LOCAL COHOMOLOGY

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These are lecture notes based on seminars and courses given by the author at the University of Michigan over a period of years. This particular version is intended for Mathematics 615, Winter 2011. The objective is to give a treatment of local cohomology that is quite elementary, assuming, for the most part, only a modest knowledge of commutative algebra. There are some sections where further prerequisites, usually from algebraic geometry, are assumed, but these may be omitted by the reader who does not have the necessary background.

This version contains the first twenty sections, and is tentative. There will be extensive revisions and additions. The final version will also have Appendices dealing with some prerequisites.

Throughout, all given rings are assumed to be commutative, associative, with identity, and all modules are assumed to be unital. By a *local* ring (R, m, K) we mean a Noetherian ring R with a unique maximal ideal m and residue field $K = R/m$.

Topics to be covered include the study of injective modules over Noetherian rings, Matlis duality (over a complete local ring, the category of modules with ACC is anti-equivalent to the category of modules with DCC), the notion of depth, Cohen-Macaulay and Gorenstein rings, canonical modules and local duality, the Hartshorne-Lichtenbaum vanishing theorem, and the applications of D-modules (in equal characteristic 0) and F-modules (in characteristic $p > 0$) to study local cohomology of regular rings. The tie-in with injective modules arises, in part, because the local cohomology of a regular local rings with support in the maximal ideal is the same as the injective huk of the residue class field. This is true for a larger class of local rings. There are many equivalent ways to define local cohomology, and it is important to make use of the fact that one has these different points

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of view. Typically, local cohomology modules are not finitely generated, and in this sense may seem "big." However, they frequently have properties that make them manageaable objects of study: some satisfy DCC, while others can be viewed as modules over a ring of differential operators or in some other category in which they turn out to have finite length.

The first local cohomology module arises in a natural way in studying the obstruction to extending a section of a sheaf on an open set to a global section. This will be explained in the notes without assuming prior knowledge from sheaf theory. Our development of the theory will be largely independent from this point of view.

Local cohomology modules can be used to measure the depth of a module on an ideal, and as a way to test the Cohen-Macaulay and Gorenstein properties. Moreover, the cohomology of coherent sheaves on projective varieties can be recovered from graded components of local cohomology modules, providing useful insights into theorems about projective varieities that were originally proved in other ways.

In the course of this treatment we shall discuss many open questions about local cohomology.

0. INJECTIVE MODULES

Our study of local cohomology will depend heavily on a detailed understanding of the behavior of injective modules. Throughout this section, let R denote a ring.

If $0 \to M \to N \to Q \to 0$ is an exact sequence of R-modules, we know that for any R-module N the sequence

$$0 \to \mathrm{Hom}_R(Q,\, N) \to \mathrm{Hom}_R(M,\, N) \to \mathrm{Hom}_R(M,\, N)$$

is exact. An R-module E is called *injective* if, equivalently, (1) $\mathrm{Hom}_R(\_\,,\, E)$ is an exact functor or (2) for any injection $M \hookrightarrow N$, the map $\mathrm{Hom}_R(N,\, E) \to \mathrm{Hom}_R(M,\, E)$ is surjective. In other words, every R-linear map from a submodule M of N to E can be extended to a map of all of N to E.

(0.1) Proposition. *An R-module E is injective if and only if for every I ideal I of R and R-linear map $\phi : I \to E$, ϕ extends to a map $R \to E$.*

Proof. "Only if" is clear, since the condition stated is a particular case of the definition of injective module when $N = R$ and $M = I$. We need to see that the condition is sufficient for injectivity. Let $M \subseteq N$ and $f : M \to E$ be given. We want to extend f to all of N. Define a partial ordering of maps of submodules M' of N to E as follows: $g \leq g'$ means that the domain of g is contained in the domain of g' and that g is a restriction of g' (thus, g and g' agree on the smaller domain, where they are both defined). The set of maps that are $\geq f$ (i.e., extensions of f to a submodule $M' \subseteq N$ with $M \subseteq M'$) has the property that every chain has an upper bound: given a chain of maps, the domains form a chain of submodules, and we can define a map from the union to E by letting is value on an element of the union be the value of any map in the chain that is defined on that element: they all agree. It is easy to see that this gives an R-linear map that is an upper bound for the chain of maps. By Zorn's lemma, there is a maximal extension. Let $f' : M' \to N$ be this maximal extension. If $M' = N$, we are done. Suppose not. We shall obtain a contradiction by extending f' further.

If $M' \neq N$, choose $x \in N - M'$. It will suffice to extend f' to $M' + Rx$. Let $I = \{i \in R : ix \in M'\}$, which is an ideal of R. Let $\phi : I \to E$ be defined by $\phi(i) = f'(ix)$ for all $i \in I$. This makes sense since every $ix \in M'$. By hypothesis, we can choose an R-linear map $\psi : R \to E$ such that $\psi(i) = \phi(i)$ for all $i \in I$. We have a map $\gamma : M \oplus R \to E$ defined by the rule $\gamma(u \oplus r) = f'(u) + \psi(r)$. We also have a surjection $M \oplus R \to M + Rx$ that sends $u \oplus r \mapsto u + rx$. We claim that γ kills the kernel of this surjection, and therefore induces a map $M' + Rx \to E$ that extends f'. To see this, note that if $u \oplus r \mapsto 0$ the $u = -rx$, and then $\gamma(u \oplus r) = f'(u) + \psi(r)$. Since $-u = rx$, $r \in I$, and so $\psi(r) = \phi(rx) = f'(-u) = -f'(u)$, and the result follows. $\square$

Recall that a module E over a domain R is *divisible* if, equivalently,

(1) $rE = E$ for all $r \in R - \{0\}$ or

(2) for all $e \in E$ and $r \in R - \{0\}$ there exists $e' \in E$ such that $re' = e$.

(0.2) Corollary. *Over a domain R, every injective module is divisible. Over a principal ideal domain R, a module is injective if and only if it is divisible.*

Proof. Consider the problem of extending a map of a principal ideal $aR \to E$ to all of R. If $a = 0$ the map is 0 and the 0 map can be used as the required extension. If $a \neq 0$, then since $aR \cong R$ is free on the generator a, the map to be extended might take any value $e \in E$ on a. To extend the map, we must specify the value e' of the extended map on 1 in such a way that the extended maps takes a to e: the condition that e' must satisfy is precisely that $ae' = e$. Thus, E is divisible if and only if every map of a principal ideal of R to E extends to a map of R to E. The result is now obvious, considering that in a principal ideal domain every ideal is principal. $\square$

It is obvious that a homomorphic image of a divisible module is divisible. In particular, $W = \mathbb{Q}/\mathbb{Z}$ is divisible $\mathbb{Z}$-module and therefore injective as a $\mathbb{Z}$-module. We shall use the fact that W is injective to construct many injective modules over many other rings. We need several preliminary results.

First note that if C is any ring and V is any C-module, we have a map

$$M \to \mathrm{Hom}_C(\mathrm{Hom}_C(M, V), V)$$

for every R-module M. If $u \in M$, this maps sends u to

$$\theta_u \in \mathrm{Hom}_C\bigl(\mathrm{Hom}_C(M, V), V\bigr),$$

define by the rule that $\theta_u(f) = f(u)$ for all $f \in \mathrm{Hom}_C(M, V)$.

Now let $\_^\vee$ denote the contravariant exact functor $\mathrm{Hom}_{\mathbb{Z}}(\_, W)$, where $W = \mathbb{Q}/\mathbb{Z}$ as above. As noted in the preceding paragraph, for every $\mathbb{Z}$-module A we have a map $A \to A^{\vee\vee}$, the double dual into W.

(0.3) Lemma. *With notation in the preceding paragraph, for every $\mathbb{Z}$-module A, A the homomorphism $\theta_A = \theta : A \to A^{\vee\vee}$ is injective.*

If A happens to be an R-module then the map $A \to A^{\vee\vee}$ is R-linear, and for every R-linear map $f : A_1 \to A_2$ we have a commutative diagram of R-linear maps

$$\begin{array}{ccc} A_1^{\vee\vee} & \xrightarrow{f^{\vee\vee}} & A_2^{\vee\vee} \\ {\scriptstyle \theta_{A_1}}\big\uparrow & & \big\uparrow{\scriptstyle \theta_{A_2}} \\ A_1 & \xrightarrow[f]{} & A_2 \end{array}$$

Proof. Given a nonzero element $a \in A$, we must show that there exists $f \in \mathrm{Hom}_{\mathbb{Z}}(A, W)$ such that the image of f under θ_a, is not 0, i.e., such that $f(a) \neq 0$. The $\mathbb{Z}$-submodule

D of A generated by a is either $\mathbb{Z}$ or else a nonzero finite cyclic module, which will be isomorphic to $\mathbb{Z}/n\mathbb{Z}$ for some $n > 1$. In either case, there will exist a surjection $D \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ for some $n > 1$, and $\mathbb{Z}/n\mathbb{Z}$ embeds in W: it is isomorphic to the span of the class of $1/n$ in $\mathbb{Q}/\mathbb{Z}$. Thus, we have a nonzero map $D \to W$, namely $D \twoheadrightarrow \mathbb{Z}/n\mathbb{Z} \hookrightarrow W$. Since $D \subseteq A$ and W is injective as a $\mathbb{Z}$-module, this map extends to a map of $f : A \to W$. Evidently, $f(a) \neq 0$.

The verifications of the remaining statements are straightforward and are left to the reader. $\square$

Before proving the next result we observe the following. Let R be a C-algebra, let M and N be R-modules, let Q be a C-module, and suppose that we are given a C-bilinear map $B : M \times N \to Q$ such that $B(ru,\, v) = B(u,\, rv)$ for all $r \in R$. Then there is a unique C-linear map $f : M \otimes_R N \to Q$ such that $f(u \otimes v) = B(u, v)$ for all $u \in M$ and $v \in N$. This is a consequence of the easily verified fact that $M \otimes_R N$ is the quotient of $M \otimes_C N$ by the span of all elements of the form $ru \otimes v - u \otimes rv$ for $r \in R$, $u \in M$ and $v \in N$. We are now ready to establish the following easy but very important result:

(0.4) Theorem (adjointness of tensor and Hom). *Let $C \to R$ be a ring homomorphism, let M be and N be R-modules, and let Q be a C-module. Then there is a natural isomorphism $\mathrm{Hom}_C(M \otimes_R N,\, Q) \to \mathrm{Hom}_R(M,\, \mathrm{Hom}_C(N,\, Q)$ as R-modules: the two sides are isomorphic as functors of the three variables M, N, and Q.*

Proof. We define mutually inverse maps explicitly. Given $f : M \otimes_R N \to Q$ as C-modules, let $\Theta(f)$ be the map $M \to \mathrm{Hom}_C(N,\, Q)$ whose value on $u \in M$ is $\beta_{f,u}$, where $\beta_{f,u}(v) = f(u \otimes v)$. Note that the value of $\Theta(rf)$ on u for $r \in R$ is $\beta_{rf,u}$, where $\beta_{rf,u}(v) = (rf)(u \otimes v) = f\big(r(u \otimes v)\big) = f\big((ru) \otimes v)\big)$, while the value of $r\Theta(f)$ on u is $\Theta(f)(ru)$, and the value of that map on $v \in N$ is $\beta_{f,ru}(v) = f\big((ru) \otimes v\big)$. The R-linearity of Θ follows.

On the other hand, given $g : M \to \mathrm{Hom}_C(N,\, Q)$, we can define a C-bilinear map $B_g : M \times N \to Q$ by letting $B_g(u,\, v) = g(u)(v)$. Note that $B_g(ru,\, v) = g(ru)(v) = \big(rg(u)\big)(v) = g(u)(rv) = B_g(u,\, rv)$. Let

$$\Lambda : \mathrm{Hom}_R(M,\, \mathrm{Hom}_C(N,\, Q) \to \mathrm{Hom}_C(M \otimes_R N,\, Q)$$

be such that $\Lambda(g)$ is the linear map corresponding to B_g. The check that Λ and Θ are mutually inverse is straightforward, as is the check of naturality: further details are left to the reader. $\square$

(0.5) Corollary. *Let R be a C-algebra, let F be a flat R-module, and let W be an injective C-module. Then $\mathrm{Hom}_C(F, W)$ is an injective R-module.*

Proof. Because of the natural isomorphism

$$\mathrm{Hom}_R(M, \mathrm{Hom}_C(F, W)) \cong \mathrm{Hom}_C(M \otimes_R F, W)$$

we may view the functor

$$\mathrm{Hom}_R(\_, \mathrm{Hom}_C(F, W))$$

as the composition of two functors: $\_ \otimes_R F$ followed by $\mathrm{Hom}_C(\_, W)$. Since F is R-flat, the first is exact, while since W is C-injective, the second is exact. Therefore, the composition is exact. $\square$

We can now put things together:

(0.6) Theorem. *Over every commutative ring R, every R-module embeds in an injective R-module. In fact, this embedding can be achieved canonically, that is, without making any arbitrary choices.*

Proof. Let M be any R-module. In this construction, $\mathbb{Z}$ will play the role of C above. We can map a free R-module F onto $\mathrm{Hom}_{\mathbb{Z}}(M, W)$, were $W = \mathbb{Q}/\mathbb{Z}$ is injective over $\mathbb{Z}$. We can do this canonically, as in the construction of Tor, by taking one free generator of F for every element of $\mathrm{Hom}_{\mathbb{Z}}(M, W)$. By the Corollary above, $F^\vee = \mathrm{Hom}_Z(F, W)$ is R-injective. Since we have a surjection $F \twoheadrightarrow M^\vee$, we may apply $\mathrm{Hom}_{\mathbb{Z}}(\_, W)$ to get an injection $M^{\vee\vee} \hookrightarrow F^\vee$. But we have injection $M \hookrightarrow M^{\vee\vee}$, and so the composite $M \hookrightarrow M^{\vee\vee} \hookrightarrow F^\vee$ embeds M in an injective R-module canonically. $\square$

While the embedding does not involve the axiom of choice, the proof that it is an embedding and the proof that $F^\vee$ is injective do: both use that W is injective. The argument for that used that divisible $\mathbb{Z}$-modules are injective, and the proof of that depended on Proposition (0.1), whose demonstration used Zorn's lemma.

1. ESSENTIAL EXTENSIONS AND INJECTIVE HULLS

(1.1) Definition-Proposition. *If R is a ring, a homomorphism of R-modules $h : M \to N$ is called an* essential extension *if it is injective and the following equivalent conditions hold:*
(a) *Every nonzero submodule of N has nonzero intersection with $h(M)$.*
(b) *Every nonzero element of N has a nonzero multiple in $h(M)$.*

(c) *If $\phi\colon N \to Q$ is a homomorphism and ϕh is injective then ϕ is injective.*

Proof. (a) and (b) are equivalent because a nonzero submodule of N will always have a nonzero cyclic submodule (take the submodule generated by any nonzero element). If (a) holds and $\operatorname{Ker}\phi$ is not zero it will meet $h(M)$ in a nonzero module. On the other hand if (c) holds and $W \subseteq N$ is any submodule, let $\phi\colon N \to N/W$. If W is not zero, this map is not injective, and so ϕh is not injective, which means that W meets $h(M)$. □

(1.2) Proposition. *Let M, N, and Q be R-modules.*

(a) *If $M \subseteq N \subseteq Q$ then $M \subseteq Q$ is essential if and only if $M \subseteq N$ and $N \subseteq Q$ are both essential.*

(b) *If $M \subseteq N$ and $\{N_i\}_i$ is a family of submodules of N each containing M such that $\bigcup_i N_i = N$, then $M \subseteq N$ is essential if and only if $M \subseteq N_i$ is essential for every i.*

(c) *The identity map on M is an essential extension.*

(d) *If $M \subseteq N$ then there exists a maximal submodule N' of N such that $M \subseteq N'$ is essential.*

Proof. (a), (b) and (c) are easy exercises. (d) is immediate from Zorn's lemma, since the union of a chain of submodules of N containing M each of which is an essential extension of M is again an essential extension of M. □

(1.3) Example. Let R be an integral domain. The fraction field of R is an essential extension of R, as R-modules.

(1.4) Example. Let (R, m, K) be a local ring and let N be an R-module such that every element of N is killed by a power of m. Thus, every finitely generated submodule of N has finite length. Let $\operatorname{Soc} N$, the socle of N, be $\operatorname{Ann}_N m$, the largest submodule of N which may be viewed as a vector space over K. The $\operatorname{Soc} N \subseteq N$ is an essential extension. To see this, let $x \in N$ be given nonzero element and let t be the largest integer such that $m^t x \neq (0)$. Then we can choose $y \in m^t$ such that $yx \neq 0$. Since $m^{t+1}x = 0$, $my \subseteq mm^t x = 0$, and so $y \in \operatorname{Soc} M$. (Exercise: show that if $S \subseteq N$ is any submodule such that $S \subseteq N$ is an essential extension, then $\operatorname{Soc} N \subseteq S$.)

(1.5) Exercise. Show that if $M_i \subseteq N_i$ is essential, $i = 1, 2$, then $M_1 \oplus M_2 \subseteq N_1 \oplus N_2$ is essential. Generalize this to arbitrary (possibly infinite) direct sums.

In the situation of Proposition (1.2d) we shall say that N' is a *maximal essential extension* of M within N. If $M \subseteq N$ is an essential extension and N has no proper essential extension we shall say that N is a *maximal essential extension* of M. It is not clear that maximal essential extensions in the absolute sense exist. However, they do exist: we shall deduce this from the fact that every module can be embedded in an injective module.

(1.6) Proposition. *Let R be a ring.*

(a) *An R-module is injective if and only if it has no proper essential extension.*

(b) *If M is an R-module and $M \subseteq E$ with E injective, then a maximal essential extension of M within E is an injective module and, hence, a direct summand of E. Moreover, it is a maximal essential extension of M in an absolute sense, since it has no proper essential extension.*

(c) *If $M \subseteq E$ and $M \subseteq E'$ are two maximal essential extensions of M, then there is a (non-canonical) isomorphism of E with E' that is the identity map on M.*

Proof. (a) It is clear that an injective R-module E cannot have a proper essential extension: if $E \subseteq N$ then $N \cong E \oplus E'$, and nonzero elements of E' cannot have a nonzero multiple in E. It follows that $E' = 0$. On the other hand, suppose that M has no proper essential extension and embed M in an injective module E. By Zorn's lemma we can choose $N \subseteq E$ maximal with respect to the property that $N \cap M = 0$. Then $M \subseteq E/N$ is essential, for if N'/N were a nonzero submodule of E/N that did not meet M then $N' \subseteq E$ would be strictly larger than N and would meet M in 0. Thus, $M \to E/N$ is an isomorphism, which implies that $E = M + N$. Since $M \cap N = 0$, we have that $E = M \oplus N$, and so M is injective.

(b) Let E' be a maximal essential extension of M within the injective module E. We claim that E' has no proper essential extension whatsoever, for if $E' \subseteq Q$ were such an extension the inclusion $E' \subseteq E$ would extend to a map $Q \to E$, because E is injective. Moreover, the map $Q \to E$ would have to be injective, because its restriction to E' is injective and $E' \subseteq Q$ is essential. This would yield a proper essential extension of E' within E, a contradiction. By part (a), E' is injective, and the rest is obvious.

(c) Since E' is injective the map $M \subseteq E'$ extends to a map $\phi : E \to E'$. Since $M \subseteq E$ is essential, ϕ is injective. Since $E \cong \phi(E) \subseteq E'$, $\phi(E)$ is injective and so $E' = \phi(E) \oplus E''$. Since $M \subseteq E'$ is essential and $M \subseteq \phi(E)$, E'' must be zero. $\square$

If $M \to E$ is a maximal essential extension of M over R we shall also refer to E is an *injective hull* or an *injective envelope* for M and write $E = E_R(M)$ of $E = E(M)$. Note that every R-module M has an injective hull, unique up to non-canonical isomorphism. Note also that if $M \subseteq E$, where E is any injective, then M has a maximal essential extension E_0 within E that is actually a maximal essential extension of M. Thus, $M \subseteq E$ will factor $M \subseteq E(M) \subseteq E$, and then $E(M)$ will split off from E, so that we can think of E as $E(M) \oplus E'$, where E' is some other injective.

(1.7) Exercise. Using (1.5), show that there is an isomorphism $E(M_1 \oplus M_2) \cong E(M_1) \oplus E(M_2)$. (The corresponding statement for infinite direct sums is false in general, because a direct sum of injective modules need not be injective. However, it is true if the ring is Noetherian.)

(1.8) Discussion. Given an R-module M we can form an injective resolution as follows: let $E_0 = E(M)$, let $E_1 = E(E_0/\operatorname{Im} M)$, let $E_2 = E(E_1/\operatorname{Im} E_0)$, and, in general, if

$$0 \to E_0 \to E_1 \to \cdots \to E_i$$

has been constructed (with $M = \operatorname{Ker}(E_0 \to E_1)$), let $E_{i+1} = E(E_i/\operatorname{Im} E_{i-1})$. Note that we have $E_i \twoheadrightarrow E_i/(\operatorname{Im} E_{i-1}) \subseteq E_{i+1} = E(E_i/\operatorname{Im} E_{i-1})$ so that we get a composite map $E_i \to E_{i+1}$ whose kernel is $\operatorname{Im} E_{i-1}$. It is evident that this yields an injective resolution of M.

We shall say that a given injective resolution

$$0 \to E_0 \to \cdots \to E_i \to \cdots$$

(with $M = \operatorname{Ker}(E_o \to E_1)$) is a *minimal injective resolution* of M if $M \to E_0$ is an injective hull for M and if for every $i \geq 0$, $\operatorname{Im}(E_i \to E_{i+1}) \subseteq E_{i+1}$ is an injective hull for $\operatorname{Im}(E_i \to E_{i+1})$. The discussion just above shows that minimal injective resolutions exist. It is quite easy to see that any two minimal injective resolutions for M are isomorphic as complexes.

2. THE NOETHERIAN CASE

(2.1) Proposition. *Let M be a finitely generated R-module and let $\{N_i\}_i$ be a possibly infinite family of modules. Then $\operatorname{Hom}_R(M, \bigoplus_i N_i) \cong \bigoplus_i \operatorname{Hom}_R(M, N_i)$. (In general, there is an injection of the right hand side into the left hand side.)*

Proof. Each inclusion $N_j \subseteq \bigoplus_i N_i$ induces a map $\operatorname{Hom}_R(M, N_j) \subseteq \operatorname{Hom}_R(M, \bigoplus_i N_i)$. It is easy to see that the various submodules of $\operatorname{Hom}_R(M, \bigoplus_i N_i)$ obtained in this way have the property that their sum inside $\mathrm{Hom}_R(M, \bigoplus_i N_i)$ is actually a direct sum. This explains the injection $\bigoplus_i \mathrm{Hom}_R(M, N_i) \subseteq \mathrm{Hom}_R(M, \bigoplus_i N_i)$. We want to see that the map is onto. Let $m_1, \cdots, m_h$ generate M. Let $\phi : M \to \bigoplus_i N_i$. Then each $\phi(m_\nu)$ has nonzero entries in only finitely many N_i. It follows that there is a finite set of indices $i(1), \cdots, i(r)$ such that every $\phi(m_\nu) \subseteq \bigoplus_{s=1}^r N_{i(s)} \subseteq \bigoplus_i N_i$, and then $\phi(M) \subseteq \bigoplus_{s=1}^r N_{i(s)}$. Since $\mathrm{Hom}_R(M, \_)$ commutes with finite direct sums, the result follows. $\square$

(2.2) Corollary. *Let R be a Noetherian ring. Then an arbitrary (possibly infinite) direct sum of injective modules is injective.*

Proof. Call the family $\{E_i\}$. It suffices to show that if I is an ideal of R, then

$$\mathrm{Hom}\,(R, \bigoplus_i E_i) \to \mathrm{Hom}(I, \bigoplus_i E_i)$$

is surjective. Using (2.1), it is easy to see that this is the direct sum of the maps $\mathrm{Hom}\,(R, E_i) \to \mathrm{Hom}\,(I, E_i)$, each of which is surjective. $\square$

When R is Noetherian we shall write $\mathrm{Ass}\, M$ for

$$\{P \in \mathrm{Spec}\, R : R/P \text{ can be embedded in } M\}$$

whether M is finitely generated or not. If $M \neq 0$, $\mathrm{Ass}\, M$ is nonempty, since it contains $\mathrm{Ass}\, Rx$ for every nonzero element $x \in M$.

(2.3) Proposition. *Let E be any injective module over a Noetherian ring R. Then E is a direct sum of modules each of which has the form $E_R(R/P)$ for some prime ideal P of R.*

Proof. Choose a maximal family $\{E_i\}_i$ of submodules $E_i \subseteq E$ such that (1) each $E_i \cong E(R/P_i)$ for some prime ideal P_i of R, and (2) the sum of the E_i in E is an internal direct sum (i.e., each E_i is disjoint from any finite sum of other E_j). Such a family exists by Zorn's lemma (it might be the empty family). Let E_1 be the sum of the modules in the family, which has the form we want. We want to prove that $E = E_1$. We know that E_1 is injective, by (2.2). Thus, we can write $E = E_1 + E'$, where the sum is an internal direct sum. We shall show that if $E' \neq 0$ then it has a submodule E'' of the form $E(R/P)$. This will yield a contradiction, since E'' can evidently be used to enlarge the supposedly maximal family $\{E_i\}_i$. Suppose that $x \in E' - \{0\}$. Then $Rx \neq 0$ is a finitely generated nonzero module: choose $P \in \operatorname{Ass} Rx$, so that we have embedding $R/P \subseteq Rx \subseteq E'$. Then we also have $E(R/P) \subseteq E'$, as wanted. $\square$

We shall soon prove a uniqueness statement for decompositions of injective modules as in (2.3). We first want to study the modules $E(R/P)$ more closely.

