text{Hom}_{R_1}(K, E_R(N))$ is essential. Since this is an inclusion of $K$-vector spaces, for it to be essential we must have $\text{Hom}_R(K, N) = \text{Hom}_{R_1}(K, E_R(N))$, since an inclusion of vector spaces always splits. The $K$-vector space dimension of the latter is the number of copies of $E_R(K)$ in a direct sum decomposition for $E_R(N)$ by a class theorem, and this is equal to the $K$-vector space dimension of the former, as required. \(\square\)

2. By the universal mapping property for base change, the functor $\text{Hom}_R(\_ , E)$ is isomorphic to the functor $\text{Hom}_S(S \otimes_R \_ , E)$. (This is true for any $S$-module $E$ viewed as an $R$-module by restriction of scalars.) This is the composition of the functors $S \otimes_R \_ , \_$, which is exact because $S$ is flat over $R$, and $\text{Hom}_S(\_ , E)$, which is exact because $E$ is injective over $S$. It follows that the functor $\text{Hom}_R(\_ , E)$ is exact and, hence, that $E$ is injective over $R$. \(\square\)

3. Note that a discrete valuation ring $(V, tV, K)$ is a PID, since every nonzero element is a unit times a power $t^k$ of $t$, with $k \in \mathbb{N}$. To show that $\mathcal{F}/V$ is injective, it suffices to show that it is a divisible $V$-module, which is clear because it is a homomorphic image of the divisible $V$-module $\mathcal{F}$. Every element of $\mathcal{F} - \{0\}$ is a unit of $V$ times a power $t^k$ of $t$, with $k \in \mathbb{Z}$. Thus, the nonzero elements of $\mathcal{F}/V$ can be represented by an element of the form $ut^{-k}$, where $u$ is a unit of $V$ and $k > 0$. If we multiply by $u^{-1}t^{k-1}$, we obtain $t^{-1}$, which is killed by $t$ and so by $tV = m$. Thus, the cyclic module generated over $V$ by the class of $t^{-1}$ is $\cong K$, and the extension is essential. Thus, $\mathcal{F}/V \cong E_V(K)$. \(\square\)

4. By a class result, $E$ is injective if and only if maps from prime ideals of $R$ to $E$ extend to $R$. [Consider a map to $E$ defined on $N \subseteq M$ which supposedly cannot be extended further. Pick $m \in M - N$. One may replace $m$ by a multiple such that the ideal multiplying $m$ into $N$ is prime. (The image of $m$ in $M/N$ has a multiple which has prime annihilator.)] This is trivial for the 0 prime ideal, and extending maps from principal ideals of a domain can be done if and only if the module is divisible. Hence, we may assume that $E$ is divisible. The only remaining prime ideal of $R$ is the maximal ideal $(x, y)R$. Thus, a divisible module $E$ is injective if every map from $(x, y)R$ to $E$ extends to a map $R \to E$. We can map $R^2 \to m$ by $(r, s) \mapsto rx + sy$. If $(r, s)$ is in the kernel, then $rx = -sy$. Since $x$ is prime and does not divide $y$, it must divide $s$, say $s = ax$. But then it follows that $r = -ay$, and $(r, s) = a(-y, x)$. Thus, $m \cong R^2/R(-y, x)$. To give a map $m \to E$ is therefore the same as to give a map $R^2 \to E$ that kills $(-y, x)$, i.e., to give $u, v \in E$ such that $-yu + xv = 0$, where $u, v \in E$ are to be the images of $x, y \in m$. The problem of extending the map to $R$ is the same as the problem of specifying the value $w$ for 1 so that $x$ will map to $u$ and $y$ to $v$, i.e., such that $xw = u$ and $yw = v$. \(\square\)
5. The map cannot be extended to $R$ because the image of 1 in the direct sum of the $E_n$ will be nonzero in only finitely many $E_n$, which implies that the image of the map is contained in only finitely many $E_n$. But each $x_j$ maps to a nonzero element in the corresponding $E_j$, a contradiction. □

6. Consider a nonzero element $v \oplus e$ in $S = V \oplus E$. If $v$ is not 0 it has the form $ut^k$ where $u$ is a unit of $V$ and $k \geq 0$. In this case we may take $(0 \oplus u^{-1}t^{-k+1})(ut^k \oplus e)$ to obtain $0 \oplus t^{-1} \in 0 \oplus K$. If the nonzero element has the form $0 \oplus e$ where $e \neq 0$, we may multiply by an element of $V \oplus 0$ to get a nonzero element of $K$, since $K \hookrightarrow E$ is essential. Thus, $K = 0 \oplus K \hookrightarrow V \oplus E$ is essential. Since every element of $E$ is killed by a power of $t$, if we localize at $t$, $E$ becomes 0, while $V_t \cong \mathcal{F}$. Thus, $S_t \cong \mathcal{F}$ is a field. The extension is no longer essential, since the submodule becomes 0 while the ambient module does not. Note that since $S_t$ is a field, it has a unique maximal ideal, which means that $S_t$ is also the localization of $S$ at the prime ideal $0 \oplus E$.

**Extra Credit 1.** Let $K[X]$ and $K[Y]$ be polynomial rings in one variable over the field $K$. $K(X)$ (resp., $K(Y)$) is divisible as a module over the principal ideal domain $K[X]$ (resp., $K[Y]$), and, hence, injective. But $M = K(X) \otimes_K K(Y)$ is not injective over $K[X] \otimes_K K[Y] \cong K[X, Y]$: one can see this because, for example, $X + Y$ has no inverse, and so $M$ is not divisible. □