1. Let $R$ be a ring of prime characteristic $p > 0$. Let $S$ denote $R$ viewed as an $R$-module by the Frobenius endomorphism $F$. If $R \to S$ splits as a map of $R$-modules, then $R$ is called F-split.

(a) Show that a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$ of positive characteristic $p$ is F-split.

(b) Show that if $R$ is F-split, then the action of $F$ on $H^*_F(R)$ is injective for every ideal $I$ of $R$.

(c) Suppose that $R$ is a finitely generated $K$-algebra, where $K$ is a field of characteristic $p > 0$, that $R$ is F-split, and that $R$ is $\mathbb{N}$-graded with $R_0 = K$. Let $m = \bigoplus_{k=1}^{\infty} R_k$ denote the homogeneous maximal ideal of $R$. Show that $[H^*_F(R)]_k = 0$ for all $i$ and all $k > 0$.

2. Let $K$ be a field of positive characteristic $p$, and let $\Delta$ be a finite simplicial complex with vertices $x_1, \ldots, x_n$. Let $R = K[x_1, \ldots, x_n]/I_\Delta$ be the Stanley-Reisner ring. Prove that $R$ is F-split.

3. Assume the same hypothesis as in 1. (c). Also assume that $K$ is perfect, that $R$ is a domain, and that $R_P$ is Cohen-Macaulay for $P \neq m$. Let $S$ denote $R$ viewed as an $R$-module by the Frobenius endomorphism $F$, so that $S = R \oplus M$ as $R$-modules. Prove that $M$ is Cohen-Macaulay.

4. Let $(R, m, K)$ be a Gorenstein local ring and let $T$ be a module-finite extension ring such that $T$ is a Cohen-Macaulay local ring. Let $x_1, \ldots, x_n$ be a system of parameters for $R$, and let $I = (x_1, \ldots, x_n)R$. Show that $R \to T$ splits if and only if $IT \cap R = I$.

5. Let $(R, m) K$ be a local ring, let $X = \text{Spec} (R)$, and let $Y = X - \{m\}$, the punctured spectrum of $R$.

(a) Show that $Y$ is disconnected if and only if there exist ideals $I, J$ in $R$ such that $I + J$ is primary to $m$, $I \cap J$ consists of nilpotents, but neither $I$ nor $J$ consists of nilpotents.

(b) Prove that if $Y$ is disconnected, then $H^1_F(R) \neq 0$. [The Mayer-Vietoris sequence may be helpful.] Hence, if depth$_m R \geq 2$ then $Y$ is connected.

6. Let $f(x)$ denote a transcendental power series (over $\mathbb{C}(x)$) in the maximal ideal of $\mathbb{C}[[x]]$, e.g., $e^x - 1$ or $\sin(x)$. Suppose that $f(x) = \sum_{i=1}^{\infty} a_i x^i$, and let $f_n(x) = \sum_{i=1}^{n} a_i x^i \in \mathbb{C}[x]$. Let $R = \mathbb{C}[x, y]_M$ be a localized polynomial ring in two variables, where $M = (x, y)$, let $m = MR$, and let $I_n = (y - f_n, x^{n+1}) \subseteq R$. Show that $I_n$ is a decreasing sequence of ideals primary to the maximal ideal $m$ of $R$ whose intersection is $(0)$, but for all $n$, $I_n \not\subseteq m^2$.

This shows that Chevalley’s Lemma fails when $R$ is not complete.

**EXTRA CREDIT 6.** Let $S$ be the $\mathbb{N}$-graded ring defined in Problem 6. of Problem Set #3. $S$ has Krull dimension 3 and is a normal domain that is not Cohen-Macaulay. Let $m$ be the homogeneous maximal ideal of $S$ and let $E = E_S(S/m)$. Find an explicit homogeneous ideal $I$ of $S$ such that $\text{Hom}_S(I, E) \cong H^3_m(S)$.