(2.4) Theorem. *Let R be a Noetherian ring and let P be a prime ideal. Let E denote an injective hull for R/P.*

(a) *The set of all elements in E that are killed by P is isomorphic with the fraction field F of R/P; $F \cong R_P/PR_P$.*

(b) *Multiplication by an element of $R - P$ is an automorphism of E. Thus E has, in a unique way, the structure of an R_P-module.*

(c) *E is a maximal essential extension of F (as described in part (a)) over R_P. Thus, E is also an injective hull for F over R_P. We may abbreviate these facts by writing $E_R(R/P) \cong E_{R_P}(R_P/PR_P)$.*

(d) *Every element of E is killed by a power of P, and $\operatorname{Ass} E = \{P\}$. The annihilator of every nonzero element of E is primary to P.*

(e) $\operatorname{Hom}_{R_P}(F, E_P) \cong F$, *while for any prime ideal Q of R different from P, we have that* $\operatorname{Hom}_{R_P}(F, E(R/Q)_P) = 0$.

Proof. Since $R/P \subseteq F$ is an essential extension, we have a copy of F in the maximal essential extension E, and we may also view $E = E_R(F)$. Let $a \in R - P$. Since multiplication by a is injective on F, it is injective on E, for E is an essential extension of F. Then aE is an injective submodule of E containing F, and will split off. Since E is an essential extension of F, we must have $aE = E$. Thus, E is a module over R_P, and contains a copy of $F \cong R_P/PR_P$. The annihilator of P in E will then be the same as the annihilator of PR_P in E, and so may be regarded as an F-vector space V, with $F \subseteq V \subseteq E$. Since E is an essential extension of F, so is V. But this is impossible unless $V = F$, since $F \subseteq V$ will split over F (and, hence, over R_P). This establishes both (a) and (b).

It is clear that $F \subseteq E$ is essential as a map of R_P-modules, since this is true even as a map of R-modules. Suppose that E has an essential extension M as an R_P-module. Let $x \in M$ be any nonzero element. Then we can choose $r \in R$, $a \in R - P$ and $e \in E$ such that $(r/a)x = e \neq 0$ in E. Then $rx = ae \neq 0$ by (b). Thus $E \subseteq M$ is an essential extension as R-modules as well, and so $E = M$, as required. This proves (c).

Now suppose that $R/Q \subseteq E$. Let x be a nonzero element of E such that $\operatorname{Ann} x = Q$, i.e., $Rx \cong R/Q$. Supposedly, x has a nonzero multiple in R/P. But every nonzero element in R/Q has annihilator Q, while every nonzero element in R/P has annihilator P. This yields a contradiction. Thus, $\operatorname{Ass} E = \{P\}$. Now, if $x \in E$, let $I = \operatorname{Ann} x$. Then $R/I \cong Rx \subseteq E$, and so $\operatorname{Ass}(R/I) \subseteq \operatorname{Ass} E = \{P\}$. Thus, P is the only associated prime of I, which implies that I is primary to P. This establishes (d).

For the first statement in part (e), note that $E_P \cong E$. Any value of a homomorphism of F into E must be killed by P, and so must lie in $\operatorname{Ann}_E P \cong F$. Thus, $\operatorname{Hom}_{R_P}(F, E) \cong \operatorname{Hom}_{R_P}(F, F) \cong F$.

Now suppose that $Q \neq P$. If P does not contain Q we have that $E(R/Q)_P = 0$, since the element of Q that is not in P acts invertibly on $E(R/Q)_P$ while, at the same time, for each element of $E(R/Q)_P$ it has a power that kills that element. If $Q \subset P$ strictly, note that a nonzero element of the image of a map of F into E_P must be killed by P. This will yield an element of $\operatorname{Ass} E_P$ that is a prime containing P. But the annihilator of each element will be QR_P-primary. $\square$

(2.5) Exercise. Show that if $M \subseteq N$ is essential then $\operatorname{Ass} M = \operatorname{Ass} N$.

(2.6) Theorem. *Let E be an injective module over a Noetherian ring R. Let P be a prime ideal of R and let F be the fraction field of $R/P \cong R_P/PR_P$. Then the number of copies of $E(R/P)$ occurring in a representation of E as direct sum of modules of this form is $\dim_F \operatorname{Hom}_{R_P}(F,\, E_P)$, and so is independent of which such representation we choose. $E(R/P)$ occurs if and only if $P \in \operatorname{Ass} E$.*

Proof. Suppose that $E = \bigoplus_i E(R/P_i)$. Then

$$\operatorname{Hom}_{R_P}(F, E_P) \cong \operatorname{Hom}_{R_P}(F, \bigoplus_i E(R/P_i)_P) \cong \bigoplus_i \operatorname{Hom}_{R_P}\big(F, E(R/P_i)\big)_P.$$

Now $\operatorname{Hom}_{R_P}\big(F, E(R/P_i)\big)_P \cong F$ if $P_i = P$ and is 0 otherwise. Thus, if J_P is the set of values for i such that $P_i = P$, then $\operatorname{Hom}_{R_P}(F, E_P) \cong \bigoplus_{i \in J_P} F$, a vector space over F whose dimension is the number of copies of $E(R/P)$ occurring in the decomposition, as required.

The final statement is left as an exercise. $\square$

(2.7) Corollary. *A nonzero injective module over a Noetherian ring R is indecomposable (not a direct sum in a non-trivial way) if and only if it isomorphic with $E(R/P)$ for some prime P.*

Proof. If $E(R/P)$ were the direct sum of two nonzero modules, these would be injective, and so decompose further into direct sums of modules of the form $E(R/Q)$. This would give two different representations for $E(R/P)$ as a direct sum of modules of the form $E(R/Q)$, one with only one term, and one with at least two terms, a contradiction. Since every injective is a direct sum of modules of the form $E(R/P)$, the only possible indecomposable injective modules are the modules $E(R/P)$ themselves. $\square$

We therefore have a bijective correspondence between the prime ideals of a Noetherian ring R and the isomorphism classes of indecomposable injective modules, where P corresponds to the isomorphism class of $E_R(R/P)$. We also note:

(2.8) Theorem. *Let R be a Noetherian ring and let S be a multiplicative system.*

(a) *The injective modules over $S^{-1}R$ coincide with the injective R-modules E with the property that for every $E(R/P)$ occurring as a summand (i.e. for every $P \in \operatorname{Ass} E$), P does not meet S.*
(b) *If E is any injective R-module then $S^{-1}E$ is an injective $S^{-1}R$-module.*
(c) *If $M \subseteq N$ is essential then $S^{-1}M \subseteq S^{-1}N$ is essential. If $M \subseteq E$ is a maximal essential extension then $S^{-1}M \subseteq S^{-1}E$ is a maximal essential extension.*

Proof. Each indecomposable injective module over $T = S^{-1}R$ has the form $E_T(T/Q)$, where Q is a prime of T. Note that $T_Q \cong R_P$, where P is the unique prime ideal of R whose expansion is Q (it is also true that P is the contraction of Q: expansion and contraction give bijections between the primes in Spec R disjoint from S and Spec T). But then $E_T(T/Q) \cong E_{T_Q}(T_Q/QT_Q) \cong E_{R_P}(R_P/PR_P) \cong E_R(R/P)$. Thus, the indecomposable injectives over T are precisely the modules $E_R(R/P)$ for P in Spec R disjoint from S. This proves (a).

For part (b), since E will be a direct sum of indecomposable injectives, we may assume that $E = E_R(R/P)$. If S does not meet P then $S^{-1}E \cong E$, while if S meets P then, since every element of $E = E_R(R/P)$ is killed by a power of P, we have that $S^{-1}E = 0$. This proves (b).

Now suppose that $M \subseteq N$ is essential. Let $x/s \in S^{-1}N$, $x \in N$, $s \in S$, be nonzero. Then $x/1$ is the same element, up to a unit. We may replace $x/1$ by a multiple whose annihilator is a prime ideal Q of $S^{-1}R$, and we then have that $Q = PS^{-1}R$ where P is the contraction of Q to R. After replacing x by $s'x$ for suitable $s' \in S$ we may assume that x is killed by P. It follows that $\operatorname{Ann} x = P$, where P is disjoint from S. Let rx be a nonzero multiple of x in M. The rx is also a nonzero element of $R/P \cong Rx$, and so $\operatorname{Ann} rx = P$.

It follows that the image of rx in $S^{-1}M$ is not zero, and this proves the first statement. If we couple this result with (b) we obtain the final statement. $\square$

(2.9) Theorem. *Let M be a finitely generated module over a Noetherian ring R, and let $0 \to E_0 \to \cdots \to E_i \to \cdots$ denote a minimal injective resolution of M (so that $M = \mathrm{Ker}\,(E_0 \to E_1)$). Then for every prime P of R, the number of copies of $E(R/P)$ occurring in E_i is finite: in fact, if $F = R_P/PR_P$, this number is equal to $dim_F \mathrm{Ext}^i_{R_P}(F, M_P)$. (This number is sometimes denoted $\mu_i(P, M)$ and called the i th Bass number of M with respect to P).*

Proof. If we localize at P we obtain a minimal resolution of M_P over R_P, by (2.7c). In this way we reduce to the case where P is the maximal ideal of a local ring (R, P, F) with residue field F: the number of copies of $E(R/P)$ is unaffected. In this situation, the number of copies of $E(F)$ in E_i is precisely $\dim_F \mathrm{Hom}_R(F, E_i)$. On the other hand, the modules $\mathrm{Ext}^i_R(F, M)$ are the homology of the complex $\mathrm{Hom}_R(F, E.)$. Thus, to prove the result it suffices to show that all the maps in the complex $\cdots \to \mathrm{Hom}_R(F, E_i) \to \cdots$ are zero. Now $\mathrm{Hom}_R(F, E_i) \cong \mathrm{Ann}_{E_i} P$ is a vector subspace of E_i: each element in it is contained in a copy of $F \subseteq E_i$. It therefore suffices to see that this copy of F must be zero, i.e., that it must be in $\mathrm{Im}\, E_{i-1}$ (where $E_{-1} = M$). But a generator of that copy of F has a nonzero multiple in $\mathrm{Im}\, E_{i-1}$, since $\mathrm{Im}\, E_{i-1} \subseteq E_i$ is essential. Since any nonzero element of F generates it as an R-module, it follows that the entire copy of F is in $\mathrm{Im}\, E_{i-1}$ and, hence, maps to zero in E_{i+1}. $\square$

3. THE INJECTIVE HULL OF THE RESIDUE FIELD OF A LOCAL RING

We have already seen that, in a certain sense, understanding the injective modules over a Noetherian ring R comes down to understanding the injective hulls $E_R(R/P) \cong E_{R_P}(R_P/PR_P)$. Thus, we are led to consider what the injective hull of the residue field of a local ring (R, m, K) is like. We already know that every element of $E(K)$ is killed by a power of the maximal ideal of R, so that $\mathrm{Ass}\, E(K) = \{m\}$. Any module M with the property that every element is killed by a power of m is automatically a module over $\widehat{R}$ (the m-adic completion of R): if $x \in M$ is killed by m^t and $s \in \widehat{R}$ we let sx be the same as rx, where r is any element of R such that $s - r \in m^t \widehat{R}$. In fact, if M is such a module then its R-submodules are the same as its $\widehat{R}-$ submodules (all submodules and quotients inherit the same property) and if $M,\ N$ are two such modules then $\mathrm{Hom}_R(M,\ N) = \mathrm{Hom}_{\widehat{R}}(M,\ N)$. Every such module over $\widehat{R}$ may be viewed as arising from itself considered as an R-module via restriction of scalars. This leads to:

(3.1) Theorem. *Let (R, m, K) be a local ring. A maximal essential extension of K over R is also a maximal essential extension of K over $\widehat{R}$. I.e., $E_R(K) \cong E_{\widehat{R}}(K)$.*

Proof. It is clear that $E_R(K)$ viewed as an $\widehat{R}$-module then W still has the property that every element is killed by a power of $m\widehat{R}$, and, hence, of m. Since the R-modules and the $\widehat{R}$-submodules of W are the same, W is an essential extension of $E_R(K)$ as an R-module. Thus, $W = E_R(K)$, as required. $\square$

This means that it suffices to "understand" $E_R(K)$ when (R, m, K) is a *complete* local ring.

We recall:

(3.2) Lemma. *If $R \to S$ is a ring homomorphism, E is injective over R, and F is S-flat, then $\mathrm{Hom}_R(F, E)$ (which has an S-module structure induced by the S-module structure on F) is an injective S-module. In particular, if E is an injective R-module, then $\mathrm{Hom}_R(S, E)$ is an injective S-module.*

The point is that the functors (from S-modules to S-modules) $\mathrm{Hom}_R(\_\,, \mathrm{Hom}_R(F, E))$ and $\mathrm{Hom}_S(\_ \otimes_S F,\ E)$ are isomorphic, by the adjointness of $\otimes$ andHom. Since $\_ \otimes_S F$ and $\mathrm{Hom}_R(\_\,, E)$ are both exact functors, so is the composite $\mathrm{Hom}_R(\_ \otimes_S F, E)$, and this implies that $\mathrm{Hom}_S(\_\,, \mathrm{Hom}_R(F, E))$ is an exact functor, which means that $\mathrm{Hom}_R(F, E)$ is S-injective. (This is how one embeds an arbitrary R-module M in an injective: let W be the injective $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ (over a PID, a module is injective if it is divisible), map a free R-module F onto $M^\vee$, where ${}^\vee = \mathrm{Hom}_{\mathbb{Z}}(\_\,,\ W)$, and so obtain a composite injection $M \to M^{\vee\vee} \to F^\vee$. $F^\vee$ is R-injective by the lemma above.) This lemma suggests a useful transition between injective hulls of residue fields when one has a local homomorphism.

(3.3) Theorem. *Let $(R, m, K) \to (S, n, L)$ be a local homomorphism of local rings and suppose that S is module-finite over the image of R. Let E be an injective hull of K over R. Then $\mathrm{Hom}_R(S,\ E)$ is an injective hull of L over S.*

Proof. First note that mS will be n-primary. If we fix $\phi \in \mathrm{Hom}_R(S, E)$ then each of its values on the finitely many generators of S will be killed by a power of m. It follows that ϕ is killed by a power of m, hence by a power of mS, and so by a power of n. By the preceding lemma, $\mathrm{Hom}_R(S, E)$ is S-injective, and since every element is killed by a power of n it must be a direct sum of copies of the injective hull of $L = S/n$ over S. It remains to see that there is only on copy, for which we need only show that $\mathrm{Hom}_S(S/n,\, \mathrm{Hom}_R(S, E))$ is one-dimensional as a vector space over L. As in the proof of (3.2) this module is $\cong \mathrm{Hom}_R(S/n \otimes_S S,\, E) \cong \mathrm{Hom}_R(S/n,\, E)$. Since S/n is killed by m, the image of any map of S/n into E must lie in $\mathrm{Ann}_E m \cong \mathrm{Hom}_R(K, E) \cong K$, the unique copy of K which is the socle in E. Thus, $\mathrm{Hom}_S(S/n, \mathrm{Hom}_R(S, E)) \cong \mathrm{Hom}_R(S/n, K) \cong \mathrm{Hom}_K(L, K)$, which is a finite-dimensional vector space over L, say of dimension δ over L. But it has dimension $d = \dim_K L$ over K, and we see that $d\delta = d$, so that $\delta = 1$. □

The situation is much more transparent when S is a homomorphic image of R:

(3.4) Corollary. *If $S = R/I$, where (R, m, K) is local, and $E = E_R(K)$, then the annihilator of I in E ($\cong \mathrm{Hom}_R(R/I, E)$ is an injective hull for K over S.*

Proof. Although this follows at once form (3.2), we give a completely different argument. Let E' be a maximal essential extension of K as an (R/I)-module. Then it is also an essential extension of K as an R-module, and so may be identified with a submodule of a maximal essential extension E of K as an R-module. Then $E' \subseteq E''$, where E'' is the set of all elements in E that are killed by I. But $K \subseteq E''$, is essential over R, and it follows that this is an essential extension of (R/I)-modules. Thus, $E' = E''$. □

We have already noticed that replacing a local ring (R, m, K) by its completion does not affect the injective hull of K. Once R is complete, we may view it as T/I where T is complete regular (even a formal power series ring). Thus, if we understand the injective hull of the residue field of T, we can think of $E_R(K)$ as the set of elements inside it killed by I. This gives us one handle on $E_R(K)$.

However, we can use (3.4) in a different way to gain insight into the structure of $E_R(K)$. The set of elements in $E = E_R(K)$ killed by m^t may be identified with $E_{R/m^t}(k)$. But every element of E is killed by some power of m. Thus, we may think of E as the union of the modules $E_{R/m^t}(K)$. We may also use a different sequence of m-primary ideals, provided it is cofinal with the powers of m (e.g., the ideals $(x_1^t, \cdots, s_n^t)R$, where $x_1, \cdots, x_n$ is a system of parameters). This suggests that to understand E we should first try to understand the injective hull of K in the case where R is an Artin local ring.

4. THE CASE OF AN ARTIN LOCAL RING

If the Artin local ring (R, m, K) contains a field, it will contain a coefficient field, i.e. a copy of K that maps isomorphically onto K when we kill m. In this situation R is a finite dimensional K-vector space. Since $E_K(K) = K$, it is immediate from (3.3) that $E_R(K) \cong \mathrm{Hom}_K(R, K)$ in this case. This R-module has the same length (or dimension over K) that R does. When R does not contain a field, we can no longer use vector space dimension, but the notion of length is still available. The situation is the same in the general case. This is easy to prove, but is nonetheless a very important theorem that is the basis for the fancy duality theory we shall obtain later.

(4.1) Theorem. *Let (R, m, K) be an Artin local ring. Then $E_R(K)$ is a module of finite length, and its length is equal to the length of R*

Proof. We use induction of the length $\ell(R)$ of R. If the length of R is 1 then $R = K$ and $E \cong K$. Now suppose that the length is positive and choose $x \in m$ in the highest nonvanishing power of m, so that $x \neq 0$ but $mx = 0$. Thus, $Rx \cong R/m$ as R-modules. We have a short exact sequence $0 \to Rx \to R \to R' \to 0$, where $R' = R/xR$ is a ring of length exactly one less than R. Let ${}^\vee = \mathrm{Hom}_R(\_\,, E)$, where $E = E_R(K)$. Then we have a short exact sequence $0 \to (Rx)^\vee \to E \to (R')^\vee \to 0$. $Rx \cong R/m, (Rx)^\vee \cong \mathrm{Hom}_R(R/m, E) \cong R/m$, and $(R')^\vee = \mathrm{Hom}_R(R', E)$ is an injective hull for R'. By the induction hypothesis, $\ell((R')^\vee) = \ell(R') = \ell(R) - 1$. It follows that $\ell(E) = \ell(R)$. $\square$

We next observe:

(4.2) Lemma. *Let (R, m, K) be any local ring and let ${}^\vee$ denote $\mathrm{Hom}_R(\_\,, E)$, where $E = E_R(K)$. Then for every finite length module M, $\ell(M^\vee) = \ell(M)$.*

Proof. First note that $K^\vee \cong K$, which takes care of the case where the length of M is one. The result now follows from an easy induction: if M has length bigger than one, there is a short exact sequence $0 \to K \to M \to M' \to 0$, where $\ell(M') = \ell(M) - 1$, and applying ${}^\vee$ yields the result. $\square$

We can now show:

(4.3) Theorem. *Let (R, m, K) be an Artin local ring and let $E = E_R(K)$. Then the obvious map $R \to \mathrm{Hom}_R(E, E)$ (which sends r to the map multiplication by r) is an isomorphism.*

Proof. By (4.2) $\mathrm{Hom}_R(E, E)$ has the same length as E, which has the same length as R by (4.1). Thus, R and $\mathrm{Hom}_R(E, E)$ have the same length. Therefore, to show that the map is an isomorphism it suffices to show that it is one-to-one. Suppose $x \in R$ kills E. Then $\mathrm{Hom}_R(R/xR, E) = E$ will be an injective hull for K over R/xR. But then $\ell(R) = \ell(E_R(K)) = \ell(E_{R/xR}(K)) = \ell(R) - \ell(xR)$, so that $\ell(xR) = 0 \Rightarrow x = 0$. $\square$

We can now classify the local rings that are injective as modules over themselves.

(4.4) Theorem. *A local ring (R, m, K) is injective as a module over itself if and only if the Krull dimension of R is zero and the socle of R is one-dimensional as a K-vector space (which means that the type of R as a Cohen-Macaulay ring is 1). Moreover, $R \cong E_R(K)$ in this case.*

Proof. Suppose that R in injective as an R-module. A local ring is always indecomposable as a module over itself (apply $K \otimes_R \_$ to see this). Thus, if R is injective then it is isomorphic to $E(R/P)$ for some prime ideal P. Since each element of $m - P$ acts invertibly on $E(R/P)$ but non-invertibly on R, we must have $P = m$. Since every element of R is killed by a power of m, including the identity element, we must have that m is nilpotent, and that $R = E(R/m)$. Since the socle in $E(R/m)$ is one-dimensional over K, this must be true for R as well.

Now suppose that R has Krull dimension zero and that the socle of R is one-dimensional. Since a module M with $\mathrm{Ass}\, M = \{m\}$ is an essential extension of its socle, R is an essential extension of $K = R/m$, and so R can be enlarged to a maximal essential extension E of K. But R and E must have the same length. Thus, $R = E$ is injective in this situation. $\square$

Our next objective is to show that Theorem (4.3) is valid for any complete local ring. This is the essential point in the proof of what is known as Matlis duality. In the course of this we shall also show that the injective hull of the residue field of a local ring has DCC, and that an R-module has DCC if and only if it can be embedded in a finite direct sum of copies of the injective hull of the residue field of R.

5. MODULES WITH DCC AND MATLIS DUALITY

If (R, m, K) is local with $E = E_R(K)$ we shall let $\_^{\vee}$ denote the exact contravariant functor $\mathrm{Hom}_R(\_\,, E)$. We have an obvious map $R \to \mathrm{Hom}_R(E,\, E)$ that sends r to the map consisting of multiplication by r. But since E is also an injective hull of K over $\widehat{R}$, and since $\mathrm{Hom}_R(E,\, E) = \mathrm{Hom}_{\widehat{R}}(E,\, E)$, this map extends to a map $\widehat{R} \to \mathrm{Hom}_R(E,\, E)$ that sends s to the map consisting of multiplication by s. Our next main result is:

(5.1) Theorem. *With notation as in the paragraph above, the map* $\widehat{R} \to \mathrm{Hom}_R(E, E)$ *is an isomorphism. Thus, if R is a complete local ring, the obvious map* $R \to \mathrm{Hom}_R(E, E) = E^{\vee}$ *is an isomorphism.*

Proof. It suffices to prove the second statement, and so we assume that R is complete. Let $R(t) = R/m^t$. Then the set of elements $E(t)$ killed by m^t in E is an injective hull for $R(t)$. Any map of E into E maps $E(t)$ into $E(t)$. We already know that the module of such maps is isomorphic with $R(t)$. Now, $E = \cup_t E(t)$, so that to give an endomorphism of E is equivalent to giving a family of endomorphisms of the $E(t)$ that fit together under restriction, i.e. an element of $\varprojlim_t R/m^t \cong R$ in the complete case. $\square$

Note that the functor $\_^{\vee} = \mathrm{Hom}_R(\_\,, E_R(K))$ is faithful when (R, m, K) is a local ring. If M is a nonzero module we can embed a nonzero cyclic module $R/I \to M$. Since $M^{\vee} \to (R/I)^{\vee}$ is onto, it suffices to see that $(R/I)^{\vee}$ is nonzero. But $R/I \to R/m$ is onto, so that $(R/m)^{\vee} \to (R/I)^{\vee}$ is injective, and $(R/m)^{\vee} \cong R/m$ is nonzero. We next observe:

(5.2) Corollary. *Let (R, m, K) be a local ring and let $E = E(K)$. The E has DCC as an R-module.*

Proof. If not we can choose an infinite strictly descending chain of submodules

$$E \subseteq E_1 \supset E_2 \supset \cdots \supset E_i \supset \cdots ,$$

and if we apply $^{\vee}$ we obtain

$$\widehat{R} \twoheadrightarrow S_1 \twoheadrightarrow S_2 \twoheadrightarrow \cdots \twoheadrightarrow S_i \twoheadrightarrow \cdots ,$$

where each of the maps $S_i \twoheadrightarrow S_{i+1}$ is a proper surjection (nonzero kernel). Each of these map is $\widehat{R}$-linear, and so the kernels $J_i = Ker(\widehat{R} \twoheadrightarrow S_i)$ form a strictly increasing chain of ideals of $\widehat{R}$. Since $\widehat{R}$ is Noetherian, this is a contradiction. $\square$

Of course, a ring with DCC must be Noetherian, but this is not at all true for modules. The modules with DCC that are also finitely generated have finite length. But there are, usually, many non-Noetherian modules with DCC over the ring R. The injective hull of K will not have finite length unless R is zero-dimensional, but it does have DCC.

Note that one cannot expect to give a very simple proof that $E_R(K)$ has DCC, since the family of its submodules is precisely as rich and complicated in structure as the family of ideals of the m-adic completion of R.

(5.3) Theorem. *Let (R, m, K) be a local ring and let M be an R-module. The following conditions are equivalent:*

(1) *Every element of M is killed by a power of m and the socle of M is a finite-dimensional vector space over K.*

(2) $\operatorname{Ass} M = \{m\}$ *and the socle of M is a finite-dimensional vector space over K.*

(3) *M is an essential extension of a finite-dimensional K-vector space.*

(4) *The injective hull of M is a finite direct sum of copies of $E = E_R(K)$.*

(5) *M can be embedded in a finite direct sum of copies of E.*

(6) *M has DCC.*

Proof. We shall prove that (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (1). Assume (1). Then each cyclic module R/I embeddable in M has the property that I contains a power of m, and so if I is prime it must be m. Thus, (1) $\Rightarrow$ (2). Now assume (2). If R/I is isomorphic with a nonzero cyclic submodule of M then $\operatorname{Ass} R/I \subseteq \operatorname{Ass} M \Rightarrow \operatorname{Ass} R/I = \{m\} \Rightarrow I$ is m-primary. Thus (1) $\Leftrightarrow$ (2). Assume that these equivalent conditions hold. We have already observed that when (1) holds then M is an essential extension of its socle. Thus, (2) $\Rightarrow$ (3). If M is an essential extension of $V = K \oplus \cdots \oplus K$, then $E(M) = E(V) \cong E(K) \oplus \cdots \oplus E(K)$. This shows that (3) $\Rightarrow$ (4), while (4) $\Rightarrow$ (5) is obvious. (5) $\Rightarrow$ (6) because E has DCC (establishing this is the hardest part of the proof of this theorem, but we have already done so), a finite direct sum of modules with DCC has DCC, and a submodule of a module with DCC has DCC. Finally (6) implies that every element of M is killed by a power of m, because each cyclic module R/I embeddable in M will have DCC, and then R/I is an Artin ring. Moreover, the socle is a vector space with DCC, and so must be finite-dimensional (if $v_1, \cdots, v_n, \cdots$ were infinitely many linearly independent element we could let W_n be the span of the vectors v_h for $h \geq n$ for each n, and then $V_1 \supset \cdots \supset V_n \supset \cdots$ is a strictly descending infinite chain of subspaces). $\square$

We shall later take a closer look at what the injective hull of the residue field of a regular local ring is like, but not until after we have begun the study of local cohomology.

The isomorphism $R \to \mathrm{Hom}_R(E, E)$ when R is a complete local ring is the key to the following result:

(5.4) Theorem (Matlis duality). *Let $R, m, K)$ be a complete local ring, let $E = E_R(K)$ and let $\_^\vee$ denote the functor $\mathrm{Hom}_R(\_\,, E)$.*

(a) *If M is a module with ACC then $M^\vee$ has DCC, while if M has DCC then $M^\vee$ has ACC. Moreover, if M has either ACC or DCC then the obvious map $M \to M^{\vee\vee}$ is an isomorphism.*

(b) *The category of R-modules with ACC is antiequivalent to the category of R-modules with DCC. The functor $\_^\vee$ with its domain restricted to modules with ACC and its codomain to modules with DCC gives the antiequivalence in one direction, and the same functor with its domain restricted to modules with DCC and its codomain to modules with ACC gives the antiequivalence the other way.*

Proof. M has ACC (respectively, DCC) if there is a surjection $R^t \to M$ (respectively, an injection $M \subseteq E^t$). Dualizing gives an injection $M^\vee \subseteq (R^\vee)^t \cong E^t$ (respectively, a surjection $(E^\vee)^t \to M$, and $E^\vee \cong R$). Moreover, if M has ACC (respectively, DCC) there is a presentation $R^s \xrightarrow{\alpha} R^t \to M \to 0$ (respectively, $0 \to M \to E^t \xrightarrow{\beta} E^s$); since E^t/M has DCC it can be embedded in a direct sum of copies of E). Here, α is a $t \times s$ (respectively, β is an $s \times t$) matrix over R (in the case of β we are making use of the identification $R \cong \mathrm{Hom}_R(E, E)$). Applying $^\vee$ once yields $0 \to M^\vee \to (R^t)^\vee \xrightarrow{\alpha^{tr}} (R^s)^\vee$ (respectively, $(E^s)^\vee \xrightarrow{\beta^{tr}} (E^t)^\vee \to M^\vee \to 0$), where we are forestalling making use of our identifications $R \cong E^\vee$ and $R^\vee \cong E$. Applying the functor $\_^\vee$ we obtain two commutative diagrams,

$$\begin{array}{ccccccc}
(R^s)^{\vee\vee} & \xrightarrow{\alpha} & (R^t)^{\vee\vee} & \longrightarrow & M^{\vee\vee} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
R^s & \xrightarrow{\alpha} & R^t & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccc}
0 & \longrightarrow & M^{\vee\vee} & \longrightarrow & (E^t)^{\vee\vee} & \xrightarrow{\beta} & (E^s)^{\vee\vee} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & M & \longrightarrow & E^t & \xrightarrow{\beta} & E^s
\end{array}$$

Now $\_^\vee$ and, hence, $\_^{\vee\vee}$ commutes with direct sum. Since cokernels (respectively, kernels) of isomorphic maps are isomorphic, it suffices to see that the maps from R^s and R^t (respectively, E^s and E^t) to their double duals are isomorphisms, and this comes down to checking the case where $M = R$ (respectively, $M = E$). But in one case we get the map $R \to \mathrm{Hom}_R(E,\, E)$ that we already know to be an isomorphism, and in the other we get the map $E \to \mathrm{Hom}_R(E^\vee,\, E) \cong \mathrm{Hom}_R(R,\, E) \cong E$. This establishes part (a).

Let $\mathcal{F}_1$ denote the functor $\_^\vee$ from modules with ACC to modules with DCC and let $\mathcal{F}_2$ denote the functor $\_^\vee$ from modules with DCC to modules with ACC. Then part (a) implies at once that both $\mathcal{F}_1 \circ \mathcal{F}_2$ and $\mathcal{F}_2 \circ \mathcal{F}_1$ are isomorphic to the identity functor (the first on modules with DCC, the second on modules with ACC), and the result follows. $\square$

(5.5) Exercise. Let (R, m, K) be a complete local ring and let notation be as in Theorem 4.4. Let M be a finitely generated R-module. Show that $\mathrm{Hom}_K(M/mM,\, K) \cong \mathrm{Hom}_R(K,\, M^\vee)$. Thus, the least number of generators of M is the same as the dimension of the socle in its dual.

(5.6) Remark. Since modules with DCC over the local ring R all have the property that every element is killed by a power of m, the category of R-modules with DCC is the same as the category of $\widehat{R}$-modules with ACC. If M is finitely generated over R, then $M^\vee$ still has DCC, while the map $M \to M^{\vee\vee}$ is isomorphic with the map $M \to \widehat{M}$. If N has DCC then $N^\vee$ is a finitely generated module over $\widehat{R}$. In general, it will not be true that every R-module with DCC "arises" from an R-module with ACC: there are more modules with ACC over $\widehat{R}$ than there are over R.

We are now ready to begin our discussion of local cohomology.

6. LOCAL COHOMOLOGY: A FIRST LOOK

(6.1) Definition. Let R be a Noetherian ring and let M be an arbitrary module. Suppose that $I \subseteq R$ is an ideal. Notice that if $I \supseteq J$ the surjection $R/J \to R/I$ induces map $\mathrm{Ext}^i(R/I, M) \to \mathrm{Ext}^i(R/J, M)$. Thus, if $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_t \supseteq \cdots$ is a decreasing sequence of ideals then we get a direct limit system

$$\cdots \to \mathrm{Ext}^i_R(R/I_t, M) \to \mathrm{Ext}^i_R(R/I_{t+1}, M) \to \cdots$$

and we may form the direct limit of these Ext 's. We define $H^i_I(M) = \varinjlim_t \mathrm{Ext}^i(R/I^t, M)$, and call this module the i th *local cohomology module of* M *with support in* I.

(6.2) Discussion. Suppose that we replace the sequence $\{I_t\}_t$ by an infinite subsequence. The direct limit is obviously unaffected. Likewise, if $\{J_t\}_t$ is another decreasing sequence of ideals which is cofinal with I_t (i.e., for all t, there exists u such that $J_u \subseteq T_t$ and v such that $I_v \subseteq J_t$), then the direct limit computed using the J's is the same. We can form a sequence $I_{a(1)} \supseteq J_{b(1)} \supseteq I_{a(2)} \supseteq J_{b(2)} \supseteq \cdots \supseteq I_{a(t)} \supseteq J_{b(t)} \supseteq \cdots$ which yield the same result, on the one hand, as $\{I_{a(t)}\}_t$ and, hence, as $\{I_t\}_t$. Similarly, it yields the same result as $\{J_t\}_t$.

In particular, if $I = (x_1, \cdots, x_n)R$, then the sequence $I_t = (x_1^t, \cdots, x_n^t)R$ is cofinal with the powers of I, and so may be used to compute the local cohomology.

We also have:

(6.3) Theorem. *If I, J are ideals of the Noetherian ring R with the same radical, then $H_I^i(M) \cong H_J^i(M)$ canonically for all i and for all R-modules M.*

Proof. Each of the ideals I, J has a power contained in the other, and it follows that the sequences $\{I^t\}_t, \{J^t\}_t$ are cofinal with one another. □

(6.4) Discussion. If $X \subseteq \operatorname{Spec} R$ is closed, then $X = V(I)$ where I is determined up to radicals: we may write $H_X^i(M)$ for $H_I^i(M)$ and refer to *local cohomology with support in* X.

(6.5) Discussion. $\operatorname{Ext}_R^i(R/I^t, M)$ is a covariant additive functor of M, and Ext has a long exact sequence. All this is preserved when we take a direct limit. Thus, each $H_I^i()$ is a covariant additive functor, and given a short exact sequence of modules

$$0 \to A \to B \to C \to 0$$

there is a long exact sequence

$$0 \to H_I^0(A) \to H_I^0(B) \to H_I^0(C) \to H_I^1(A) \to H_I^1(B) \to H_I^1(C) \to$$

$$\cdots \to H_I^{i-1}(A) \to H_I^i(A) \to H_I^i(B) \to H_I^i(C) \to H_I^{i+1}(A) \to \cdots$$

which is functorial in the given short exact sequence. Moreover, if M is injective, $H_I^i(M) = 0$ for all $i \geq 1$. It is also worth noting that if $x \in R$ then the map $M \xrightarrow{x} M$ induces the map $H_I^i(M) \xrightarrow{x} H_I^i(M)$ on local cohomology.

Note, however, that even when M is finitely generated, the modules $H_I^i(M)$ need note be finitely generated, except under special hypotheses. However, we shall see that when I is a maximal ideal of R, they do have DCC.

(6.6) Discussion of H^0_I. Note that $\mathrm{Hom}_R(R/I, M)$ may be identified with $\mathrm{Ann}_M I$ and that the map $\mathrm{Hom}_R(R/I, M) \to \mathrm{Hom}_R(R/J, M)$ when $I \supseteq J$ may then be identified with the obvious inclusion $\mathrm{Ann}_M I \subseteq \mathrm{Ann}_M J$. This means that $H^0_I(M)$ may be identified with the functor which assigns to M its submodule $\cup_t \mathrm{Ann}_M I^t$, the submodule of M consisting of all elements that are killed by a power of I.

(6.7) Discussion. A minor variation on the definition of the local cohomology functors is as follows: First define $H^0_I(M) = \{x \in M : x \text{ is killed by some power of } I\}$. The define $H^i_I(M)$ as the ith right derived functor of H^0_I. Thus, to compute H^i_I one would choose an injective resolution of M, say $0 \to E_0 \to \cdots \to E_i \to \cdots$, where $M = \mathrm{Ker}\,(E_0 \to E_1)$, and then take the cohomology of $0 \to H^0_I(E_0) \to \cdots \to H^0_I(E_i) \to \cdots$. In the original definition one first takes the cohomology of the complex $\mathcal{C}^\bullet_t$:

$$0 \to \mathrm{Hom}_R(R/I^t,\, E_0) \to \cdots \to \mathrm{Hom}_R(R/I^t, E_i) \to \cdots$$

and then takes the direct limit of the cohomology. In the second definition, up to isomorphism, one takes the direct limit of the complexes $\mathcal{C}^\bullet_t$ and then takes cohomology. Since calculation of homology or cohomology commutes with taking direct limits, these two definitions are simply minor variations on one another.

(6.8) Proposition. *Let R be a Noetherian ring, $I \subseteq R$ and let M be any R-module. Then every element of $H^i_I(M)$ is killed by a power of I.*

Proof. Every element is in the image of some $\mathrm{Ext}^i_R(R/I^t, M)$ for some t, and I^t kills that Ext. $\square$

We now prove that local cohomology can be used to test depth.

(6.9) Theorem. *Let I be an ideal of a Noetherian ring R and let M be a finitely generated R-module. Then $H^i_I(M) = 0$ for all i if and only if $IM = M$. If $IM \neq M$ then the least value d of i such that $H^i_I(M) \neq 0$ is the depth of M on I, i.e., the length of any maximal M-sequence contained in I.*

Proof. If $IM = M$ then $I^tM = M$ for all t, and then $I^t + \mathrm{Ann}\, M = R$ for all t. Since $I^t + \mathrm{Ann}\, M$ kills $\mathrm{Ext}^i_R(R/I^t, M)$, it follows that every one of these Ext 's is zero, and so all the local cohomology modules vanish.

Now suppose that $IM \neq M$ and let $x_1, \ldots, x_d$ be a maximal M-sequence in I. We shall show by induction on d that $H^i_I(M) = 0$ if $i < d$ while $H^d_I(M) \neq 0$. If $d = 0$ this is clear, since then some element of $M - \{0\}$ will be killed by I and will be nonzero in $H_I^0(M)$. If $d > 0$ the short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ with $x = x_1$ yields a long exact sequence for local cohomology:

$$\cdots \to H_I^{i-1}(M/xM) \to H_I^i(M) \xrightarrow{x} H_I^i(M) \to H_I^i(M/xM) \cdots .$$

For $i < d$ the induction hypothesis shows that x is a nonzerodivisor on $H_I^i(M)$, which must vanish, since every element is killed by a power of $x \in I$. When $i = d$ the sequence also shows that $H_I^{d-1}(M/xM)$, which we know from the induction hypothesis is nonzero, injects into $H_I^d(M)$ (we already have $H_I^{d-1}(M) = 0$). $\square$

Our next objective is to give quite a different method of calculating local cohomology: equivalently, we may use either a direct limit of Koszul cohomology or a certain kind of Cech cohomology. In order to present this point of view, we first discuss the tensor product of two or more complexes, and then define Koszul homology and cohomology. We subsequently explain how to set up a direct limit system and, after a while, prove that we can obtain local cohomology in this way.

One of the virtues of having this point of view is that it will enable us to prove a very powerful theorem about change of rings. One of the virtues of local cohomology is that it is “more invariant,” in some sense, than other theories that measure some of the same qualities. Its disadvantage is that it usually produces modules that are not finitely generated.

7. TENSOR PRODUCTS OF COMPLEXES AND KOSZUL COHOMOLOGY

(7.1) Discussion. We shall discuss Koszul cohomology, using the notion of the tensor product of two complexes to define it. Let $K_\bullet$ and $L_\bullet$ be complexes of R-modules with differentials d, d', respectively. Then we let $M_\bullet = K_\bullet \otimes_R L_\bullet$ denote the complex such that:

(1) $M_h = \bigoplus_{i+j=h} K_i \otimes L_j$ and

(2) $d(a_i \otimes b_j) = da_i \otimes b_j + (-1)^i a_i \otimes d' b_j$ when $a_i \in K_i$ and $b_j \in L_j$

It is easy to check that this does, in fact, give a complex. If there are n complexes then we may define the tensor product

$$K_\bullet^{(1)} \otimes_R \ldots \otimes_R K_\bullet^{(n)}$$

recursively as

$$\left(K_\bullet^{(1)} \otimes_R \cdots \otimes_R K_\bullet^{(n-1)}\right) \otimes_R K_\bullet^{(n)},$$

or we may take it to be the complex $M_\bullet$ such that

$$M_h = \bigoplus_{i(1)+\cdots+i(n)=h} K_{i(1)}^{(1)} \otimes_R \cdots \otimes_R K_{i(n)}^{(n)}$$

and such that if $a_{i(j)}^j \in K_{i(j)}^{(j)}$ for each j, then

$$d\left(a_{i(1)}^1 \otimes \cdots \otimes a_{i(n)}^n\right) = \sum_{t=1}^{n} (-1)^{i(1)+\cdots+i(t-1)} a_{i(1)}^1 \otimes \cdots \otimes d^t a_{i(t)} \otimes \cdots \otimes a_{i(n)}^n,$$

where d^t denotes the differential on $K_\bullet^{(t)}$.

Given a sequence of n elements of a ring R, say $\underline{x} = x_1, \ldots, x_n$, we may define the (homological) Koszul complex $K_\bullet(\underline{x}; R)$ as follows: If $n = 1$ and $x_1 = y$, it is the complex $0 \to K_1 \xrightarrow{y} K_0 \to 0$ where $K_1 = K_0 = R$ and the middle map is multiplication by y. Then, in general, $K_\bullet(\underline{x}; R) = K_\bullet(x_1; R) \otimes_R \cdots \otimes_R K_\bullet(x_n; R)$.

For the cohomological version we proceed slightly differently: We let $K^\bullet(y; R)$ (with one element, y, in the sequence) be the complex

$$0 \to K^0 \xrightarrow{y} K^1 \to 0$$

in which $K^0 = K^1 = R$ and the middle map is multiplication by y. We then let

$$K^\bullet(\underline{x}; R) = K^\bullet(x_1; R) \otimes_R \cdots \otimes_R K^\bullet(x_n; R).$$

We may then define $K_\bullet(\underline{x};\, M) = K_\bullet(\underline{x}; R) \otimes_R M$ and $K^\bullet(\underline{x};\, M) = K^\bullet(x; R) \otimes_R M$, which is isomorphic with $\mathrm{Hom}_R(K_\bullet(\underline{x}; R), M)$. We are mainly interested in the cohomological version here.

(7.2) Discussion. Let M be an R-module and let $x \in R$ be any element. We may form a direct limit system

$$M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} \cdots M \xrightarrow{x} \cdots .$$

Let N be the set of all elements in N killed by some power of x, i.e., $N = \mathrm{Ker}\,(M \to M_x)$. Let $M' = M/N$. The copy of N (notice, by the way, that $N = H^0_{xR}(M)$) inside each copy of M is killed in the direct limit. Thus, the system above has the same direct limit as

$$M' \xrightarrow{x} M' \xrightarrow{x} M' \xrightarrow{x} \cdots M' \xrightarrow{x} \cdots .$$

This system is isomorphic with an increasing union, as indicated in the commutative diagram below:

$$\begin{array}{ccccccccccccc}
M' & \hookrightarrow & M'\cdot\frac{1}{x} & \hookrightarrow & M'\cdot\frac{1}{x^2} & \hookrightarrow & \cdots & \hookrightarrow & M'\cdot\frac{1}{x^t} & \hookrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
M' & \xrightarrow{x} & M' & \xrightarrow{x} & M' & \xrightarrow{x} & \cdots & \xrightarrow{x} & M' & \xrightarrow{x} & \cdots
\end{array}$$

where $M'\cdot\frac{1}{x^t}$ denotes $\{m'/x^t : m' \in M'\} \subseteq M'_x$, and the map $M' \to M'\cdot\frac{1}{x^t}$ is the R-isomorphism sending m' to m'/x^t for every $m' \in M'$. Since the union of the modules in the top row is $M'_x \cong M_x$, it follows that the direct limit of the system in the bottom row is also M_x, and so the direct limit of the original system $M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} \cdots M \xrightarrow{x} \cdots$ is M_x as well (where the map from the tth copy of M into M_x sends m to m/x^t). The case where $M = R$ is of particular interest.

(7.3) Discussion. If $\underline{x} = x_1, \dots, x_n$ is a sequence of elements of R, we let $\underline{x}^t$ denote the sequence $x_1^t, \dots, x_n^t$. We next want to describe how to form a direct limit system, indexed by t, from the Koszul complexes $K_\bullet(\underline{x}^t; M)$, where M is an R-module.

We begin with the case where $n = 1$, $x_1 = x$, and $M = R$. Then the map from $K^\bullet(x^t; R) \to K^\bullet(x^{t+1}; R)$ is as indicated by the vertical arrows in the diagram below:

$$(\#_t) \qquad \begin{array}{ccccccc}
 & & K^0 & & K^1 & & \\
0 & \longrightarrow & R & \xrightarrow{x^t} & R & \longrightarrow & 0 \\
 & & {\scriptstyle \mathrm{id}_R}\big\downarrow & & \big\downarrow{\scriptstyle x} & & \\
0 & \longrightarrow & R & \xrightarrow{x^{t+1}} & R & \longrightarrow & 0
\end{array}$$

When we have maps of complexes $K_1^\bullet \xrightarrow{f} L_1^\bullet$, $K_2^\bullet \xrightarrow{g} L_2^\bullet$ there is an induced map $K_1^\bullet \otimes K_2^\bullet \to L_1^\bullet \otimes L_2^\bullet$ (such that the element $x \otimes y$ is sent to $f(x) \otimes g(y)$), and a similar observation applies to the tensor product of several complexes. Thus, the maps $K^\bullet(x_i^t; R) \to K^\bullet(x_i^{t+1}; R)$ that we constructed above may be tensored together over R to produce a map $K^\bullet(\underline{x}^t; R) \to K^\bullet(\underline{x}^{t+1}; R)$, and we may tensor over R with an R-module M to obtain a map $K^\bullet(\underline{x}^t;\, M) \to K^\bullet(\underline{x}^{t+1};\, M)$.

This leads to two equivalent cohomology theories. On the one hand, we may use the induced maps $H^\bullet(\underline{x}^t; M) \to H^\bullet(\underline{x}^{t+1} M)$ and take the direct limit.

On the other hand, we may form the complex $\varinjlim_t K^\bullet(\underline{x}^t; M)$, which we shall denote $K^\bullet(\underline{x}^\infty; M)$, and then take its cohomology, which we shall denote $H^\bullet(\underline{x}^\infty; M)$. This gives the same result as taking the direct limit of Koszul cohomology, since the calculation of cohomology commutes with direct limits.

Our main result along these lines, whose proof we defer for a while, is this:

(7.4) Theorem. *Let R be a Noetherian ring and let $x_1, \ldots, x_n$ be elements of R. Let $I = (x_1, \ldots, x_n)R$. Then $H^j_I(M) \cong H^\bullet(\underline{x}^\infty; M)$ canonically as functors of M.*

The idea of our proof is this: we establish the result when $j = 0$ by an easy calculation, we note that both $H^\bullet_I(\_)$ and $H^\bullet(\underline{x}^\infty; \_)$ give rise to functorial long exact sequences given short exact sequences of modules, and also that both vanish in higher degree when the module M is injective. The result will then follow from very general considerations concerning cohomological functors. Before giving the details of the argument, we want to analyze further the complexes $K^\bullet(\underline{x}^\infty; M)$.

When the x's form a regular sequence there is quite a different explanation of why this complex ought to give the local cohomology. In that case $K^\bullet(\underline{x}^t; R)$ is a projective resolution of $R/(\underline{x}^t)R$. Applying $\mathrm{Hom}_R(\_, M)$ yields the same result as forming $K^\bullet(\underline{x}^t; R) \otimes_R M = K^\bullet(\underline{x}^t; M)$, and so $H^\bullet(\underline{x}^t; M)$ is $\mathrm{Ext}^\bullet_R(R/(\underline{x}^t), M)$ in this case, and the direct limit system of complexes $K^\bullet(\underline{x}^t; M)$ is the correct one for calculating the direct limit of these Ext's. What is somewhat remarkable is that the direct limit of Koszul cohomology gives the local cohomology whether the x_i form a regular sequence or not.

(7.5) Description of the direct limit of cohomological Koszul complexes. We first consider the case where there is only one x and $M = R$. We refer to the diagrams $(\#_t)$ above that were used to define the direct limit system. The direct limit of the K^0's, each of which is a copy of R, and where each map is the identity map on R, is R. Thus, $K^0(x^\infty; R) = R$. The direct limit of the K^1's is

$$\varinjlim_t \,(R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots R \xrightarrow{x} \cdots) \cong R_x.$$

Moreover, the limit of the maps is the standard map $R \to R_x$ (which sends 1_R to 1_{R_x} and is injective when x is not a zerodivisor in R). Since $\otimes_R$ commutes with direct limits, it is easy to see that $K^\bullet(\underline{x}^\infty; R) \cong \bigotimes_R{}_{i=1}^{n} (0 \to R \to R_{x_i} \to 0)$. The term in degree 0 is simply the tensor product of n copies of R, and may be identified with R. The term in degree 1 is the direct sum of n terms, each of which is tensor product of $i-1$ copies of R, R_{x_i}, and then $n-i$ copies of R. Thus, the term in degree 1 is $R_{x_1} \oplus \cdots \oplus R_{x_n}$. The term in degree j, $0 \le j \le n$, is the sum of $\binom{n}{j}$ terms, one for each j element subset $S = \{i(1), \ldots, i(j)\}$ of the integers from 1 to n, where the term corresponding to $\{i(1), \ldots, i(j)\}$ consists of the tensor product of n terms, such that the h th term is a copy of R if $h \notin S$ and is a copy of $R_{x_{i(\nu)}}$ if $h = i(\nu) \in S$. Since $R_x \otimes_R R_y \cong R_{xy}$ (with the obvious generalization to tensor products of several such terms), we may use the following description: For each set $S \subseteq \{1, \ldots, n\}$, let $x(S) = \prod_{i \in S} x_i$ (note that $x(\emptyset) = 1$). Then $K^j(\underline{x}^\infty; R) \cong \bigoplus_{S \subseteq \{1, \ldots, n\}, |S| = j} R_{x(S)}$.

When there are just two elements x, y the direct limit complex looks like:

$$0 \to R \to R_x \oplus R_y \to R_{xy} \to 0$$

while in the case where there are three elements x, y, z the direct limit complex looks like:

$$0 \to R \to R_x \oplus R_y \oplus R_z \to R_{yz} \oplus R_{xz} \oplus R_{xy} \to R_{xyz} \to 0.$$

Moreover, the map from each term to the next is easy to describe: it suffices to explain how $R_{x(S)}$ maps to $\bigoplus_{|T| = |S|+1} R_{x(T)}$: we then take the direct sum of all these maps. If we think of the sum $\bigoplus_{|T| = |S|+1} R_{x(T)}$ as a product, we see that this map will be given by component maps $R_{x(S)} \to R_{x(T)}$, where T has one more element in it than S does. The map is zero unless $S \subseteq T$. If $S \subseteq T$ then $R_{x(T)}$ is, up to isomorphism, the localization of $R_{x(S)}$ at the single element corresponding to the index that is in T and not in S. The map $R_{x(S)} \to R_{x(T)}$ is, except for sign, the obvious map of the ring $R_{x(S)}$ into its localization at the additional element. The only issue is what sign to attach, and the definition for tensor products of complexes tells us that is done with the same pattern as in the cohomological Koszul complex. To be completely explicit, the sign attached is $(-1)^a$, where a is the number of elements of S that precede the element of T that is not in S.

It is worth noting that the first map $R \to R_{x_1} \oplus \cdots \oplus R_{x_n}$ simply sends the element $r \in R$ to $r/1 \oplus \cdots \oplus r/1$, where the i th copy of $r/1$ is to be interpreted as an element of R_{x_i}.

We next want to discuss $K^\bullet(\underline{x}^\infty; M)$. The key point is that every $K^\bullet(\underline{x}^t; M) \cong K^\bullet(\underline{x}^t; R) \otimes_R M$, and it readily follows that $K^\bullet(\underline{x}^\infty; M) \cong K^\bullet(\underline{x}^\infty; R) \otimes_R M$. Thus, $K^j(\underline{x}^\infty; M) \cong \oplus_{|S|=j} M_{x(S)}$ and the maps are constructed from the ones in the case where $M = R$ by applying $\_ \otimes_R M$: thus, they are direct sums of maps whose components are maps induced by “localizing further,” but with suitable signs attached. For example, in case the sequence of elements is x, y, z, the direct limit complex is

$$0 \to M \to M_x \oplus M_y \oplus M_z \to M_{yz} \oplus M_{xz} \oplus M_{xy} \to M_{xyz} \to 0.$$

We should also note that, in complete generality, the first map in the complex

$$M \to M_{x_1} \oplus \cdots \oplus M_{x_n}$$

simply sends m to $m/1 \oplus \ldots \oplus m/1$, where the i th copy of $m/1$ is to be interpreted as an element of M_{x_i}

We shall write $H^j(\underline{x}^\infty; M)$ for $H^j(K^\bullet(\underline{x}^\infty; M))$. We note the following facts:

(7.6) Proposition. *Let $\underline{x} = x_1, \ldots, x_n$ be a sequence of elements in any ring R. Let $I = (x_1, \ldots, x_n)R$. Let M, M', M'', M_λ, etc., be arbitrary R-modules.*

(a) *$K^\bullet(\underline{x}^\infty; R)$ is a complex of flat R-modules.*

(b) *$H^0(\underline{x}^\infty; M)$ is the submodule of M consisting of all elements killed by a power of I. Thus, if R is Noetherian, it coincides with $H^0_I(M)$.*

(c) *Given a short exact sequence $0 \to M' \to M \to M'' \to 0$ of R-modules there is a functorial long exact sequence of cohomology*

$$0 \to H^0(\underline{x}^\infty; M') \to H^0(\underline{x}^\infty; M) \to H^0(\underline{x}^\infty; M'')$$

$$\to H^1(\underline{x}^\infty; M') \to H^1(\underline{x}^\infty; M) \to H^1(\underline{x}^\infty; M'') \to \ldots$$

$$\to H^i(\underline{x}^\infty; M') \to H^i(\underline{x}^\infty; M) \to H^i(\underline{x}^\infty; M'') \to \ldots$$

$$\to H^n(\underline{x}^\infty; M') \to H^n(\underline{x}^\infty; M) \to H^n(\underline{x}^\infty; M'') \to 0.$$

(d) *If $\{M_\lambda\}_\lambda$ is any direct limit system of R-modules then*

$$H^j(\underline{x}; \varinjlim_\lambda M_\lambda) \cong \varinjlim_\lambda H^j(\underline{x}^\infty; M_\lambda).$$

In particular, $H^j(\underline{x}^\infty; \_)$ commutes with arbitrary direct sums.

(e) *For every value of j, every element of $H^j(\underline{x}^\infty; M)$ is killed by Ann M.*

(f) *Let $R \to S$ be a homomorphism, let $x_1, \ldots, x_n \in R$, and let $\underline{y} = y_1, \ldots, y_n$ denote their images in S. Let M be an S-module viewed as an R-module by restriction of scalars. Then $H^j(\underline{x}^\infty; M) \cong H^j(\underline{y}^\infty; M)$ as S-modules. (The first is an S-module because multiplication by any element of S gives an R-endomorphism of M which induces an R-endomorphism of the module $H^j(\underline{x}^\infty; M)$.)*

Proof. (a) is obvious, since each module in the complex is a direct sum of localizations of R. Now $H^0(\underline{x}^\infty; M)$ is the kernel of the map $M \to M_{x_1} \oplus \cdots \oplus M_{x_n}$ sending m to

$m/1 \oplus \cdots \oplus m/1$, and m will be in the kernel if and only if it is killed by a power of x_i for each i : this is equivalent to the assertion that m is killed by a power of I, since I is finitely generated by the x_i. The short exact sequence $0 \to M' \to M \to M'' \to 0$ may be tensored with the flat complex $K^\bullet(\underline{x}^\infty; R)$. Because of that flatness, we get a short exact sequence of complexes:

$$0 \to K^\bullet(\underline{x}^\infty; M') \to K^\bullet(\underline{x}^\infty; M) \to K^\bullet(\underline{x}^\infty; M'') \to 0$$

which, by the snake lemma, yields the long exact sequence of cohomology we want. (d) is clear from the fact that both $\otimes$ and calculation of (co)homology commute with formation of direct limits. (e) is immediate from the fact $H^j(\underline{x}^\infty; M)$ is a direct limit of Koszul cohomology $H^j(\underline{x}^\infty; M)$ (and this is the same as Koszul homology numbered backwards).

Finally, (f) follows from the fact that the action of any x_i or any product x of the x_i on M is the same as the action of the corresponding y_i or product y of y_i. This means that we may identify each M_x with the corresponding M_y, and so the complex $K^\bullet(\underline{x}^\infty; M)$ may be identified with the complex $K^\bullet(\underline{y}^\infty; M)$. The cohomology is then evidently the same. $\square$

We next observe:

(7.7) Lemma. *Let $\underline{x} = x_1, \ldots, x_n$ be a sequence of elements of the ring R and let M be an R-module of finite length. Then $H^j(\underline{x}^\infty; M) = 0$ for all $j \geq 1$.*

Proof. Since M has finite length, it has a finite filtration in which all the factors have the form $R/m = K$, where m is a maximal ideal of R. By induction on the length of the filtration and the long exact sequence provided by (7.6c), it suffices to handle the case where $M = K$. But then, by (f), we may replace R by $S = R/\mathrm{Ann}M = R/m = K$. I.e., we may assume that $R = K$ is a field and that $M = K$. Here, the x_i are replaced by their images in K. If any x_i is nonzero, the x_i generate the unit ideal and the result follows from the fact that every element of every H^j is killed by a power of the unit ideal of K. If every x_i is 0 the result follows from the fact that the complex is zero in all positive degrees, since in each summand one is localizing at 0 and, for any module N over any ring, $N_x = 0$ when $x = 0$. $\square$

(7.8) Theorem.. *Let R be a Noetherian ring and let E be an injective module. Let $x_1, \ldots, x_n \in R$. Then $H^j(\underline{x}^\infty; E) = 0$ for all $j \geq 1$*

Proof. Since E is a direct sum of modules $E = E(R/P)$, where P is prime, we assume by (7.6d) that $E = E(R/P) = E_{R_P}(R_P/RR_P)$. By (7.6f) we may replace R by R_P. Thus, we may assume that (R, P, K) is local. Then every element of E is killed by a power of P. Since E is the directed union of its finitely generated submodules, each of which has finite length, the result follows at once from (7.7) and (7.6d). $\square$

We are now ready to go back and give the proof of (7.4).

(7.9) Discussion: the proof of Theorem 7.4. Fix a Noetherian ring R and a sequence of elements $\underline{x} = x_1, \ldots, x_n$ in R. Let $I = (x_1, \ldots, x_n)R$. We already know that the sequences of functors $H^i_I(\_)$ and $H^i(\underline{x}^\infty; \_)$ from R-modules to R-modules behave similarly in three respects:
(1) $H^0_I(\_)$ and $H^0(\underline{x}^\infty; \_)$ are canonically isomorphic functors: in both cases their values on M may be identified with the submodule of M consisting of all elements that are killed by a power of I.
(2) Both $H^i_I(\_)$ and $H^i(\underline{x}^\infty; \_)$ vanish on injective R-modules for $i \geq 1$.
(3) Both the sequence of functors $H^i_I(\_)$ and the sequence of functors $H^i(\underline{x}^\infty; \_)$ have functorial long exact sequences induced by a given short exact sequence of modules.

These three properties are sufficient to enable us to give a canonical isomorphism between these two cohomology theories. The argument is very general: it makes no use of any properties of these functors other than (1), (2), (3) above. Both for typographical convenience and to illustrate the degree of generality of the proof, we change notation and write, simply, H^i for H^i_I and $\underline{H}^i$ for $H^i(\underline{x}^\infty; \_)$.

Now, given any R-module M we can embed M in an injective module E and so construct a short exact sequence $0 \to M \to E \to C \to 0$. This gives rise to two long exact sequences, one for H and one for $\underline{H}$. Since both vanish on injectives, we obtain exactness of the top and bottom rows in the diagram below, while the vertical arrows are provided by the identification of H^0 and $\underline{H}^0$:

$$(\#) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \underline{H}^0(M) & \longrightarrow & \underline{H}^0(E) & \longrightarrow & \underline{H}^0(C) & \longrightarrow & \underline{H}^1(M) & \longrightarrow & 0 \\ & & \Big\uparrow{\scriptstyle\cong} & & \Big\uparrow{\scriptstyle\cong} & & \Big\uparrow{\scriptstyle\cong} & & & & \\ 0 & \longrightarrow & H^0(M) & \longrightarrow & H^0(E) & \longrightarrow & H^0(C) & \longrightarrow & H^1(M) & \longrightarrow & 0 \end{array}$$

Thus, we get an induced isomorphism $H^1(M) \cong \underline{H}^1(M)$, since cokernels of isomorphic maps are isomorphic. This identification is independent of the choice of the embedding of M into E. To see this, it suffices to compare what happens when E is an injective hull of M with what happens with an embedding into some other injective. The second embedding may then be taken into $E \oplus E'$ (where $M \subseteq E$), and E' is injective. C is then replaced by $C \oplus E'$. We leave the details to the reader. It is also not hard to check that the identification of H^1 with $\underline{H}^1$ is an isomorphism of functors. The long exact sequences that yield the rows of (#) also give isomorphisms of $H^{i+1}(M) \cong H^i(C)$ for $i \geq 1$, and similarly for $\underline{H}$. Thus, once we have established the isomorphisms $H^i \cong \underline{H}^i$ for some $i \geq 1$, we may use the isomorphisms coming from the long exact sequences to get $H^{i+1}(M) \cong H^i(C) \cong \underline{H}^i(C) \cong \underline{H}^{i+1}(M)$. Again, one can check easily that the isomorphism $H^{i+1}(M) \cong \underline{H}^{i+1}(M)$ that one obtains in this way is independent of the choice of the embedding $0 \to M \to E$. It is also not difficult to check that it is an isomorphism of functors.

Finally, one can also check that our identification of $H^\bullet$ with $\underline{H}^\bullet$ is compatible with the connecting homomorphisms in long exact sequences, so that, in each instance, one gets an isomorphism of long exact sequences. The details are not difficult, and we omit them here.

This completes our discussion of the proof of Theorem 7.4. $\square$

When we speak of *the number of generators of an ideal I up to radicals* we mean the least integer n such that $\operatorname{Rad} I$ is also the radical of an ideal generated by n elements. By taking powers, we may always arrange for the n elements to be in I. We note:/medskip

(7.10) Corollary.. *Let I be an ideal of a Noetherian ring R. If I is generated by n elements up to radicals, then $H^i_I(M) = 0$ for $i > n$.*

Proof. Suppose that $\operatorname{Rad} I = \operatorname{Rad} J$, where $J = (x_1, \dots, x_n)R$. Then $H^i_I(M) = H^i_J(M) = H^i(\underline{x}^\infty; M)$, and the last is obviously zero when $i > n$, since the complex $K^\bullet(\underline{x}^\infty; M)$ is zero in degree bigger than n. $\square$

(7.11) Corollary. *Let $R \to S$ be a homomorphism of Noetherian rings, let $I \subseteq R$ be an ideal and let M be an S-module. Then $H^i_I(M) \cong H^i_{IS}(M)$ as S-modules.*

Proof. Let $I = (x_1, \dots, x_n)R$ and let $y_1, \dots, y_n$ be the images of the x's in S. Then $H^i_I(M)$ may be identified with $H^i(\underline{x}^\infty; M)$, and since $IS = (y_1, \dots, y_n)S$, $H^i_{IS}(M)$ may be identified with $H^i(\underline{y}^\infty; M)$, and $H^i(\underline{x}^\infty; M) \cong H^i(\underline{y}^\infty; M)$, as noted earlier, because each x_j acts on M in the same way that y_j does. $\square$

Moreover, since we already know the corresponding fact for $H^i(\underline{x}^\infty; \_\,)$ we have:

(7.12) Corollary. *Let R be Noetherian, $I \subseteq R$, and let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a direct limit system of R-modules. Then* $H^i_I(\varinjlim_\lambda M_\lambda) \cong \varinjlim_\lambda H^i_I(M_\lambda)$. $\square$

(7.13) Exercise. Let S be a flat Noetherian R-algebra, where R is Noetherian, and let I be an ideal of R and M an R-module. Then $S \otimes_R H^i_I(M) \cong H^i_I(S \otimes_R M) \cong H^i_{IS}(S \otimes_R M)$. In particular, this holds when S is a localization of R or when S is the m-adic completion of the local ring (R, m, K).

8. LOCAL COHOMOLOGY WITH SUPPORT IN A MAXIMAL IDEAL

We now want to study the case where I is a maximal ideal m of a Noetherian ring R. We first note:

(8.1) Proposition. *Let m be a maximal ideal of the Noetherian ring R and let M be an R-module. Let $(A, \mu) = (R_m, mR_m)$.*

(a) $H^i_m(M) \cong H^i_\mu(M_m)$.

(b) *If (R, m, K) is local and $(\widehat{R}, m\widehat{R}, K)$ is its completion, then $H^i_m(M) \cong H^i_{m\widehat{R}}(\widehat{R} \otimes_R M)$. (When M is finitely generated, $\widehat{R} \otimes_R M$ is isomorphic with the completion $\widehat{M}$ of M).*

Proof. (a) Since every element of $H^i_m(M)$ is killed by a power of m, the elements of $R - m$ already act invertibly on this module, and so $H^i_m(M) \cong (R - m)^{-1} H^i_m(R) \cong A \otimes_R H^i_m(M) \cong H^i_\mu(A \otimes_R M)$ by Exercise (7.13) above, and this is $H^i_\mu(M_m)$.
(b) Since every element of $H^i_m(M)$ is killed by a power of m, we have that $H^i_m(M)$ is $\widehat{R} \otimes_R H^i_m(M)$, and now the result is immediate from Exercise (7.13). $\square$

Thus, the study of local cohomology modules with support in a maximal ideal reduces, in a sense, to the case where the ring is local or even complete local.

(8.2) Exercise. Let R be any ring and let $\to G_i \to \cdots \to G_0 \to 0$ be a left resolution of M by flat modules (i.e., every G_i is R-flat, the complex is acyclic, and Coker $(G_1 \to G_0) \cong M$). Show that for every R-module N, $H_i(G_\bullet \otimes_R N) \cong \mathrm{Tor}_i^R(M, N)$. In other words, flat resolutions, and not merely projective resolutions, may be used to calculate Tor.

The next result is a critical step in the proof of what is known as "local duality."

(8.3) Lemma.. *Let R be a Noetherian ring and let I be an ideal which is the radical of $(x_1, \dots, x_n)R$, where the x_i form a regular sequence in R. Then $H^i_m(M) \cong \mathrm{Tor}^R_{n-i}(M, H^n_I(R))$ functorially in M.*

Proof. We may identify $H^\bullet_I(M) \cong H^\bullet(\underline{x}^\infty; M) \cong H^\bullet\big(K_\bullet(\underline{x}^\infty; R) \otimes_R M\big)$ which will give the required Tor, by (7.8), provided that the complex $K^\bullet(x^\infty;\, R)$, numbered backwards, is a flat resolution of $H^n_I(R)$. Since the $H^i_I(R)$ vanish for $i < \mathrm{depth}\,_I R = n$, the complex has $H^n_I(R)$ as its only nonvanishing cohomology module, and the result follows from Exercise (8.2), since the complex consists of R-flat modules. $\square$

If (R, m, K) is Cohen-Macaulay and $I = m$ is the maximal ideal, we may choose the x's to be a system of parameters. We therefore have:

(8.4) Corollary. *If (R, m, K) is a Cohen-Macaulay local ring of dimension n, then for every R-module M, $H^i_m(M) \cong \mathrm{Tor}^R_{n-i}(M,\, H^n_m(R))$.* $\square$

This means that we shall have considerable interest in understanding $H^n_m(R)$. It turns out that when (R, m, K) is regular, and, more generally, when (R, m, K) is Cohen-Macaulay of type 1 (or *Gorenstein*) and has Krull dimension n, then $H^n_m(R)$ is the injective hull of the residue field of K over R !!

9. SOME CONNECTIONS WITH THE THEORY OF SCHEMES

Suppose that R is a Noetherian ring, that $X = \mathrm{Spec}\, R$ (as a scheme), that $I = (x_1, \ldots, x_n)R$ is an ideal, that $Z = V(I)$, a closed subscheme of X, and that $U = X - Z = \bigcup_i U_i$ where $U_i = D(x_i)$. Note that the U_i form an open cover, by affine schemes, of U. Let M be any R-module, let $M^\sim$ be the corresponding quasicoherent sheaf, and let $\mathcal{M} = M^\sim|_U$ be its restriction to U. Thus, $\mathcal{M}(U_i) \cong M_{x_i}$. More generally, if $S \subseteq \{1, \ldots, n\}$ then if $U_S = \bigcap_{i \in S} U_i$ then $\mathcal{M}(U_S) = M_{x(S)}$ where $x(S) = \prod_{i \in S} x_i$. It follows at once that if we drop the zero degree term M from the complex $H^\bullet(\underline{x}^\infty; M)$ we obtain precisely the Cech complex of $\mathcal{M}$ with respect to the affine open cover $\{U_i\}_i$, whose cohomology is the same as $H^i(U, \mathcal{M})$.

From this data one readily obtains that

$$0 \to H^0(\underline{x}^\infty; M) \to M \to H^0(U, \mathcal{M}) \to H^1(\underline{x}^\infty; M) \to 0$$

is exact, while $H^i(U; \mathcal{M}) \cong H^{i+1}(\underline{x}^\infty; M)$ for all $i \geq 1$.

Thus,

$$0 \to H^0_I(M) \to M \to H^0(U; \mathcal{M}) \to H^1_I(M) \to 0$$

is exact while $H^i(U; \mathcal{M}) \cong H^{i+1}_I(M)$ for all $i \geq 1$. Here, one may think of M as $H^0(X, M^\sim)$. The elements of $H^0_I(M)$ correspond to sections s of $M^\sim$ supported only on $V(I)$ (i.e. the germ s_P of s at P is 0 unless $P \supseteq I$). Such sections are the kernel of the restriction map from sections of $M^\sim$ on X to sections on U. The cokernel $H^1_I(M)$ measures the obstruction to extending a section of $M^\sim$ on U to a section on all of X.

(9.1) Corollary. *If R is a Noetherian ring, M is a finitely generated R-module, and $I \subseteq R$, and if depth$_I M \geq 2$, then every section of $M^\sim$ on $U = \operatorname{Spec} R - V(I)$ extends to a section of $M^\sim$ on Spec R.*

10. A MORE GENERAL VERSION OF LOCAL COHOMOLOGY

Let $(X, \mathcal{O}_X)$ be a ringed space. There are enough injectives in the category of $\mathcal{O}_X$-Modules (sheaves of modules over $(X, \mathcal{O}_X)$): if one fixes a sheaf $\mathcal{F}$ and chooses an embedding $\mathcal{F}_x \to E_x$ for each $x \in X$, where E_x is an injective $\mathcal{O}_{X,x}$-module, then $\mathcal{F}$ embeds in $\mathcal{E}$ defined by $\mathcal{E}(U) = \prod_{x \in X} E_x$ in an obvious way, and $\mathcal{E}$ is easily checked to be injective. It is also easy to see that $\mathcal{E}$ is flasque (restriction maps are onto). Since every injective can be embedded in a flasque sheaf (and, hence, is a direct summand of a flasque sheaf), it follows that every injective in the category of $\mathcal{O}_X$-Modules is flasque. In particular, there are enough injectives in the category of abelian sheaves on X (sheaves of abelian groups: this corresponds to taking $\mathcal{O}_X$ to be the sheaf whose value on U is the ring of locally constant integer-valued functions on U).

Suppose that $\mathcal{F}$ is an $\mathcal{O}_X$-Module and $Z \subseteq X$ is closed. We write U for $X - Z$. (There is also a version of the theory when Z is only locally closed.) We define the sections of $\mathcal{F}$ supported on Z, which we denote $H^0_Z(X, \mathcal{F})$, as the kernel of the restriction map $H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F})$. Then $H^0_Z(X, \_)$ is right exact, and we define $H^i_Z(X, \_)$ as the ith right derived functor of $H^0_Z(X, \_)$ in the category of abelian sheaves on X (take an injective resolution of $\mathcal{F}$, apply H^0_Z, and take cohomology). There is an exact sequence $0 \to H^0_Z(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F})$ which is also exact on the right when $\mathcal{F}$ is injective, because injectives are flasque. Applying this to the terms of an injective resolution of $\mathcal{F}$ we may use the snake lemma to obtain a long exact sequence:

$$0 \to H^0_Z(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \to H^1_Z(X, \mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(U, \mathcal{F}) \to$$

$$\cdots \to H^{i-1}(U, \mathcal{F}) \to H^i_Z(X, \mathcal{F}) \to H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \to H^{i+1}(X, \mathcal{F}) \to \cdots$$

$H^i_Z(\mathcal{F})$ is called the ith *local cohomology of $\mathcal{F}$ with support in Z*. H^0_Z gives sections supported on Z, while H^1_Z calculates the obstruction to extending sections of $\mathcal{F}$ on U to all of X. See [GrH]. When $X = \operatorname{Spec} R$ is affine, $\mathcal{F} = M^\sim$, and $Z = V(I)$, we have $H^i(X, \mathcal{F}) = 0$ for $i \geq 1$. (This can be proved easily when R is Noetherian from the fact that if E is injective over R, then $E^{\sim}$ is flasque. Thus, an injective resolution of M gives rise to a flasque resolution of $M^{\sim}$, which can be used to compute cohomology, since flasque sheaves have vanishing higher cohomology.) From this we can conclude that $H^i_Z(X, \mathcal{F}) \cong H^i_I(M)$ for all i in this case. Thus, in the case, the sheaf-theoretic notion of local cohomology agrees with the notion that we have already defined by purely algebraic means.

When X is a paracompact and locally contractible topological space, G is an abelian group, and $\boldsymbol{G}$ denotes the sheaf of locally constant functions to G, then $H^i_Z(X, \boldsymbol{G}) \cong H^i(X, X - Z; G)$, the relative singular cohomology, which "discards" what happens in $X - Z$. For example, if X is an n-manifold and Z is one point, one gets the same cohomology as if everything not very close to the point Z were squeezed to a single point. This gives the same result as taking a small n-ball around Z and collapsing its boundary to a point, which produces an n-sphere. This situation motivates the term *local cohomology.*

11. GORENSTEIN RINGS AND LOCAL DUALITY

(11.1) Discussion and definitions. The *type* of a (finitely generated) Cohen-Macaulay module M over a local ring (R, m, K) is equivalently, the dimension as a K-vector space of $\mathrm{Ext}^d_R(K, M)$, where $d = \dim M$, or the dimension of the socle in $M/(x_1, \dots, x_d)M$, where $d = \dim M$ and $x_1, \dots, x_d$ is any maximal M-sequence in m. Note that if x is a nonzerodivisor on M, M, and M/xM have the same type. The type is unaffected by completion. If R is regular, we leave it as an exercise to show that the type is the least number of generators of $\mathrm{Ext}^{n-d}(M, R)$, where $n = \dim R$ (so that $n - d = \mathrm{pd}_R M$); it is also the last nonvanishing rank of a free module in a minimal free resolution of M over R. The type cannot increase when one localizes.

Thus, a Cohen-Macaulay local ring has a type, and the type can only decrease as one localizes.

It follows that if a Cohen-Macaulay local ring has type one, then all of its localizations have type one. Such a local ring is called *Gorenstein.* A Noetherian ring is called *Gorenstein* if all of its localizations at prime (equivalently, maximal) ideals are Gorenstein. (We shall eventually show that a local ring is Gorenstein if and only if it has finite injective dimension as a module over itself — this is sometimes taken to be the definition.)

(11.2) Proposition. *Let (R, m, K) be a local ring.*

(a) *If $x_1, \ldots, x_k$ is a regular sequence in R, then the lcoal ring R is Gorenstein if and only if $R/(x_1, \ldots, x_k)R$ is Gorenstein.*

(b) *If R is Artin, then R is Gorenstein if and only if R is an injective R-module (we have already seen that this holds if and only if the socle of R is a one-dimensional K-vector space).*

(c) *R is Gorenstein if and only if its completion is Gorenstein.*

(d) *If R is regular then R is Gorenstein. Hence, if R is regular then and $x_1, \ldots, x_k$ is part of a system of parameters, then $R/(x_1, \ldots, x_k)R$ is Gorenstein. (Rings of this form are called complete intersections. Some authors include in the definition rings whose completion is of this form.)*

Proof. (a) This holds because both the Cohen-Macaulay property and the type are unaffected.

(b) This part is immediate: zero-dimensional rings are Cohen-Macaulay and the type is the K-dimension of the socle.

(c) This is clear, since both the Cohen-Macaulay property and the type are unaffected by completion.

(d) We know that fields are Gorenstein, and so the Gorenstein property for a regular ring R of dimension n in which $x_1, \ldots, x_n$ generate the maximal ideal follows from the fact that $R/(x_1, \ldots, x_n)R$ is Gorenstein: the x_i form an R-sequence, and we may apply the "if" part of (a). The second statement then follows from the "only if" part of (a). $\square$

(11.3) Discussion. If $x_1, \ldots, x_n$ form a regular sequence on the R-module M, it is easy to see that if $(x_1 \cdots x_n)u \in (x_1^{t+1}, \ldots x_n^{t+1})M$, then $u \in (x_1^t, \ldots, x_n^t)M$. For if $(x_1 \cdots x_n)u = \sum_{j=1}^n x_j^{t+1} v_j$ then $x_n(x_1 \cdots x_{n-1}u - x_n^t v_n) \in (x_1^{t+1}, \ldots, x_{n-1}^{t+1})M$, and so $x_1 \cdots x_{n-1}u \in (x_1^{t+1}, \ldots, x_{n-1}^{t+1}, x_n^t)M$. A straightforward induction on $n - i$ enables one to establish that $x_1 \ldots x_i u \in (x_1^{t+1}, \ldots, x_i^{t+1}, x_{i+1}^t, \ldots, x_n^t)M$ for $i = n, n-1, \ldots, 0$ (it is the case $i = 0$ that we want). We are using that any powers of the x's also form an M-sequence, and also that if $y_1, \ldots, y_n$ form an M-sequence, then each y_i is a nonzerodivisor on $M/(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)M$, even though the regular sequence may not be permutable.

Phrased slightly differently, this result asserts that multiplication by $x = x_1 \ldots x_n$ on M induces an injective map from $M_t \to M_{t+1}$ for each t, where $M_t = M/(x_1^t, \ldots, x_n^t)M$.

Now $M_t = H^n(\underline{x}^t; M)$, and the direct limit system $\cdots \to M_t \xrightarrow{x} M_{t+1} \to \cdots$ is precisely the one we need to calculate $H^n_{(\underline{x})R}(M)$. Thus, when the x's form an M-sequence, we can view $H^n_{(\underline{x})R}(M)$ as the direct limit of the M_t via injective maps: a sort of increasing union of the M_t. We next observe:

(11.4) Lemma. *Let (R, m, K) be local and let $q_1 \supseteq \cdots q_t \supseteq \cdots$ be a non-increasing sequence of m-primary ideals cofinal with the powers of m. Let $R_t = R/q_t$. Let E be an R-module which is an increasing union of submodules E_t such that for every t, $E_t \cong E_{R_t}(K)$. Then $E \cong E_R(K)$.*

Proof. Fix a copy G of $E_R(K)$ and let G_t denote the annihilator of q_t in G. Then the G_t form an increasing sequence of submodules of G whose union is G. We know that for each fixed t there is an isomorphism $\alpha_t : g_t \to E_t$. We shall prove by induction on t that there is a sequence of isomorphisms $\beta_t : E_t \to G_t$ such that for all t, $\beta_{t+1}|_{G_t}$ is β_t. This will suffice to prove the theorem, since the union of the graphs of the β_t then gives an isomorphism of E with G. We take $\beta_1 = \alpha_1$. Suppose that $\beta_1, \ldots, \beta_t$ have been constructed. Note that β_t determines all its predecessors. It will suffice to show that we can extend β_t to an isomorphism of E_{t+1} with G_{t+1}. Let γ denote the restriction of α_{t+1} to E_t. Since E_t is killed by q_t, so is $\gamma(E_t)$, and so it must be contained in G_t. Since α_{t+1} is injective, $\gamma(E_t)$ has the same length as E_t, and, hence, the same length as G_t. It follows that $\gamma(E_t) = G_t$. Let γ^{-1} denote the inverse of γ as a map from E_t to G_t. Then $\gamma^{-1}\beta_t : E_t \cong E_t$ is an automorphism of the injective hull of R_t, and so coincides with multiplication by a unit ζ of R_t. We may lift that unit to a unit η of R_{t+1}. We then take $\beta_{t+1} = \alpha_{t+1}\eta$. Restricting to E_t gives $\gamma\zeta = \gamma(\gamma^{-1}\beta_t) = \beta_t$, as required. $\square$

(11.5) Theorem. *Let (R, m, K) be a Gorenstein local ring of dimension n and suppose that $x_1, \ldots, x_n$ is any system of parameters. Then $H^n_m(R) = \varinjlim_t R/(x_1^t, \ldots, x_n^t)R$ (where the successive maps are included by multiplication by $x = x_1 \ldots x_n$) is an injective hull for K. If u generates the socle in $R/(x_1, \ldots, x_n)R$, then $x_1^t \ldots x_n^t u$ generates the socle in $R/(x_1^{t+1}, \ldots, x_n^{t+1})R$.*

In particular, this is true when R is regular, and in that case we may choose the x_i to be a minimal set of generators of the maximal ideal of R.

Proof. We apply (11.4) with $q_t = (x_1^t, \ldots, x_n^t)R$. The first statement is immediate if we know that $R/(\underline{x}^t)R$ is an injective hull for K over itself. But since R is Gorenstein and $\underline{x}^t$ is a system of parameters, $R/(\underline{x}^t)T$ is a zero-dimensional Gorenstein ring, and hence it itself is the injective hull of K over itself. It follows that there is a one-dimensional socle in each of the modules $R/(\underline{x}^t)R$. Since the map $R/(\underline{x})R \to R/(\underline{x}^{t+1})R$ induced by multiplication by $x_1^t \dots x_n^t$ is injective, it must send the copy of K in $R/(\underline{x})R$ to the copy of K in $R/(\underline{x}^{t+1})R$. This establishes the second statement. The statement about the case of a regular ring is then obvious. □

(11.6) Remarks. Let $x_1, \dots, x_n \in R$ and let M be any R-module. Let $I = (x_1, \dots, x_n)R$. Then the right exactness of $\otimes_R$ implies that $H_I^n(M) \cong H_I^n(R) \otimes_R M$, for $H_I^n(M) \cong \varinjlim_t M/(\underline{x}^t M) \cong \varinjlim_t \Big(\big(R/(\underline{x}^t R)\big) \otimes_R M\Big) \cong \Big(\varinjlim_t \big(R/(\underline{x}^t R)\big)\Big) \otimes_R M \cong H_I^n(R) \otimes_R M$.

While, from one point of view, $H_I^n(R)$ is a direct limit, we may take the different point of view that it is $H^n(\underline{x}^\infty; R)$, i.e., that is is the cokernel of the map $\bigoplus_j R_{y(j)} \to R_x$, where x denotes the product $x_1 \dots x_n$ and $y(j)$ denotes the product of the x_i with x_j omitted. The maps on the various summands are induced by the obvious maps $R_{y(j)} \to R_x$, each with a certain sign, but the signs don't affect the image, which is $\Sigma_j \mathrm{Im}\, R_{y(j)}$. This means that $H_I^n(R) \cong R_x / \sum_j R_{y(j)}$, where $x = x_1 \cdots x_n$. Likewise, it may also be identified with the tensor product over R of the R-modules R_{x_i}/R (and note that $R_{x_i}/R \cong H^1_{x_iR}(R)$).

In particular, when R is a discrete valuation ring, $H_m^1(R) \cong R_x/R \cong L/R$, where L is the fraction field of R. This is also the injective hull of K.

When $R \cong K[[x_1, \dots, x_n]]$, R_x may be identified with formal power serious involving monomials with both positive and negative exponents, where there is a lower bound for all negative exponents. When one kills the sum of the $R_{y(j)}$ one kills all series such that in every term, the exponent on one or more variables is not strictly negative. Thus, $H_m^n(R) \cong E_R(K)$ may be identified with all polynomials in $x_n^{-1}, \dots, x_n^{-1}$ with the property that in every term, every variable occurs. One multiplies such a polynomial by a power series by forcing a formal distributive law: there are *a priori* infinitely many terms, but all but finitely many turn out to have a nonnegative exponent on some variable, and so are zero. The socle in this module is is generated by $x_1^{-1} \cdots x_n^{-1}$. (There is a variant formulation in which one uses $xR_x/\big(x(\Sigma_j \mathrm{Im}\, R_{y(j)})\big)$ instead: this is isomorphic to the module discussed just previously, and is the same as $R_x/(x\Sigma_j \mathrm{Im}\, R_{y(j)})$. The quotient may now be identified with $K[x_1^{-1}, \dots, x_n^{-1}]$, so that one winds up with the nonpositive monomials instead of the strictly negative monomials. In this version, the image of 1 represents a generator of the socle.)

Once one understands $E_R(K)$ when R is a complete regular local ring, one may argue that one understands $E_A(K)$ for any local ring A: completing A doesn't change the injective hull of K, and the completed ring A may be written as R/I with R complete, regular, and local. Now think of $E_A(K)$ as the annihilator of I in $E_R(K)$. This is satisfactory for some purposes but there are problems for which it is not a very useful point of view.

(11.7) Lemma. *Let P be a finitely generated projective module and W, N be any modules over R. Then $P \otimes \operatorname{Hom}(W, N) \cong \operatorname{Hom}\big(\operatorname{Hom}(P, W), N\big)$ functorially in P, W, M.*

Proof. There is a map $\theta_P : P \otimes \operatorname{Hom}(W, N) \to \operatorname{Hom}_R(\operatorname{Hom}_R(P, W), N)$ that sends $u \otimes \alpha$ to the map that takes $f \in \operatorname{Hom}(P, W)$ to $\alpha f(u)$. Since $\theta_{P \oplus Q} \cong \theta_P \oplus \theta_Q$ the problem of showing that this map is an isomorphism when P is finitely generated projective reduces to the case where P is a finitely generated free module and then to the case $P = R$, which is easily checked. $\square$

(11.8) Theorem (local duality for Gorenstein rings). *Let (R, m, K) be a local Gorenstein ring of dimension n, let $E = H^n_m(R)$, which is an injective hull for K, and let $^\vee$ be $\operatorname{Hom}_R(\_\,, E)$. For finitely generated modules M, $H^i_m(M) \cong \operatorname{Ext}^{n-i}_R(M, R)^\vee$ as functors in M.*

Proof. We have already observed that $K^\bullet(\underline{x}^\infty; R)$ is a flat resolution of $H^n_m(R)$ even if the hypothesis is only that the ring is Cohen-Macaulay. In the present case, we know that the ring is Gorenstein, which carries the additional information that $H^n_m(R) \cong E_R(K)$. Thus $H^i_m(M) \cong \operatorname{Tor}^R_{n-i}(M, E)$. On the other hand, let $P_\bullet$ be free resolution of M by finitely generated free modules. Then $\operatorname{Ext}^\bullet(M, R)$ is the cohomology of $\operatorname{Hom}(P_\bullet, R)$, and since E is injective, $\operatorname{Ext}^\bullet(M, R)^\vee$ is the cohomology of $\operatorname{Hom}_R(\operatorname{Hom}_R(P_\bullet, R), E)$, and this complex is isomorphic with $P_\bullet \otimes_R E$ by Lemma (11.7), with $W = R$ and $N = E$, whose homology gives the modules $\operatorname{Tor}_\bullet(M, E)$. Thus, $\operatorname{Ext}^{n-i}(M, R)^\vee \cong \operatorname{Tor}_{n-i}(M, E)$. $\square$

(11.9) Corollary. *Let R be a Noetherian ring and m a maximal ideal. Then for every finitely generated R-module M, $H^i_m(M)$ (which is actually a module over R_m) has DCC.*

Proof. The issues are unaffected by localization at m. Thus, we may assume that (R, m, K) is local. They are likewise unaffected by replacing R, M by their completions. Thus, we may assume that R is a complete local ring. But then R is a homomorphic image of a complete regular local ring (S, n, K). Now, $H^i_m(M) \cong H^i_n(M)$, and thus it suffices to prove the result when R is a regular local ring. But then $H^i_m(M)$ is the Matlis dual of $\operatorname{Ext}^{d-i}_R(M, R)$ and so has DCC. $\square$

Before we prove our next major result, we want to establish:

(11.10) Lemma. *Let M be a module of dimension d over a Noetherian ring R, and suppose that every proper homomorphic image of M has smaller dimension than M. Then $S = R/\mathrm{Ann}_R M$ is a domain, and M is a torsion-free module of rank one over S.*

Proof. We might as well replace R by S and assume that M is faithful. Choose P to be a minimal prime of R such that $\dim R/P = \dim R$. Then $P \in \mathrm{Ass}\, M$, since M is faithful. We can map M_P/PM_P onto a copy of R_P/PR_P. Let N be the kernel of the composite map $M \to M_P/PM_P \to R_P/PR_P$. Then M/N embeds into R_P/PR_P, and the image is not zero, since it generates R_P/PR_P as an R_P-module. It follows that $M = M/N$, since M/N is non-zero and torsion-free over R/P. Moreover, it is now clear that $M = M/N$ has rank one. □

(11.11) Theorem. *Let M be a finitely generated module over a local ring (R, m, K) and let $d = \dim M$ ($= \dim R/(\mathrm{Ann} M)$). Then $H^d_m(M) \neq 0$, while $H^i_m(M) = 0$ for $i > d$.*

Proof. First, we may replace R by $R/\mathrm{Ann}_R M$ without affecting any relevant issues. Thus, we may assume that M is faithful and so $\dim M = \dim R$. Since m is then the radical of an ideal with d generators (use a system of parameters), it follows at once that $H^i_m(M) = 0$ for $i > d$. It remains to see that $H^d_m(M) \neq 0$.

To this end, first replace R, M by their completions: this does not affect any relevant issue. Second, choose $N \subseteq M$ maximal such that $\dim M/N = d$. The tail end of the long exact sequence for local cohomology yields that $H^d_m(M) \to H^d_m(N, M) \to H^{d+1}_m(N)$ is exact, and since H^{d+1}_m is zero for all R-modules, the first map is surjective. Thus, it suffices to show that $H^d_m(M/N) \neq 0$ and so we may replace M by M/N and therefore assume that killing any nonzero submodule of M lowers its dimension. Again, replace R by $R/\mathrm{Ann}_R M$. By Lemma (11.10) above, R is then a domain and M is a torsion-free R-module of rank one, i.e., M is isomorphic with an ideal of R. But then R is module-finite over a regular ring A of the same dimension, and M is also torsion-free as an A-module. The maximal ideal μ of A extends to an m-primary ideal of R, and so $H^d_\mu(M) \cong H^d_m(M)$.

Thus, it will suffice to show that if M is a nonzero torsion-free module over a regular local ring (R, m, K) of dimension d, then $H^d_m(M) \neq 0$. Suppose that M is torsion-free of rank r. Choose r elements of M that are linearly independent over R. These yield an embedding $R^r \to M$ so that the cokernel is a torsion module: suppose it is killed by the nonzero element c. Then $M \cong cM \subseteq R^r$ is an embedding $M \to R^r$ such that the cokernel C is torsion (c kills it). The long exact sequence for local cohomology yields

$H^d_m(M) \to H^d_m(R^r) \to H^d_m(C) = 0$ (because $\dim C \le d-1$). Thus, it will suffice to see that $H^d_m(R^r) = H^d_m(R)^r \neq 0$, which we already know since $H^d_m(R) \cong E_R(K)$. $\square$

12. COHOMOLOGICAL DIMENSION

We now have considerable information about the modules $H^\bullet_m(M)$ when m is a maximal ideal and M is a finitely generated module. They have DCC, and their study reduces to the case where R is local or, even, complete local. The first nonvanishing $H^i_m(M)$ occurs when $i = \text{depth } M$, and the last when $i = \dim M$. When I is not a maximal ideal, it is much harder to study the modules $H^i_m(M)$ even if $M = R$. The first nonvanishing one still occurs at depth ${}_I M$. Where the highest nonvanishing one occurs is a difficult problem. Certainly, all of the local coholomogy modules vanish if i exceeds the least number of generators of I up to radicals. We first prove a lemma and then note that one at least has the statements in Proposition (12.3) below.

(12.1) Lemma. *Let (R, m, K) be a local ring of Krull dimension n and let $I \subseteq m$ be any ideal. Then I is the radical of an ideal generated by at most n elements.*

Proof. Choose $x_1 \in I$ so that x_1 is not in any minimal prime of R which does not contain I (we can do this, since there are only finitely many). Suppose that $x_1, \dots, x_k \in I, k \ge 1$, have been chosen so that any prime of height $k-1$ or less that contains $(x_1, \dots, x_k)R$ must contain I. If $k < n$, first note that any prime of height k that contains $(x_1, \dots, x_k)R$ and not I must be a minimal prime of $(x_1, \dots, x_k)R$: otherwise, it contains a minimal prime which will contain $(x_1, \dots, x_k)R$ and be of height $k-1$ or less and still not contain I. Thus, the set of height k primes containing $(x_1, \dots, x_k)R$ and not I is finite. Now choose x_{k+1} in I and not in any of them. The recursion stops when $k = n$. Since the only height n prime in the ring is the maximal ideal, every prime that contains $(x_1, \dots, x_k)R$ contains I. Since $(x_1, \dots, x_k)R \subseteq I$, it follows that the two ideals have the same radical. $\square$

(12.2) Remark. A slight variation of the proof shows that every ideal in a Noetherian ring of Krull dimension n is the radical of an ideal generated by $n+1$ elements.

(12.3) Proposition. *Let R be a Noetherian ring, and $I \subseteq R$ an ideal.*

(a) *If $H^i_I(R) = 0$ for $i > d$, then $H^i_I(M) = 0$ for every R-module M for $i > d$.*

(b) *If $H^i_I(R/P) = 0$ for all $i > d$ then for every R-module M such that every element is killed by a power of P, $H^i_I(M) = 0$ for all $i > d$.*

(c) *Let S be a set of primes of R such that if $P \in \operatorname{Ass} M$ then P contains a prime in S. Suppose that for all $P \in S$, $H^i_I(R/P) = 0$ for $i > d$. Then $H^i_I(M) = 0$ for all $i > d$. If M is a finitely generated module we may choose S to be the set of all minimal primes of R.*

(d) *For every R-module M, $H^i_I(M) = 0$ for $i > \dim R$.*

(e) *For every R-module M, $H^i_I(M) = 0$ for $i > \dim(R/\mathrm{Ann} M)$.*

Proof. (a) Assume not and let j be the least integer $> d$ such that $H^j_I(M) \neq 0$ for some R-module M (note that $j \leq$ the number of generators of I). There is a short exact sequence $0 \to M' \to F \to M \to 0$ where F is free. Since $H^{j+1}_I(M') = 0$, the long exact sequence for local cohomology shows that $H^j_I(F) \to H^j_I(M)$ is onto, and $H^j_I(F) = 0$, since F is R-free.

(b) By a direct limit argument, we may assume that M is finitely generated and so killed by a power of P itself. Then M has a finite filtration in which each factor is killed by P, so that each factor is an (R/P)-module. By the long exact sequence for local cohomology, it suffices to prove the result for each factor, and this follows from (a).

(c) Again, we may replace M by a finitely generated submodule. Then M has a finite filtration in which each factor is killed by a prime in $\operatorname{Ass} M$, and hence by a prime of S. The result now follows from (b).

(d) We may replace M by R. If there is a counterexample we may localize to produce a local counterexample. Thus, we may assume that R is a local ring. But then the result follows from Lemma 9.3, since the expansion of I to the local ring is generated up to radicals by at most n elements (or else it is the unit ideal, which makes all local cohomology vanish).

(e) This is immediate, since we may also view M as a module over $R/(\mathrm{Ann} M)$ (replacing I by its expansion) without affecting the local cohomology. $\square$

We shall refer to the least integer d such that $H^i_I(M) = 0$ for all $i > d$ and all R-modules M as the *cohomological dimension* of the pair $(R; I)$. This implies that $H^i(X, \mathcal{F}) = 0$ for every quasicoherent sheaf $\mathcal{F}$ on $X = \operatorname{Spec} R - V(I)$ if $i > d - 1$, provided $d > 1$.

Thus, if (R, m, K) is local, the cohomological dimension of the pair $(R; m)$ is $\dim R$.

13. MORE ABOUT GORENSTEIN RINGS

Our next objective is to show that a local ring has finite injective dimension as a module over itself if and only if it is Cohen-Macaulay of type 1, i.e., iff it is Gorenstein.

(13.1) Theorem. *Let (R, m, K) be a Gorenstein local ring of Krull dimension n. Then the minimal injective resolution $E^\bullet$ of R has length n, and for $0 \leq i \leq n$, the module E^i is the direct sum of the injective hulls of all the primes P of R of height i. Thus, $id_R R = dim\, R$.*

Proof. Let $P \subseteq R$ be a prime of height i. It suffices to show that $\dim_{\,\kappa}\mathrm{Ext}^j_{R_P}(\kappa, R_P) = 1$ if $j = i$ and 0 otherwise, where $\kappa = R_P/PR_P$. Replacing R by R_P, we see that we might as well assume that $P = m, \;\; \kappa = K$. The modules $\mathrm{Ext}^j_R(K, R)$ are the Matlis duals of the local cohomology modules $H^{n-j}_m(K)$. Since $\dim K = 0$, only one of these is nonvanishing, when $j = n$, and that one is $H^0_m(K) \cong K$. Since $K^\vee \cong K$, this proves the result. $\square$

We eventually want to prove the converse. We first note:

(13.2) Proposition. *let (R, m, k) be local and let M be a finitely generated R-module. Then $id_R M \leq d$ iff $\mathrm{Ext}^i(K, M) = 0$ for $i > d$.*

Proof. $\mathrm{id}_R M \leq d \Leftrightarrow \mathrm{id}_R \mathrm{cosyz}^{\,d} M \leq \Leftrightarrow \mathrm{Ext}^j(R/I, \mathrm{cosyz}^{\,d} M) = 0$ for all ideals I of R and all $i > d$. Thus, it will suffice to show that $\mathrm{Ext}^i_R(N, M) = 0$ for all $i > d$ and all finitely generated R-modules N. If not, we may choose N of smallest dimension giving a counterexample. Since N has a prime filtration we obtain a counterexample of that dimension in which N has the form R/P for some prime P of R. We know $P \neq m$: thus, we may choose $x \in m - P$, and then we have a short exact sequence

$$0 \to R/P \xrightarrow{x} R/P \to C \to 0$$

where $\dim C < \dim R/P$. For $i > d$ the long exact sequences for Ext then yields

$$0 = \mathrm{Ext}^i_R(C, M) \to \mathrm{Ext}^i_R(R/P, M) \xrightarrow{x} \mathrm{Ext}^i_R(R/, M) \to \mathrm{Ext}^{i+1}_R(R/P, C) = 0,$$

and Nakayama's lemma then implies that $\mathrm{Ext}^i_R(R/P, M) = 0$, a contradiction. $\square$

(13.3) Lemma. *Let (R, m, K) be a local ring such that $id_R R$ is finite and* $\dim R \geq 1$.

(a) *$depth\, R \geq 1$.*

(b) *If $x \in R$ is not a zerodivisor and $S = R/xR$, then $id_S S$ is finite.*

Proof. (a) If not, then there is a short exact sequence $0 \to K \to R \to R \to 0$. Let $d = \mathrm{id}\, R$. Then $d \geq 1$ (if R itself is injective, we already know that $\dim R = 0$), and applying $\mathrm{Hom}(\_\,,\, R)$ we obtain

$$0 = \mathrm{Ext}^d_R(R,\, R) \to \mathrm{Ext}^d_R(K,\, R) \to \mathrm{Ext}^{d+1}_R(R,\, R) = 0$$

which shows that $\mathrm{Ext}^j_R(K,\,R) = 0$ for $j \geq d$. But then the preceding proposition implies that $\mathrm{id}_R R < d$, a contraction.

(b) Let $E^\bullet$ be an injective resolution of R of length, say, d. Then the cohomology of $\mathrm{Hom}_R(S, E^\bullet)$ is $\mathrm{Ext}^\bullet_R(S, R)$, and this may be calculated from the free resolution

$$0 \to R \xrightarrow{x} R \to S \to 0.$$

We have that $\mathrm{Hom}_R(S, R) = 0$, that $\mathrm{Ext}^1_R(S, R) \cong S$, and that $\mathrm{Ext}^i_R(S, R) = 0$ if $i \geq 2$. Next note that $E'_i = \mathrm{Hom}_R(S, E_i)$ is an injective S-module. Now, E'_0 injects into E'_1 and since it is injective over S, it is a direct summand. Thus, E'_1/E'_0 is injective over S, and the sequence

$$0 \to E'_1/E'_0 \to E'_2 \to \ldots \to E'_d \to 0$$

is exact except possibly at E'_1/E'_0 and consists of S-injective modules. Thus, it is an injective resolution over S of

$$\mathrm{Ker}\,(E'_1/E'_0 \to E'_2) \cong (\mathrm{Ker}\,E'_1 \to E'_2)/E'_0 \cong H^1(\mathrm{Hom}_R(S, E^\bullet)) \cong \mathrm{Ext}^1_R(S, R) \cong S.$$

□

(13.4) Theorem. *A local ring (R, m, K) has finite injective dimension as a module over itself iff it is Cohen-Macaulary of type 1, i.e., Gorenstein, in which case $\mathrm{id}_R R = \dim R$ and the resolution is as described in Theorem (13.1).*

Proof. All we need to show is that if $\mathrm{id}_R R$ is finite then R is Cohen-Macaulay of type 1. We proceed by induction on $\dim R$.

First suppose that $\dim R = 0$. Consider a minimal injective resolution of R. By calculating the Bass numbers from the finite resolution, it follows that the minimal resolution is at least as short. There is only one indecomposable injective, $E = E_R(K)$, and each module in the minimal resolution is a finite direct sum of copies of E. Applying $\mathrm{Hom}_R(\_\,, E)$ we obtain a finite projective resolution of E over R, which shows that $\mathrm{pd}_R E$ is finite. Since $\operatorname{depth} R = 0$, we have that E must be R-free, i.e. $E \cong R^t$ for some t, and since E and R have the same length we must have that $t = 1$. Thus, $R \cong E$, which means that R is self-injective. In the zero-dimensional case we already know that this is equivalent to the condition that R have type 1, and we are done.

Now suppose that $\dim R > 0$. By part (a) of the lemma above, we know that $\operatorname{depth} R > 0$, and so we can choose a nonzerodivisor $x \in R$. By part (b) of the lemma, $S = R/xR$

also has finite injective dimension as a module over itself. By the induction hypothesis, S is Cohen-Macaulay of type 1. But then, since x is a nonzerodivisor in R, R is Cohen-Macaulay type 1. $\square$

(13.5) Exercise. Show that for a Noetherian ring R, $\mathrm{id}_R R$ is finite iff all of its local rings (respectively, its local rings at maximal ideals) are Gorenstein *and* $\dim R$ is finite, in which case $\mathrm{id}_R R = \dim R$.

14. CANONICAL MODULES AND LOCAL DUALITY OVER COHEN-MACAULARY RINGS

(14.1) Definition. We shall say that a finitely generated module M over a Cohen-Macaulay local ring (R, m, K) of dimension d is a *canonical module* for R if its Matlis dual is $\cong H^d_m(R)$. If R is complete then there is always a canonical module, namely the Matlis dual of $H^d_m(R)$. If R is Gorenstein local, then R itself is a canonical module for R.

There are pathological examples of Cohen-Macaulay local rings which have no canonical module. (Their completions have a canonical module, but in the bad cases this module is not the completion of a finitely generated module over the original ring.) We shall see later that the canonical module is unique up to non-unique isomorphism. However, before that, we want to explain its use in establishing a form of local duality for a Cohen-Macaulay ring R. We shall usually write ω_R for a canonical module over the ring R, or, simply, ω, if R is understood.

(14.2) Theorem (local duality over a Cohen-Macaulay ring). *Let (R, m, K) be a Cohen-Macaulay local ring of dimension d, let $E = E_R(K)$ be an injective hull of K over R, let ω_R be a canonical module, and fix an isomorphism of $\omega_r^\vee \cong H^d_m(R)$, where $\_^\vee$ denotes $\mathrm{Hom}_R(\_, E)$. Then for all finitely generated R-modules M, there is an isomorphism $H^i_m(M) \cong \mathrm{Ext}^{d-i}_R(M, \omega_R)^\vee$, functorial in M.*

Proof. Recall that we know over any Cohen-Macaulay ring that

$$H^i_m(M) \cong \mathrm{Tor}^{d-i}_R(M, H^d_m(R)) \cong \mathrm{Tor}^{d-i}_R(M, \omega^\vee).$$

The rest of the proof is exactly the same as in the case where R is Gorenstein: if $P_\bullet$ is a resolution of M by finitely generated projectives we may identify $P_\bullet \otimes \omega^\vee$ with

$\mathrm{Hom}_R\,(\mathrm{Hom}_R(P_\bullet, \omega),\, E) = \mathrm{Hom}_R(P_\bullet, \omega)^\vee$ using Lemma (11.7), with $W = \omega$ and $N = E$, and since $\_^\vee$ commutes with the calculation of (co)homology we may identify $\mathrm{Tor}_R^{d-i}(M, \omega^\vee)$ with $\mathrm{Ext}_R^{d-i}(M, \omega^\vee)$. $\square$

We also have:

(14.3) Theorem. *Let (R, m, K) be Cohen-Macaulay and a homomorphic image of a Gorenstein ring S. Then S may be chosen to be local, and if $h = \dim S - \dim R$ then $\mathrm{Ext}_S^h(R, S)$ is a canonical module for R.*

More generally, if $(S, n, L) \to (R, m, K)$ is a local homomorphism of local rings such that R is module-finite over the image of S, R, S are Cohen-Macaulay, S has canonical module ω_S, and $h = \dim S - \dim R$ then $\mathrm{Ext}_S^h(R,\, S)$ is a canonical module for R.

In particular, if S is Gorenstein (e.g., if S is regular) and R is a local module-finite extension of S that is Cohen-Macaulay, then $\mathrm{Hom}_S(R, S)$ is a canonical module for R over S. E.g. if R is complete, Cohen-Macaulay, local and either equicharacteristic or a domain, then we may choose $S \subseteq R$ regular such that R is module-finite over S, then $\mathrm{Hom}_S(S,\, R)$ is a canonical module for R.

Proof. In the situation of the first paragraph, one may replace S by its localization at the contraction on m. We now assume that we are in the situation of the second paragraph. Let $d = \dim R$, $s = \dim S$. Then $\mathrm{Ext}_S^h(R, \omega_S)$ is a finitely generated S-module and also, because of the R-module structure of the first variable, is an R-module and, hence, a finite generated R-module. By local duality over S we have that the dual of $\mathrm{Ext}_S^h(R, \omega_S)$ into $E_S(L)$ is $H_n^{s-h}(R) = H_n^d(R) \cong H_m^d(R)$. But for any R-module M,

$$\mathrm{Hom}_S(M, E_S(L)) \cong \mathrm{Hom}_S(M \otimes_R R, E_S(L))$$

$$\cong H_R(M, \mathrm{Hom}_S\big(R, E_S(L)\big) \cong \mathrm{Hom}_R\big(M, E_R(K)\big),$$

since $\mathrm{Hom}_S(R, E_S(L)) \cong E_R(K)$, and applying this with $M = \mathrm{Ext}_S^h(R,\, \omega_S)$, we obtain $\mathrm{Hom}_R(\mathrm{Ext}_S^h(R,\, \omega_S),\, E_R(K)) \cong H_m^d(R)$, as required.

The statements of the third paragraph are then immediate. $\square$

Since the local rings that come up in algebraic geometry, number theory, several complex variables, etc. are almost always homomorphic images of regular rings, the Cohen-Macaulay rings that come up will almost always have canonical modules.

(14.4) Lemma. *Let M and N be finitely generated R-modules, where (R,m,K) is local. If their m-adic completions are isomorphic, then M and N are isomorphic.*

Proof. $\mathrm{Hom}_{\widehat{R}}(\widehat{M},\,\widehat{N}) \cong \widehat{R}\otimes_R \mathrm{Hom}_R(M,\,N) \cong \mathrm{Hom}_R(M,N)\widehat{\ }$. If $\phi:\widehat{M}\to\widehat{N}$ is onto we may choose $f:M\to N$ such that $\phi\equiv\widehat{f} \mod m\mathrm{Hom}_R(M,N)\widehat{\ }$. It follows easily that $N\subseteq \mathrm{Im}\,f+mN$, since ϕ is onto, and then, by Nakayama's lemma, f is onto. Since each of $\widehat{M}$, $\widehat{N}$ can be mapped onto the other, the same is true for each of M, N. Say $f:M\to N$ and $g:N\to M$ are both onto. Then $gf:M\to M$ is onto, and hence, since M is finitely generated, gf is one-to-one. Hence, f is one-to-one and so must be an isomorphism. $\square$

The following result summarizes much of the behavior of canonical modules.

(14.5) Theorem. *Let (R,m,K) be a Cohen-Macaulary local ring of Krull dimension d.*

(a) *A finitely generated R-module ω is a canonical module for R if and only if $\widehat{\omega}$ is a canonical module for $\widehat{R}$.*

(b) *If ω,ω' are canonical modules for R, then $\omega\cong\omega'$.*

(c) *If ω is a canonical module for R and $x_1,\dots,x_k$ is a regular sequence in R, then $x_1,\dots,x_k$ is a regular sequence on ω and $\omega/(x_1,\dots,x_k)\omega$ is a canonical module for $R/(x_1,\dots,x_k)R$. In particular, depth$\,\omega=\dim R$, and ω is a Cohen-Macaulay module.*

(d) *If $\dim R=0$, then a canonical module for R is the same as an injective hull for K.*

(e) *A finitely generated R-module ω is a canonical module for R if and only if depth ω $=\dim R$ and for some (equivalently, every) system of parameters $x_1,\dots,x_d$ of R, $\omega/(x_1,\,\dots,x_d)\omega$ is an injective hull for $R/(x_1,\,\dots,x_d)R$*

(f) *If ω is a canonical module for R, then the ring $R\oplus_R\omega$ (where the product of $r\oplus w$ and $r'\oplus w'$ is defined to be $rr'\oplus(rw'+r'w)$) is a Gorenstein local ring that maps onto R.*

(g) *If ω is a canonical module for R, then for every prime ideal P of R, ω_P is a canonical module for R_P.*

(h) *If ω is a canonical module for R, then the obvious map $R\to\mathrm{Hom}_R(\omega,\omega)$ is an isomorphism. Moreover, $H^n_m(\omega)\cong E_R(K)$, so that $E_R(K)\cong\varinjlim_t\omega/(x_1^t,\dots,x_d^t)/\omega$, where the successive maps are induced by multiplication by $x_1\cdots x_d$.*

(i) *The minimal number of generators of ω is the type of R. R is Gorenstein iff If ω is cyclic, in which case $\omega\cong R$.*

Proof. Let $E=E_R(K)$ be an injective hull of K and let $\_^{\vee}$ denote $\mathrm{Hom}_R(\_,\,E)$.

(a) We have that $\mathrm{Hom}_R(\omega, E) \cong \mathrm{Hom}_{\widehat{R}}(\widehat{\omega}, E)$. Because the module in the first variable is finitely generated, the image of a specific homomorphism in either Hom is a finitely generated (and, hence, finite length) submodule of E, and for all t, $\omega/m^t\omega \cong \widehat{\omega}/m^t\widehat{\omega}$. Thus, $\omega^\vee \cong H^d_m(R)$ iff $\mathrm{Hom}_{\widehat{R}}(\widehat{\omega}, H_{\widehat{R}}(\widehat{\omega}, E) \cong H^d_m(\widehat{R})$ $(\cong H^d_m(R))$.
(b) In the complete case, the condition $\omega^\vee \cong H^d_m(R)$ (the latter has DCC) implies that $\omega \cong \omega^{\vee\vee} \cong H^d_m(R)^\vee$. Thus, any canonical module is isomorphic to $H^d_m(R)^\vee$, which is a canonical module (we are using tacitly that ω has ACC). In general, given two canonical modules, their completions are isomorphic, and so the modules are isomorphic by Lemma (14.4).
(c) It suffices to do the case $k = 1$. The issues are unaffected by completion. Thus, we may assume that $R = T/I$ where T is regular. Suppose that $\dim T - \dim R = j$: this is also ht I. Then $\omega \cong \mathrm{Ext}^h_T(T/I, T)$. Let x be a nonzerodivisor in R, and let $S = R/xR$, which is also Cohen-Macaulay. The unique nonvanishing $\mathrm{Ext}^j(R, T)$ occurs for $j = h$, while the unique nonvanishing $\mathrm{Ext}^j(S, T)$ occurs for $j = h + 1$. The long exact sequence for Ext coming from $0 \to R \xrightarrow{x} R \to S \to 0$ yields:

$$0 \to \mathrm{Ext}^h_T(R, T) \xrightarrow{x} \mathrm{Ext}^h_T(R, T) \to \mathrm{Ext}^{h+1}_T(S, T) \to 0$$

and since $\omega_R \cong \mathrm{Ext}^h_T(R, T)$ while $\omega_S \cong \mathrm{Ext}^{h+1}_T(S, T)$, the result follows.
(d) When $d = 0$, $\omega_R \cong \mathrm{Hom}_R(H^0_m(R), E) \cong \mathrm{Hom}_R(R, E) \cong E$.
(e) We have already established that if ω is a canonical module for R then $\mathrm{depth}\, \omega = \dim R$. Moreover, if $x_1, \ldots, x_d$ is a system of parameters then $\omega/(x_1, \ldots, x_d)\omega$ is a canonical module for $R/(x_1, \ldots, x_d)R$, which, by part (d), is an injective hull for K over $R/(x_1, \ldots, x_d)R$.

Now suppose that M is a module of depth d and for the one system of parameters $\underline{x} = x_1, \ldots, x_d$, we have that $M/(\underline{x})M$ is an injective hull for $R(\underline{x})$. This implies that M has type 1, and so the socle in $M/(\underline{y})M$ will be one-dimensional for any system of parameters $\underline{y}$: thus, every $M/(\underline{y})M$ is an esential extension of K. We next want to show that if $\underline{y} = \underline{x}^t$ then $M/(\underline{x}^t)M$ has the same length as $R/(\underline{x}^t)R$. This will imply that $M/(\underline{x}^t)M$ is an injective hull for K over $R/(\underline{x}^t)$ for all t.

One way to see this is to note that $R(\underline{x}^t)R$ has a filtration

$$R/(\underline{x}^t)R \supseteq I_1(\underline{x}^t)R \supseteq \cdots \supseteq I_j/(\underline{x}^t)R \supseteq \cdots \supseteq I_N(\underline{x}^t)R = 0$$

where $N = t^d$, each of the ideals I_j is generated by a set of monomials in the x's, $I_0 = R$, $I_N = (\underline{x}^t)R$, and $I_j/I_{j+1} \cong R/(\underline{x})R$ for $0 \le j \le N - 1$. One can build the sequence of ideals recursively by beginning with $I_N = (\underline{x}^t)R$, and, at the recursive step (constructing I_j from I_{j+1}) adjoining to I_{j+1} a monomial $\mu = x_1^{a_1} \cdots \boldsymbol{x}_d^{a_d}$ such that each $x_s\mu$ is already in I_{j+1}, $1 \le s \le d$. Then $I_j/I_{j+1} \cong (I_{j+1} + R\mu)/I_{j+1} \cong R\mu/(I_{j+1} \cap R\mu) \cong R/(I_{j+1} : R\mu)$ (quite generally, if μ is not a zerodivisor in R, then $J \cap R\mu = (J : R\mu)\mu \cong J : R\mu)$, and it is not difficult to see that, because the x's are a regular sequence in R, $I_{j+1} : R\mu = (\underline{x})R$. (It is clear, by construction, that $I_{j+1} : R\mu$ contains $(\underline{x})R$. To get the other inclusion, recall that the exponent of x_j in μ is a_j. We may enlarge $I = I_{j+1}$ so that it contains $x_j^{a_j+1}$ for each j, and once this is done the resulting ideal I is actually generated by the elements $x_j^{a_j+1}$, $1 \le j \le d$. Choose $t \ge \max_j a_j$. Suppose that $r\mu \in I$. Then if we multiply through by $\prod_{j=1}^d x_j^{t-a_j}$, we see that $r(x_1^t \cdots x_d^t) \in (x_1^{t+1}, \ldots, x_d^{t+1})R$, and we already know that in this case, $r \in (x_1, \ldots, x_d)R$ when the x 's form a regular sequence.)

(We note two closely related facts. When $\underline{x}$ is an R-sequence, the associated graded ring $\mathrm{gr}_{(\underline{x})R}R$ is a polynomial ring in d variables over $A = R/(\underline{x})$. Also, $A[X_1, \ldots, X_d]/(\underline{X}^t)$ is a free A-module whose generators are the images of the t^d monomials in the variables X_j in which all exponents occurring are $< t$.)

One has a corresponding filtration of $M/(\underline{x}^t)M$, namely

$$M/(\underline{x}^t)M \supseteq I_1M/(\underline{x}^t)M \supseteq \cdots \supseteq I_jM/(\underline{x}^t)M \supseteq \cdots \supseteq I_NM/(\underline{x}^t)M = 0,$$

and because the x's form a regular sequence on M one can show similarly that every factor is isomorphic with $M/(\underline{x})M$. By hypothesis, $M/(\underline{x})M$ is an injective hull of K over $R/(\underline{x})R$, and so $\ell(M/(\underline{x})M) = \ell(R/(\underline{x})R)$. But then $\ell(M/(\underline{x}^t)M = t^d\ell(M/(\underline{x})M) = t^d\ell(R/(\underline{x})R = \ell(R/(\underline{x}^t)R$. Since $M/(\underline{x}^t)M$ is an essential extension of K over $R/(\underline{x}^t)R$ whose length is the same as the length of the injective hull of K over $R/(\underline{x}^t)R$, it must be the injective hull of K over $R/(\underline{x}^t)R$.

The question of whether M is a canonical module is unaffected by completion. Suppose that ω is a canonical module. We then have that for every t there is an isomorphism $M/(\underline{x}^t)M \cong \omega/(\underline{x}^t)\omega$. It will suffice to show that we can choose these isomorphisms α_t compatibily, so that for all t, α_{t+1} induces α_t (for then we obtain an induced isomorphism of $M\varprojlim_t M/(\underline{x}^t)M$ with $\omega \cong \varprojlim_t \omega/(\underline{x}^t)\omega)$. We can do this recursively. Suppose that α_t has been chosen. Choose an arbitrary isomorphism $\beta : M(\underline{x}^{t+1})M \cong \omega/(\underline{x}^{t+1})\omega$. Then β induces an isomorphism of $M/(\underline{x}^t)M \cong \omega(\underline{x}^t)\omega$ when we apply $T/(\underline{x}^t) \ \otimes_R \ \_$: call it γ. Then $\alpha_t\gamma^{-1}$ is an automorphism of $\omega/(\underline{x}^t)\omega)$, which is an injective hull of K over $R/(\underline{x}^t)R$. Thus, $\alpha_t\gamma^{-1}$ coincides with multiplication by a certain unit η of $R/(\underline{x}^t)R$, and

η lifts to a unit ζ of $R/(\underline{x}^{t+1})R$. Let $\alpha_{t+1} = \zeta\beta$. When we look at the induced map $M/(\underline{x}^t M) \to \omega/(\underline{x}^t\omega)$ we get $\eta\gamma = \alpha_t\gamma^{-1}\gamma = \alpha_t$, as required.

(f) It is easy to check that $S = R \oplus \omega$ is a commutative associative R-algebra in which ω is an ideal whose square is zero. Evidently, S is local and module-finite over R. Any system of parametrs in R is a regular sequence on both R and on ω. Thus, S is Cohen-Macaulay, and if one kills a maximal regular sequence in R, one reduces to proving that the quotient has type 1. The quotient has the form $R \oplus E$, where R is an Artin local ring and E is the injective hull of its residue field. Suppose that $r \oplus e$ kills the maximal ideal $m \oplus E$ of S. Then $r \oplus E$ kills E, which implies that r kills E. Since E is faithful, r must be zero. Thus, we must have $0 \oplus e$, where e kills $m \oplus E$. But this simply means that e kills m. Thus, the socle of S is $0 \oplus \operatorname{Soc} E$, which shows that is a one-dimensional K-vector space, as required.

Finally, note that $S/\omega \cong R$.

(g) By part (f), there is no loss of generality in assuming that $R \cong S/I$, where S is a local Gorenstein ring. Then $\omega \cong \operatorname{Ext}^h_S(R, S)$, where h is the height of I. The result now follows from the fact that, for any prime ideal $P = Q/I$ of R, where Q is a prime ideal of S containing I, the height of I does not change when we localize at Q (R is Cohen-Macaulay), and localization at Q commutes with the calculation of Ext.

(h) The issue of whether $R \to Hom_R(\omega, \omega)$ is an isomorphism is unaffected by completion. Thus, we assume that R is complete. Any endomorphism of ω induces an endomorphism of $\omega_t = \omega/(\underline{x}^t)\omega$ for all t by applying $R/(\underline{x}^t)R \otimes_R \_$, and, conversely, given a family of endomorphisms β_t of ω_t for each t, which are compatible (so that β_{t+1} induces β_t for each t), they induce an endomorphism of $\varprojlim_t \omega_t \cong \omega$ which gives rise to all of them. Thus, $\operatorname{Hom}_R(\omega, \omega) = \varprojlim_t \operatorname{Hom}_R(\omega_t, \omega_t)$. Here, it does not matter whether we take R-endomorphisms of ω_t or $(R/(\underline{x}^t)R)$-endomorphisms. Now, ω_t is an injective hull for K over $R/(\underline{x}^t)R$ for each t, and so its endomorphisms may be identified with $R/(\underline{x}^t)R$ by the obvious map. This identifies $\operatorname{Hom}_R(\omega, \omega)$ with $\varprojlim_t R/(\underline{x}^t)R \cong R$, as required.

But then local duality over R yields $H^n_m(\omega) \cong \operatorname{Ext}^0_R(\omega, \omega)^\vee = \operatorname{Hom}_R(\omega, \omega)^\vee \cong R^\vee = E$.

(i) For the first statement we may complete, and so assume that R is a homomorphic image of a regular ring T. Then ω is the Ext dual of R over T, and so the minimal number of generators of ω is the type of R by a Math 615 problem. EXPAND ON THIS. It is then clear that R is Gorenstein iff ω is cyclic. But if R is Gorenstein we know that $\omega \cong R$. □

(14.6) Exercises. Let (R, m, K) be a Cohen-Macaulay ring of Krull dimension d with canonical module $\omega = \omega_R$.

(a) Show that $\mathrm{id}_R\,\omega = d$, and that the ith module in a minimal injective resolution of ω is the direct sum over all prime ideals P of height i of the modules $E(R/P)$ (one copy of each).

(b) Show that if M is a finitely generated module of finite projective dimension over R, then $\mathrm{Tor}_i^R(M,\omega) = 0$ for $i \geq 1$. (This only depends on the fact that ω is a faithful Cohen-Macaulay module over R, so that for every ideal I of R, $\mathrm{depth}_I\,\omega = \mathrm{depth}_I\,R = htI$. Apply the Bucsbaum-Eisenbud criterion. EXPAND)

(c) Prove that if M is a finitely generated R-module of finite projective dimension, then $M \otimes_R \omega$ is a finitely generated R-module of finite injective dimension.

(d) Prove that if M is a finitely generated module of finite injective dimension then $\mathrm{Ext}_R^i(\omega, M) = 0$ for $i \geq 1$. (Again, this only depends on the fact that ω is a faithful Cohen-Macaulay module.)

(e) (R.Y. Sharp: see [Sh]) Prove that the category of finitely generated R-modules of finite projective dimension is equivalent to the category of finitely generated R-modules of finite injective dimension (the map in one direction is given by $\_\ \otimes_R \omega$, and in the other direction by $\mathrm{Hom}_R(\omega,\ \_)$. (If R is Gorenstein, $\omega = R$ and the two categories coincide.)

(14.7) Exercises. Let (R, m, K) be any local ring.

(a) Prove that if M is any R-module of finite injective dimension, and x is not a zero divisor on M, then M/xM has finite injective dimension. Conclude the same for $M(\underline{x})M$, where $\underline{x} = x_1, \dots, x_k$ is a regular sequence on M.

(b) Show that if R is Cohen-Macaulay and $\underline{x}$ is a system of parameters, then the finite length module E which is the injective hull of K over $R/(\underline{x})R$ has finite injective dimension over R. (Reduce to the case where R is complete, and so has a canonical module ω. Then $E \cong \omega/(\underline{x})\omega$.) Conclude that every Cohen-Macaulay local ring possesses a finitely generated nonzero module of finite injective dimension.

(14.8) Remark. Bass asked whether a local ring which possesses a finitely generated nonzero module M of finite injective dimension must be Cohen-Macaulay. See his paper [B]. Peskine and Szpiro answered this affirmatively in [PS] in characteristic p and in many other cases in characteristic 0. The result can be deduced from the intersection theorem discussed by Peskine and Szpiro in [PS], and from this it can be deduced that it follows in the equicharacteristic case from the existence of big Cohen-Macaulay modules. Paul Roberts proved the intersection theorem in mixed characteristic in [Ro?], and so the question of Bass has been answered affirmatively in all cases. Thus, a local ring is Cohen-Macaulay iff it possesses a nonzero finitely generated module of finite injective dimension. Roberts' work depends on a theory of Chern classes developed by Fulton, Macpherson and Baum. See [Ful], [Ro book], [Ro MSRI exposition]

(14.9) Theorem. *Let (R, m, K) be a Cohen-Macaulay local ring with canonical module ω.*

(a) *If R is a domain, or if R is reduced, or, much more generally, if the localization of R at every minimal prime is Gorenstein (in this case we say that R is generically Gorenstein) then ω is isomorphic with an ideal of R that contains a nonzerodivisor.*

(b) *If $\omega \cong I \subseteq R$, then every associated prime of I has height one. More generally, if R is any ring that is S_2 and I is an ideal of R containing a nonzerodivisor such that I is S_2 as an R-module, then every associated prime of I has height one.*

Proof. (a) Let S be the multiplicative system of R consisting of all nonzero divisors. Since R is Cohen-Macaulay, S is simply the complement of the union of the minimal primes of R. We shall show that iff R is generically Gorenstein then $S^{-1}\omega \cong S^{-1}R$. The restriction of the isomorphism to ω then yields an injection $\eta : \omega \to S^{-1}R$ (note that the elements of S are also nonzerodivisors on ω). The images of a finite set of generators of ω can be written r_i/s_i, $1 \le i \le h$, and the if $s = s_1 \cdots c_n$, we have that $s\eta : \omega \to R$ is an injection.

Since $S^{-1}R$ is zero-dimensional, it is isomorphic with $\Pi_P R_P$ as P runs through the minimal primes of R, and $S^{-1}\omega \cong \Pi_P \omega_P$. Thus, it suffices to show that $\omega_P \cong R_P$ when P is minimal. But ω_P is a canonical module for R_P, and R_P is Gorenstein.

Of course, when R is a domain or reduced, the localization at any minimal prime is a field, and so reduced rings are generally Gorenstein.

Note that the issue of whether I contains nonzero divisor is unaffected by localization at S. The argument just given implies that $S^{-1}I \cong S^{-1}\omega \cong S^{-1}R$ is free, and an element of I which generates must correspond to a nonzerodivisor.

(b) Suppose that Q is an associated prime of I of height two or more. Then we replace R, I by R_Q , IR_Q. Thus we may assume that (R, m, K) is local, that depth $R \ge 2$, and that the maximal ideal of R is associated to I. Then depth $R/I = 0$, and since depth $R = \dim R \ge 2$, we must have depth$_m I = 1$. But depth$_m I \ge 2$, because, since I contains a nonzerodivisor, it has a submodule isomorphic with R, and this means that $\dim I$ (as an R-module is at least two. This is a contradiction. □

(14.10) Definition and discussion. Let R be a normal Noetherian domain. If $a \neq 0$ in

R, then the primary decomposition of aR has the form $P_1^{(n_1)} \cap \cdots \cap P_k^{(n_k)}$, where P_i are height one primes and n_i is the order of a as an element of the DVR R_{P_i}. We may form the free abelian group whose generators are the height one primes of R. The element $\Sigma_i\ n_i P_i$ is called the *divisor* of a in that group. If we kill subgroup generated by all the divisors of nonzero elements of R in the free abelian groups generated by the height one primes, then we obtain what is called the *divisor class group* of R. (This coincides with the notion that one has for Dedekind domains, which number theorists have studied intensely for rings of integers in finite algebraic extensions of the rationals.) It is sometimes denoted $\mathrm{Cl}\,R$.

Every element of $\mathrm{Cl}\,R$ can be represented by an ideal of pure height one (i.e., a nonzero ideal whose primary decomposition involves only height one primes). Note that the only primary ideals for a height one prime in a normal ring are its symbolic powers. Thus, we let the ideal $P_1^{(n_1)} \cap \cdots \cap P_k^{(n_k)}$, correspond to $\Sigma_i\ n_i P_i$. If some n_i is negative, we may always add the divisor of some element $a \in R - \{0\}$ to obtain an equivalent element in which all n_i are nonnegative.

It turns out that any ideal of pure height one in a normal domain is a (torsion-free) reflexive module of rank one. In fact, there is a bijective correspondence between elements of $\mathrm{Cl}\,R$ and the isomorphism classes of rank one reflexive modules: each rank one reflexive is isomorphic to a nonzero ideal, which must be of pure height one (because both it and the ring are S_2). It turns out that the ideals I, J are isomorphic as modules if and only if there are elements $a, b \in R - \{0\}$ such that $bI = aJ$, and this is the class if and only if $\mathrm{Cl}\,R$ corresponds to tensoring the reflexive modules and taking the double dual (or multiplying the representative ideals I, J and then taking the intersection of the primary components of IJ that correspond to height one primes). The inverse of I corresponds to $\mathrm{Hom}_R(I, R)$. The elements $\mathrm{Cl}\,R$ are called *divisor classes.* It is worth noting that R is a UFD iff $\mathrm{Cl}\,R = 0$.

The point we want to make here is that ω is a rank one reflexive module when R is normal Cohen-Macaulay, and so represents a divisor class.

(14.11) Theorem (Murthy). *Let R be a Cohen-Macaulay ring which is a homomorphic image of a Gorenstein ring and suppose that the local rings of R are UFD's. Then R is Gorenstein.*

Proof. We may assume that R is local. The hypothesis implies that R has a canonical module ω. Since R is a UFD, the ideal corresponding to ω must be principal. Thus, ω is cyclic, which implies that $\omega \cong R$ and so R is Gorenstein. $\square$

There is an example of a two-dimensional local UFD ($\Rightarrow$ normal and, hence, Cohen-Macaulay in dimension 2) that is not Gorenstein. This ring is consequently not a homomorphic image of a Gorenstein ring.

(14.12) Example. Let S be a subsemigroup of $\mathbb{N}$, i.e., a subset containing 0 and closed under addition. Assume that the greatest common divisor of the elements of S is 1. E.g., we might have $S = \{0, 2, 3, 4, 5, \dots\}$ (2 and 3 generate) or we might have that $S = \{0, 7, 9, 10, 11, 12, 13, \dots\}$. Let K be a field and let $K = [[t^S]]$ denote the subring of $K[[t]]$ consisting of all power series in which the exponents occurring are in S. For any choice of S as above, $K[[t^S]]$ is one-dimensional integral domain, and there Cohen-Macaulay. The integral closure of any of these rings is $K[[t]]$. There is always a largest element $a \in \mathbb{N}$ such that $a \notin S$. A corollary of one of the results of Gorenstein's thesis (he was a student of Zariski before he converted to group theory) asserst that $K = [[t^S]]$ is Gorenstein (Bass coined the term later) if and only if the number of elements in $\mathbb{N} - S$ is equation to the number of elements in S that are $< a$. In the first example, $a = 1$, and since $\{0\}, \{1\}$ have the same cardinality the ring is Gorenstein. In the second example, $a = 8$, and since $\{0, 7\}, \{1, 2, 3, 4, 5, 6, 8\}$ have different cardinalities, the ring is not Gorenstein.

(Consider a complete local domain of dimension one, R, and let A be the integral closure of R, which is a DVR. Then $J = \{r \in R : rA \subseteq R\}$ is the largest ideal of R which is also an ideal of A and is called the *conductor* of A in R. Since J is a nonzero ideal of A, $J \cong A$ as an A-module. Consider the exact sequence $0 \to J \to R \to R/J \to 0$. If R is Gorenstein and we apply $\mathrm{Hom}_R(\_\,, R)$ we obtain

$$0 \to 0 \to \mathrm{Hom}_R(R, R) \to \mathrm{Hom}_R(J, R) \to \mathrm{Ext}^1_R(R/J, R) \to 0.$$

$J \cong A$ and $\mathrm{Hom}_R(A, R)$ is a canonical module for A and therefore is $\cong A$. The Ext^1 terms is dual to $H^0_m(R/J) \cong R/J$. But the sequence can be identified with

$$0 \to 0 \to R \to A \to A/R \to 0$$

which shows that A/R is the Matlis dual of R/J, and so the two have the same length. Applying this with $R = K[[t^S]]$, $A = K[[t]]$, and $J = t^{a+1}A$ gives half the result (assuming that R is Gorenstein).

The rest of the proof is left as an excercise.

15. GLOBAL CANONICAL MODULES

(15.1) Definition and discussion. Let R be a Cohen-Macaulay ring that is not necessary local. We may then define a canonical module for R to be any finitely generated module ω such that ω_P is a canonical module for R_P for every prime ideal P of R. Of course, it suffices if the condition holds when P is maximal. If R is Gorenstein, then R itself is a canonical module, but, in general, there are many others: any module locally free of rank one (i.e., any rank one projective) will also be a canonical module. More generally, if ω is a canonical module and N is a rank one projective, then $\omega \otimes_R N$ is, evidently, also a canonical module. Pleasantly, this is the only kind of nonuniqueness that one can have, as the next result will establish.

(15.2) Theorem. *Let R be a Cohen-Macaulay ring.*

(a) *R has a canonical module if and only if R is a homomorphic image of a Gorenstein ring.*

(b) *If ω, ω' are two canonical modules for R, then $\mathrm{Hom}_R(\omega, \omega')$ is a rank one projective over R, and the natural map $\omega \otimes_R \mathrm{Hom}_R(\omega, \omega') \to \omega'$ sending $w \otimes f$ to $f(w)$ is an isomorphism.*

Proof. (a) If R has a canonical module ω, then $R \bigoplus \omega$ (where multiplication is given by $(r \oplus w)(r' \oplus w') = rr' \oplus (rw' + r'w)$) is Gorenstein: since ω is nilpotent, the primes correspond bijectively with primes P of R (P corresponds to $P \bigoplus \omega$), and the localization of $R \bigoplus \omega$ at $P \bigoplus \omega$ at $P \bigoplus \omega$ is the same as its localization at P, i.e, $R_P \bigoplus \omega_{P'}$ since this ring is already local. (Our discussion of the local case shows that $R_P \bigoplus \omega_P$ is Gorenstein.)

To establish the converse, first note that we may assume that Spec R is connected: if not R is a finite product of rings with connected Spec, each of which is a homomorphic image of R and, hence, a homomorphic image of a Gorenstein ring, and if we have a canonical module for each of these, then the product is a canonical module for the product ring.

Now suppose that $R = S/J$ where Spec R is connected and S is Gorenstein. We claim that all minimal primes of J have the same height. This is clear if the two minimal primes are both contained in the same maximal ideal m of S: we can localize at m without affecting any relevant issues, and then, since R is Cohen-Macaulay local, the quotient by any minimal prime has the same dimension as R, while since S is Cohen-Macaulay local, it follows that the two primes have the same height. The result now follows from the observation that for a Noetherian ring R, Spec R is connected iff for any two minimal primes P, P' there is a sequence of ideals $P = P_0, m_1, P_1, m_2, \dots P_{k-1}, m_k, P_h = P'$ such that all the P's are minimal primes, all the m_i are maximal ideals, and each m_j contains both P_{j-1} and P_j, $1 \le j \le k$. We leave this statement as an exercise.

Let h be the common height of all minimal primes of J, which, of course, is the height of J. Then JS_Q has height h for every prime $Q \supseteq J$. It follows from the local case that $\mathrm{Ext}^h_S(S/J, S)$ is a canonical module for $R = S/J$.

(b) Both statements reduce at once to the case where R is local. But then $\omega \cong \omega'$ and $R \to \mathrm{Hom}_R(\omega, \omega) \cong \mathrm{Hom}_R(\omega, \omega')$ is an isomorphism, and the verification is trivial. $\square$

The canonical module is a "first nonvanishing Ext." Such Ext's have a kind of uniqueness which can be establishing using local cohomology theory.

(15.3) Theorem. *Let $R \to S$ be a homomorphism of Noetherian rings, let N be a finitely generated R-module and let M be a finitely generated S-module. Then if $I = \mathrm{Ann}_R N, \mathrm{Ext}^i_R(N, M)$ and $\mathrm{Ext}^i_S(S \otimes_R N, M)$ vanish for $i < d = \mathrm{depth}_I M = \mathrm{depth}_{IS} M$ (the annihilator of $S \otimes_R N$ is the same as IS up to radicals), and if d is finite then $\mathrm{Ext}^d_R(N, M) \cong \mathrm{Ext}^d_S(S \otimes_R N, M)$ as S-modules (and are not zero).*

We shall prove this using:

(15.4) Theorem. *Let R, S, N, and M be as in (15.3), let I be any ideal of R that has a power which kills N, and suppose that $d \le \mathrm{depth}_I M$. Then*

$$\mathrm{Ext}^d_R(N, M) \cong \mathrm{Hom}_R(N,\ H^d_I(M))$$

as S-modules.

Proof that (15.4) implies (15.3). We have the isomorphism of (15.4). But we may also apply the result to the case where $R = S$ and N is replaced by $S \otimes_R N$ to obtain

$$\mathrm{Ext}^d_S(S \otimes_R N, M) \cong \mathrm{Hom}_S(S \otimes_R N, H^d_J(M))$$

where $J = \mathrm{Ann}_S(S \otimes_R N)$. Since J has the same radical as IS we have $H^d_J(M) \cong H^d_I(M)$. Thus

$$\mathrm{Ext}^d_S(S \otimes_R N, M) \cong \mathrm{Hom}_S(S \otimes_R N, H^d_J(M)).$$

But by the universal property of extension of scalars, for any S-module W we have that $\mathrm{Hom}_S(S \otimes_R N, W) \cong \mathrm{Hom}_R(N, W)$. Thus, $\mathrm{Ext}^d_S(S \otimes_R N, M) \cong \mathrm{Hom}_R(N, H^d_I(M)) \cong \mathrm{Ext}^d_R(N, M)$ (by the more transparent initial application of Theorem 15.4). $\square$

Proof of (15.4). Let N be any R-module that is killed by I^t. Then there is a map

$$N \otimes_R \mathrm{Ext}^d_R(N, M) \to \mathrm{Ext}^d_R(R/I^t, M)$$

obtained as follows: given $v \in N$ and $\theta \in \mathrm{Ext}^d_R(N, M)$ we have a map $R/I^t \to N$ sending r to rv, and this induces a map $\lambda_v : \mathrm{Ext}^d_R(N, M) \to \mathrm{Ext}^d_R(R/I^t, M)$. We send $v \otimes \theta$ to $\lambda_v(\theta)$. By the adjointness of $\otimes$ and Hom this yields a map

$$(*_t) \qquad \mathrm{Ext}^d_R(N,\, M) \to \mathrm{Hom}_R(N,\, \mathrm{Ext}^d_R(I^t,\, N)).$$

We claim that this map is an isomorphism. Given N we can take a presentation over R/I^t, say $G \to F \to N \to 0$, where each of G, F is a finite direct sum of copies of R/I^t. Both sides are left exact functors of the variable N (so long as N is killed by a power of I): the lower Ext's vanish. If we abbreviate $\mathcal{E}(W) = \mathrm{Hom}_R(N,\, \mathrm{Ext}^d(R/I^t, M))$ then we have a commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Ext}^d_R(N,\, M) & \longrightarrow & \mathrm{Ext}^d_R(F,\, M) & \longrightarrow & \mathrm{Ext}^d_R(G,\, M) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{E}(M) & \longrightarrow & \mathcal{E}(F) & \longrightarrow & \mathcal{E}(G)
\end{array}$$

The fact that we have an isomorphism now follows from the five lemma (or from the fact that isomorphic maps have isomorphic kernels) once we have shown that F, G yield isomorphisms. This comes down to the case where N is R/I^t, which is obvious.

If we increase t to t' we also get a map

$$(*_{t'}) \qquad \mathrm{Ext}^d_R(N, M) \to \mathrm{Hom}_R(N, \mathrm{Ext}^d(R/I^{t'}, M)).$$

Now $H^d_I(M) \cong \varinjlim_t \mathrm{Ext}^d_R(R/I^t, M)$ and so we have a map

$$\mathrm{Ext}^d_R(N, M) \to \mathrm{Hom}_R(N, H^d_I(M))$$

obtained by composing $(*_t)$ with the induced map

$$\mathrm{Hom}_R(N, \mathrm{Ext}^d_R(R/I^t, M)) \to \mathrm{Hom}_R(N, H^d_i(M)).$$

Since N is finitely presented, Hom commutes with direct limits in the second variable, and since $(*_t)$ is an isomorphism for all sufficiently large t, it follows that the induced map $\mathrm{Ext}^d_R(N, M) \to \mathrm{Hom}_R(N, H^d_I(M))$ is an isomorphism as well. $\square$

Our next objective is to connect the global theory of canonical modules with differentials.

16. CANONICAL MODULUS AND DIFFERENTIAL FORMS

(16.1) Discussion. In the case of Cohen-Macaulay algebras R finitely generated over a field K, one can define a global canonical module that is unique up to isomorphism: one does not need to worry about tensoring with some rank one projective. Since the case where the ring is a product can be handled by dealing with each component ring separately, we shall assume that Spec R is connected.

Then we may map a polynomial ring S over K onto R. Suppose that $R \cong S/I$, and that $\alpha : S \to R$ is the quotient surjection. Then all minimal primes of I have the same height, say h, and we may take $\omega_\alpha \cong \mathrm{Ext}^h_S(S/I, S)$.

Suppose that one choose a different polynomial ring T and maps it onto R, say $R = T/J$ where height $J = k$, and let β be the quotient surjection $T \to R$. We want to see that $Ext^k_T(T/J, T) \cong \mathrm{Ext}^h_S(S/I, S)$, i.e. that $\omega_\alpha \cong \omega_\beta$. First note that there is also a map $\gamma : S \otimes_K T \to R$ that sends $s \otimes t$ to $\alpha(s)\beta(t)$. It will suffice to show that $\omega_\alpha \cong \omega_\gamma$ since, by symmetry, we then also have that $\omega_\beta \cong \omega_\gamma$. Suppose that the indeterminates $z_1, \ldots, z_n$ generating T over K map to $r_1, \ldots, r_n$ in R. We can choose $s_1, \ldots, s_n$ in S mapping to $r_1, \ldots, r_n$ in R, and there is then a rtraction of S-algebras $\delta : S \otimes T \to S$ that sends each z_i to s_i. Then $\alpha \circ \delta = \gamma$. Thus, we may think of γ as arising as the composition of a surjection from the polynomial ring $V = S \otimes_K T$ to S with a surjection from S to R. Now, $V \cong S[z_1, \ldots, z_n]$. By a change of indeterminates sending z_i to $z_i - s_i$, we reduce to the case where the map $V \to S$ simply kills the z_i. The kernel of $V \to R$ is then $I + (z_i)$, and the isomorphism we want is that

$$\mathrm{Ext}^h_S(S/I, S) \cong \mathrm{Ext}^{h+n}_V(V/(I + (z_i)), V).$$

Now, when $z_1, \ldots, z_n$ is an R-sequence on M in the annihilator of N we have that

$$\mathrm{Ext}^{h+n}_V(N, M) \cong \mathrm{Ext}^h_V(N, M/(z_i)M).$$

Applying this to the second module just above, we have that $\mathrm{Ext}_V^{h+n}(R, V) \cong \mathrm{Ext}_V^h(R, S)$, which in turn is isomorphic with $\mathrm{Ext}_S^h(R, S)$, since the calculation of the first nonvanishing Ext is independent of which ring we work over.

Thus, in dealing with Cohen-Macaulay rings R finitely generated over K, we always let ω_R be ω_α for some α, and the resulting canonical module is unique up to isomorphism.

Recall that a finitely generated algebra R over a field K is *geometrically regular* or *smooh* over K if and only if for every field $L \supseteq K$, $L \otimes_K R$ is regular. It suffices if this holds for every finite purely inseparable field extension L of K, or if it holds when L is the smallest perfect field containing K, or if it holds for any larger field that than, e.g., for the algebraic closure of K. Thus, if K has characteristic 0 or if K is perfect, or if K is algebraically closed, then R is smooth over the field K iff R is regular.

Now suppose that R is a smooth K-algebra, with $\mathrm{Spec}\, R$ connected (which, in the regular case, simply means that R is a comain). Then R is Gorenstein, and so ω_R is locally free of rank one, i.e, it is a rank one projective. It is not true, in general, that ω_R is free. The result below explains why global canonical modules are connected with differentials. If R is an A-algebra, we write $\Omega_{R/A}$ for the universal module of Kähler differentials of R over A (thus, there is a universal A-derivation $d\colon R \to \Omega_{R/A}$ and for every R-module M, $\mathrm{Hom}_R(\Omega_{R/A},\, M) \cong \mathrm{Der}_A(R,\, M)$ via the map that sends R to $T \circ d$). We write $\Omega^i_{R/A}$ for the i th exterior power $\bigwedge^i \Omega_{R/A}$. When R is smooth over A, the module $\Omega_{R/A}$ is locally free over R, and hence so are its exterior powers. In particular, when R is a geometrically regular domain that is finitely generated over a field K and has dimension d, then $\Omega_{R/K}$ is locally free of rank d, which implies that $\Omega^d_{R/K}$ is locally free of rank one.

(16.2) Theorem. *If R is a finitely generated K-algebra that is a K-smooth domain of dimension d, then $\Omega^d_{R/K} \cong \omega_R$ (calculated using a surjection of a polynomial ring onto R).*

We defer the proof for a moment. This result suggests that for a smooth K algebra R of dimension d which is a domain, one should think of ω_R as $\Omega^d_{R/K}$. This gives a completely choice free notion of what the canonical module is which will have some functorial properties. If $R = K[x_1, \ldots, x_d]$, then $\Omega^d_{R/K}$ is isomorphic to R: it is the free module on the generator $dx_1 \wedge \cdots \wedge dx_d$. However, this generator is not canonical. Thus, thinking of ω_R as $\Omega^d_{R/K}$ is different from thinking of it as R: R has the canonical generator 1, while viewing R as a polynomial ring in different variables will produce a different generator for $\Omega^d_{R/K}$. In other words, there is no canonical identification of $\Omega^d_{R/K}$ with R. We now establish (16.2) by proving a stronger result:

(16.3) Theorem. *Let S, R be domains finitely generated and smooth over K, and suppose that $S \to R$ is surjective. Let $n = \dim S$ and $d = \dim R$. Then $\mathrm{Ext}_S^{n-d}(R, \Omega^n_{S/K}) \cong \Omega^d_{R/K}$ canonically, and this isomorphism commutes with localization at elements of S.*

Remarks. If we apply this with S a polynomial ring mapping onto R, then we obtain (16.2), since $\Omega^n_{S/K} \cong S$. Note that $n - d$ will be the height h of the kernel ideal.

The fact that one has this canonical isomorphism locally immediately globalizes. Given any scheme X over K there is a quasicoherent sheaf $\Omega_{X/K}$ whose sections on any open affine $\mathrm{Spec}\, S$ coincide with $\Omega_{S/K}$ (he point is that this construction commutes with localization). When X is of finite type this sheaf is coherent. When X is smooth it is locally free. We may similarly define $\Omega^i_{X/K}$ for $i \geq 0$.

Now suppose that X is a smooth scheme of finite type over K all of whose components have dimension n (corresponding to $\mathrm{Spec}\, S$) and let Z be a smooth closed subscheme all of whose components have dimension d (corresponding locally to $\mathrm{Spec}\, R$, where there is a surjection from S to R). The canonical isomorphisms on open affines provided by Theorem (16.3) which commute with localization obviously patch together to give a canonical global isomorphism

$$\mathcal{E}\mathrm{xt}^{n-d}_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^n_{X/K}) \cong \Omega^d_{Z/K}.$$

Here. $\mathcal{E}\mathrm{xt}$ denotes sheaf Ext: when $(X, \mathcal{O}_X)$ is a Noetherian scheme $\mathcal{F}$ is coherent, and $\mathcal{G}$ is quasicoherent, $\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a sheaf whose sections on the open set U may be identified with $\mathrm{Ext}^i_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U))$.

The scheme-theoretic discussion is not a luxury here. In order to prove the purely affine statement (16.3) we need to think scheme-theoretically. We shall construct the canonical isomorphism on a sufficiently small affine neighborhood of each point. These will then patch to give the global fact we want.

We first need two preliminary results.

(16.4) Lemma. *Let $0 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} F \to 0$ be a short exact sequence of free modules over the ring R, where H, G, F have finite ranks h, n, d respectively (so that $n = h + d$). Let γ be any splitting for β. Then the map $(\bigwedge^h \alpha) \wedge (\bigwedge^d_\gamma) : \bigwedge^h H \otimes \bigwedge^d F \to \bigwedge^n G$ is an isomorphism. This isomorphism is independent of the choice of the splitting γ, and is compatible with localization of R (and every other base change).*

Proof. $G = \alpha(H) \bigoplus \gamma(F)$ and we have the standard identification of $\bigwedge^n G$ with the direct sum of all the terms $\bigwedge^i \alpha(H) \otimes \bigwedge^j \gamma(F)$, all of which are zero unless $i = h,\ j = d$. This gives the isomorphism $\bigwedge^h H \otimes \bigwedge^d F \cong \bigwedge^h \alpha(H) \otimes \bigwedge^d \gamma(H) \supseteq \bigwedge^h G$. We need to see that this map is independent of the choice of γ. Suppose that $u_1, \ldots, u_h$ is a free basis for H, and that $v_1, \ldots, v_d$ is a free basis for F, and that $w_1, \ldots, w_d$ are the values of γ on the v_j. Then the isomorphism takes the generator $(u_1 \wedge \cdots \wedge u_h) \otimes (v_1 \wedge \cdots \wedge v_d)$ to $z = \alpha(u_1) \wedge \cdots \wedge \alpha(u_h) \wedge w_1 \wedge \cdots \wedge w_d)$. If we change γ we alter the w's by adding linear combinations of the $\alpha(u_\nu)$. To see that this does not change the value of z, it suffices to check it if we change a single w, say w_1, by adding a linear combination of the $\alpha(u_\nu)$. The result is now clear. $\square$

(16.5) Proposition. *Let I be an ideal of the ring S generated by a regular sequence of length h, and let $R = S/I$. Let M be any S-module. Then there is a canonical isomorphism*

$$\mathrm{Ext}^h_S(S/I, M) \cong \mathrm{Hom}_R(\bigwedge^h(I/I^2),\ R \otimes_S M),$$

functorial in M, which is compatible with localization of S (in fact, with arbitrary flat base change from S.

Proof. Choose a set of generators $f_1, \ldots, f_r$ for I. We construct an isomorphism using these generators. We then check that it is actually independent of the choice of generators. The first key point is that the fact that I is generated by a regular sequence of length h implies that the Koszul complex $K_\bullet(\underline{f}; S)$ is a free resolution of R over S. Let $d(\underline{f})_h$ be the last non-zero map $S \to S^h$ in the Koszul complex. Then $\mathrm{Ext}^h_S(S/ImN) \cong \mathrm{Coker}\,\mathrm{Hom}(d(\underline{f})_h), M) \cong (\mathrm{Coker}\, d(\underline{f})_h) \otimes M) \cong S/I \otimes M = R \otimes M$. Call this isomorphism $\theta_{\underline{f}}$. If we use the generator $F = \overline{f}_1 \wedge \cdots \overline{f}_h$ for $\bigwedge^h(I/I^2)$ (here $\overline{f}$ denotes the class of $f \in I$ in I/I^2; I/I^2 has the $\overline{f}_j$ as an R-free basis), we get an isomorphism $\mathrm{Hom}_R(\bigwedge^h(I/I^2), R \otimes M)$ with $\mathrm{Hom}_R(R, R \otimes M)$ and, hence, with $R \otimes M$: this map $\lambda_{\underline{f}'}$, takes ϕ to $\phi(F)$. We claim that the map $\lambda_{\underline{f}}^{-1}\theta_{\underline{f}}\mathrm{Ext}^h_S(S/I, M)\mathrm{Hom}_R(\bigwedge^h(I/I^2), M)$ is actually independent of the choice of r generators for I. To see this, suppose that $\underline{g} = g_1, \ldots, g_h$ is some other set of generators. If $[f], [g]$ denote the $h \times 1$ column vectors whose entries are the f_i and g_i, respectively, then there is an $h \times h$ matrix $A = [a_{ij}]$ such that $A[f] = [g]$. Then $A\colon K_1(\underline{f}; S) \to K_1(\underline{g}; S)$ (both K_1's are simply S^h) gives a commutative diagram:

$$\begin{array}{ccc} K_1(\underline{f};\, S) & \longrightarrow & K_0(\underline{f}; S) \\ {\scriptstyle A}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathrm{id}} \\ K_1(\underline{g};\, S) & \longrightarrow & K_0(\underline{g}; S) \end{array}$$

Note that $[f], [g]$ respectively give the matrices of the maps $K_1 \to K_0$ in the respective Koszul complexes and that both K_0's are simply S. Also note $\mathrm{id} = \mathrm{id}_S$ is $\bigwedge^0 A$ and that A is $\bigwedge^1 = A$. It is then easy to check that one gets a map of complexes if one uses $\bigwedge^i A$ as the map $K_i(\underline{g}; R) \to K_i(\underline{f}; R)$ for every value of i. The map at the hth spot from S to S may therefore be identified with multiplication by $\det A$. Recalling that the arrows are reversed when we apply $\mathrm{Hom}_R(\_\,, M)$, we find that $\theta_{\underline{g}} = (\det A)\theta_{\underline{f}}$. Note that the image $\delta = \overline{\det A}$ of $\det A$ in R must be a unit (although $\det A$ need not be a unit of S). We can write $\theta_{\underline{g}} = \delta\theta_{\underline{f}}$.

The image of A mod I evidently gives a map from I/I^2 to itself carrying the generators $\overline{f}_i$ to the generators $\overline{g}_i$: it follows that $G = (\overline{\det A})F = \delta F$, where $G = \overline{g}_1 \wedge \cdots \wedge \overline{g}_h$. It is then immediate that $\lambda_{\underline{g}} = \delta\lambda_{\underline{f}}$. Thus,

$$(\lambda_{\underline{g}})^{-1}\theta_{\underline{g}} = (\delta\lambda_{\underline{f}})^{-1}(\delta\theta_{\underline{f}}) = \delta^{-1}\delta(\lambda_{\underline{f}}^{-1}\theta_{\underline{f}}) = \lambda_{\underline{f}}^{-1}\theta_{\underline{f}}$$

as claimed. (The statement about base change is an easy exercise.) $\square$

Proof of (16.3). Fix P in $\mathrm{Spec}\, R$ and let Q be its inverse image in S. We shall construct the isomorphism on a sufficiently small Zariski neighborhood of P. Since S_Q maps onto R_P and both are regular, the kernel is generated by a regular sequence (part of a minimal set of generators for QS_Q) in S_Q. Choose $f_1, \ldots, f_h \in S$ whose images in S_Q are a regular sequence generating the kernel: we shall construct the map using these f's, but we shall eventually show that it is independent of the choice of the f's. After localizing at a single element $g \in S - Q$, we can assume that the f's form a regular sequence generating the kernel of the map $S_g \to R_g$. We also assume that we have localized so much that Ω_S is free of rank n over S and that Ω_R is free of rank d over R We shall construct the isomorphism for the corresponding affines. For simplicity we change notation and omit the subscript $_g$. Then we have that $R = S/I$, where $I = (f_1, \ldots, f_h)S$.

Quite generally, given a surjection $S \twoheadrightarrow R$, there is a surjection $\Omega_S \twoheadrightarrow \Omega_R$ (we omit $/K$ from the subscripts) and, hence, $R \otimes_S \Omega_S \twoheadrightarrow \Omega_R$ whose kernel is generated by the images of the elements du for $u \in I$, where $I = \mathrm{Ker}\,(S \twoheadrightarrow R)$. The map from $I \to R \otimes_S \Omega_S$ sending u to du is S-linear ($d(su) = sdu + uds$, and the image of uds is 0 in $R \otimes_S \Omega_S$) and kills I^2. Thus, we have an exact sequence of R-modules:

$$(*) \qquad\qquad I/I^2 \to R \otimes_S \Omega_S \to \Omega_R \to 0\ .$$

From the fact that I is generated by a regular sequence of length h, it is easy to prove that I/I^2 is R-free of rank h. If we tensor with the fraction field of R it is clear that the sequence becomes exact: since I/I^2 has rank h, the corresponding vector space, after we tensor, has the same dimension as the kernel at the middle spot, and maps onto it. But this implies that the first map is injective, since I/I^2 is torsion-free over R. Thus, the squence (*) is an exact sequence of free modules. By Lemma (16.4) above, we have an induced isomorphism $\bigwedge^h(I/I^2)\otimes_R\bigwedge^d\Omega_R \cong \bigwedge^n(R\otimes_S\Omega_S)\cong R\otimes_S\bigwedge^n\Omega_S = R\otimes_S\Omega_S^n$. If N is finitely presented and locally free of rank one and M is any module we have a canonical isomorphism $M \cong \mathrm{Hom}_R(N, N\otimes_R M)$ that sends m to the map that sends v to $v\otimes m$ (to check that the map is an isomorphism we may localize, and then we have reduced to the case where $N = R$). Thus

$$\Omega_R^d \cong \mathrm{Hom}_R(\bigwedge^h(I/I^2), \bigwedge^h(I/I^2)\otimes_R\Omega_R^d) \cong \mathrm{Hom}_R(\bigwedge^h(I/I^2), R\otimes_S\Omega_S^n).$$

Notice that in setting up this isomorphism, while we had to pass a small Zariski neighborhood of P, we made no other choices, and the isomorphism is compatible with further localization. The result now follows from Lemma (16.5). $\square$

Cf. [AK], Chapter I.

(16.6) Theorem. *Let R be a finitely generated Cohen-Macaulay K-algebra, where K is a field, and suppose that the non-smooth locus in* $\mathrm{Spec}\,R$ *(over K) has codimension at least two (this condition corresponds to normality if the characteristic is zero of if K is perfect). Suppose that all components of* $\mathrm{Spec}\,R$ *have domension d. Then ω_R, calculated using a homomorphism of a polynomial ring onto R, is isomorphic with the double dual, into R, of $\Omega^d_{R/K}$.*

Proof. It suffices to consider one component of $\mathrm{Spec}\,R$, and so we may suppose that R is a domain, since the condition implies that the non-regular locus has codimension at least two: this implies, in the presence of the Cohen-Macualay property, that the depth of the ring on its defining ideal of the non-regular locus is least two, and so the ring is normal. Map a polynomial ring S of of dimension n onto R, and fix $\omega_R = \mathrm{Ext}_S^{n-d}(R,\Omega_S^n)$ (recalling that $\Omega_S^n \cong S$). Let U be the open subset of $\mathrm{Spec}\,R$ where R is K-smooth, and let Z be its complement, which has codimension at least two. The sheaf $(\omega_R\tilde{\ })|_U$ is then isomorphic with the sheaf Ω_U^d (we omit $/K$ from the subscript), since for each open affine in a cover there is a canonical isomorphism, and these commute with localization. Thus, given an element of Ω_R^d we may restrict it to U. It then gives a section of ω_R on U. Since the defining ideal I of Z has depth at least two, and since ω_R is a faithful Cohen-Macaulay module, we have that $H_I^0(\omega_R) = H_I^1(\omega_R) = 0$, and so this section extends uniquely to a section of $\omega_R{}^{\sim}$, i.e., to an element of ω_R. This defines a map $\Omega_R^d \to \omega_R$. Moreover, if we think of these modules as sheaves, the map is an isomorphism on U (where both modules are locally free of rank one). There is an induced map of double duals. But ω_R is S_2 and the ring is normal, whence ω_R is its own double dual. If * denotes $\mathrm{Hom}_R(\_\,, R)$, we obtain a map $(\Omega_R^d)^{**} \to \omega_R$ which is an isomorphism when the corresponding sheaves are restricted to U. In particular,, it becomes an isomorphism if we tensor with the fraction field of the ring. Since these are torsion-free modules of rank one there cannot be any kernel, and the only issue is whether this map is onto. Because it becomes an isomorphism on U, we see that the cokernel cannot be supported at any height one prime. Thus, if $w \in \omega_R$ is not in $\Omega = (\Omega_R^d)^{**}$ it is multiplied into Ω by an ideal of height at least two. But then, since Ω, itself a dual, is reflexive, w must be in Ω, a contradiction. $\square$

This gives a valuable computational tool, even in the local case, whose proof depends heavily on "global" ideas.

(16.7) Exercise. Let R be the quotient of a polynomail ring S over a field in the x_{ij} of an $n \times n+1$ matrix of indeterminates by the ideal generated by the size n minors. Show either using a resolution of R over S or by the method of differentials that ω_R may be identified with the ideal of size $n-1$ minors of the first $n-1$ columns of the matrix (or any other $n-1$ columns). What is the type of R?

If we kill the $t \times t$ minors of an $r \times s$ matrix of indeterminates with $2 \le t \le r \le s$ then it turns out that the canonical module is given by the $s-r$ power of the ideal generated by the $t-1$ size minors of any $t-1$ columns. The ideal generated by the $t-1$ size minors of any $t-1$ columns is a height one prime in this ring, and this prime turns out to represent a generator of the divisor class group, which can be shown to be $\mathbb{Z}$ is in this case. Note that the ideals coming from different choices of $t-1$ columns are distinct ideals all of which are isomorphic as modules. A detailed treatment is given in [BrV], Ch. 8.

17. CONNECTIONS OF LOCAL COHOMOLOGY WITH PROJECTIVE GEOMEGRY

(17.1) Discussion: a review of some facts about projective space. Let K be a field. Let $R = K[x_0, \dots, x_n]$ be a polynomial ring. Let $[M]_i$ denote the i th graded piece of the graded module M. Then $[R_{x_i}]_0$ is readily verified to be the polynomial ring over K in the n variables x_j/x_i, $0 \le j \le n$, $j \ne i$, and the $n+1$ affinte schemes $U_i = \operatorname{Spec}[R_{x_i}]_0$ together form an open cover of projective n space, $\mathbb{P}^n_K$, over K. We often omit the subscript K. These affines fit together so that $(U_i)_{x_j/x_i} = (U_j)_{x_i/x_j}$. In fact, for every form F, $\operatorname{Spec}[R_F]_0$ may be identified with an open set in $\mathbb{P}^n$. The elements of R are not functions on $\mathbb{P}^n$, but it does make sense to refer to the set $V(F)$ where F vanishes when F is homogeneous (and, likewise, one may refer to $V(I)$ when I is a homogeneous ideal of R).

A sheaf of modules $\mathcal{F}$ on a scheme $(X, \mathcal{O}_X)$ is *quasicoherent* if for every open affine $U \cong \operatorname{Spec} T$, $\mathcal{F}_{|U}$ agrees with $M^{\sim}$ (with $^{\sim}$ used in the affine sense) for some T-module M. It is enough to know this for some cover by open affines. A quasicoherent sheaf is *coherent* if, roughly speaking, it is locally finitely presented. For a Noetherian scheme this simply means that it is quasicoherent and that the module of sections on any open affine is finitely generated.

There is also a projective version of $^{\sim}$: to distinguish it from the affine version, we adopt the nonstandard convention of writing it before the module. Let M be a finitely generated $\mathbb{Z}$-graded R-module. (Note: each graded piece will then be a finite dimensional vector space. There may be nonzero negatively graded compoents, but the finite generation of M guarantees there there will only be finitely many.) Then we write $^{\sim}M$ for the sheaf whose sections on the open set where the form F does not vanish are $[M_F]_0$: this is a finitely generated module over $[R_F]_0$. With this notation, $^{\sim}R$ is the structure sheaf for $\mathbb{P}^n_K$.

Somewhat surprisingly, every coherent sheaf on $\mathbb{P}^n_K$ has the form $^{\sim}M$ from some finitely generated $\mathbb{Z}$-graded R-module M. In fact, the entire category of coherent sheaves on $\mathbb{P}^n$ can be described by studying a certain category whose objects are the finitely generated $\mathbb{Z}$-graded, R-modules. The maps are different from the usual ones, however: we shall describe them shortly.

Before doing so, we want to point out that which sheaf one gets depends on the grading of M, not just the module structure. Isomorphic modules with different gradings will, in general, give non-isomorphic sheaves. The most important examples are the so-called "twists." If M is a finitely generated $\mathbb{Z}$-graded R-module we let $M(t)$ be the same module graded so that $[M(t)]_i = M_{t+i}$.

We let $\mathcal{O}_X(t) = {}^{\sim}R(t)$, which is a locally free sheaf of rank one. Now the tensor product of two coherent sheaves is a coherent sheaf (on any affine, the sections are found by simply tensoring the sections of the respective sheaves over the ring of sections on that affine). It turns out that if M, N are finitely generated $\mathbb{Z}$-graded R-modules that ${}^{\sim}M \otimes_{\mathcal{O}_X} {}^{\sim}N \cong {}^{\sim}(M \otimes_R N)$, where $M \otimes_R N$ is graded so that if $u \in M$ and $v \in N$ are homogeneous then $\deg(u \otimes n) = \deg u + \deg v$. Thus,

$$(M \otimes_R N)_h = \Sigma_{i+j=h} \operatorname{Im}(M_i \otimes N_j).$$

We can define the t th *twist*, $\mathcal{F}(t)$, of any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, where t is any integer, by letting

$$\mathcal{F}(T) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(t).$$

Since the graded tensor product $M \otimes_R R(t) \cong M(t)$, if one has that $\mathcal{F} \cong {}^{\sim}M$ then $\mathcal{F}(t) \cong {}^{\sim}M(t)$.

We shall also write $M_{|\geq j}$ for the graded submodule $\bigoplus_{i \geq j} [M]_i$ of the finitely generated graded $\mathbb{Z}$-module M. Then $M/(M_{|\geq j}) \cong \bigoplus_{i<j} [M]_i$ has finite length, and so is called by a power of every x_j. Now, whenever M has finite length we have that ${}^{\sim} = 0$. Moreover, it is easy to see that ${}^{\sim}(M_{|\geq j}) \cong {}^{\sim}M$ for all j. In order to compare the category of coherent sheaves on $\mathbb{P}^n_K$ with the category of finitely generated $\mathbb{Z}$-graded R-modules one has to take account of this in a suitable manner.

For this reason, we define a *morphism* between two finitely generated $\mathbb{Z}$-graded modules M, N to be a degree 0 graded R-linear map from $M_{|\geq j}$ to $N_{|\geq j}$ for some sufficiently large j, modulo equivalence, where two maps are equivalent if they agree on $[M_i]$ for all sufficiently large i. Thus, Mor (M, N) may be viewed as $\varinjlim_j \operatorname{Hom}_{\mathrm{gr}}(M_{|\geq j}, N_{|\geq j})$. where $\operatorname{Hom}_{\mathrm{gr}}$ indicates degree preserving R-linear maps. With this definition, the inclusion map of $M_{|\geq j} \subseteq M$ is an isomorphism.

One of Serre's main results in this direction (cf. [Se]) is that, with this notion of morphism, the assignment of ${}^{\sim}M$ to M yields an equivalence of the category of $\mathbb{Z}$-graded R-modules with the category of coherent sheaves on $\mathbb{P}^n$. One recovers the graded pieces

$[M]_t$ of the modules M for sufficiently large t from the sheaf $\mathcal{F} = {}^\sim M$ as the global sections of $\mathcal{F}(t)$. (One only expects, and one only needs, to be able to recover $[M]_t$ for sufficiently large t.)

(17.2) Discussion: the calculation of cohomology. Now suppose that $\mathcal{F} = {}^\sim M$ is a coherent sheaf on $\mathbb{P}^n_K$, where M is a finitely generated $\mathbb{Z}$-module. We want to understand, and calculate, the cohomology of the sheaf $\mathcal{F}$ from a cohomology theory for R-modules. We can do so by using the Cech complex with respect to the affine open cover U_i of $X = \mathbb{P}^n$, where $U_i = \operatorname{Spec}[R_{x_i}]_0$. This leads to a Cech complex:

$$0 \to \bigoplus_i \mathcal{F}(U_i) \to \bigoplus_{i<j} \mathcal{F}(U_i \cap U_j) \to \cdots \mathcal{F}(U_1 \cap \cdots \cap U_n) \to 0$$

But $\mathcal{F}(U_{i_1} \cap \cdots \cap U_{i_k})$ consists of the sections of $\mathcal{F}$ on the open set where $y = x_{i_1} \cdots x_{i_k}$ does not vanish, and these sections may be identified with 0th graded piece of M_y. Thus, the Cech complex above is simply the 0 th graded piece of the complex

$$0 \to \bigoplus M_{x_i} \to \bigoplus_{i<j} M_{x_i x_j} \to \cdots \to M_{x_0 \cdots x_n} \to 0$$

which is almost the same as the complex $K^\bullet(\underline{x}^\infty;\, M)$: one only has to drop the first the first (0 th) term of the complex $K^\bullet(\underline{x}^\infty;\, M)$ and shift the numbering by one.

Moreover, if one takes the t th graded piece of this complex one obtains the Cech complex for $\mathcal{F}(t)$ with respect to the same open cover. Now there is an obvious map of $[M_0]$ into the global sections of ${}^\sim M$ (since $u \in [M]_0$ represents an element of every $[M_x]_0$: this corresponds to the map $M \to \bigoplus_i M_{x_i}$) and, likewise, of $[M]_t$ into the global sections of ${}^\sim M(t)$ for every t.

We should also note that if M is a graded module over a graded ring and I is a homogeneous ideal then $H^i_I(M)$ is graded: one can see this by choosing homogenous generators f_j for I and using the fact $K^\bullet(\underline{f}^\infty; M)$ is graded.

(17.3) Theorem. . *Let M be a finitely generated $\mathbb{Z}$-graded module over the polynomial ring $R = K[x_1, \ldots, x_n]$ and let $\mathcal{F} = {}^\sim M$ be the corresponding coherent sheaf on $X = \mathbb{P}^n_K$. Let m be the maximal ideal of R generated by the x's. Then:*

(a) *For $i \geq 1$, $H^i(X, \mathcal{F}(t)) \cong [H^{i+1}_m(M)]_t$, so that $H^{i+1}_m(M) \cong \bigoplus_t H^i(X, (t))$.*

(b) *The map $\Theta\colon M \to \bigoplus_{t\in\mathbb{Z}} H^0(X, \mathcal{F}(t))$ sending M_t into $H^0(X, \mathcal{F}(t))$ in the obvious way is injective if and only if $\operatorname{depth}_m M \geq 1$ and bijective if and only if $\operatorname{depth}_m M \geq 2$.*

(c) *In consequence,* $\operatorname{depth} M \geq d \geq 3$ *if and only if* Θ *is bijective and* $H^i(X, \mathcal{F}(t)) = 0$ *for all* $t \in \mathbb{Z}$ *and all* i *with* $1 \leq i \leq d-1$.

Proof. We have already seen that $\bigoplus_t H^i(X, \mathcal{F}(t))$ may be identified with the cohomology of the complex $K^\bullet(\underline{x}^\infty; M)$ truncated at the beginning. This implies the isomorphism in (a) at once and also yields an exact sequence:

$$0 \to H^0_m(M) \to M \xrightarrow{\Theta} \bigoplus_t H^0(X, \mathcal{F}(t)) \to H^1_m(M) \to 0$$

Parts (b) and (c) now follow from the fact that the first non-vanishing $H^i_m(M)$ occurs for $i = \operatorname{depth}_m M$. $\square$

(17.4) Exercise. Let $X = \mathbb{P}^n_K$. Then

(a) $H^0(X, \mathcal{O}_X(t)) \cong [K[x_0, \dots, x_n]]_t$, the vector space of monomials of degree t in the x_j.

(b) $H^i(X, \mathcal{O}_X(t)) = 0$ if $i < n$.

(c) $H^n(X, \mathcal{O}_X(t))$ is isomorphic with the vector space of monomials of degree $-t$ in $y_0 \cdots y_n K[y_0, \dots, y_n]$ (if one gives the y_i degree 1). Thus, it vanishes if $t > -n-1$ and has dimension one when $t = -n-1$. (One may think of y_i as x_i^{-1}, and identify $x_1^{-1} \cdots x_n^{-1}\, K[x_1^{-1}, \dots, x_n^{-1}]$ with $H^{n+1}_{(\underline{x})R}(R)$.)

(17.5) Discussion. If Z is a projective scheme (i.e. a closed subscheme of some $\mathbb{P}^n_K$) and we fix the embedding of Z in $\mathbb{P}^n = X$, the pullback of $\mathcal{O}_X(t)$ to Z is a locally free sheaf of rank one on Z: we denote it $\mathcal{O}_Z(t)$. (The sheaf $\mathcal{O}_Z(1)$ is called a *very ample line bundle on* Z). Given a coherent sheaf $\mathcal{F}$ on Z we may then define $\mathcal{F}(t) = \mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{O}_Z(t)$.

Note that any coherent sheaf $\mathcal{F}$ on Z may be "extended by 0" to all of $\mathbb{P}^n_K$: the sheaf obtained is technically the direct image $\iota_*\mathcal{F}$ with respect to the inclusion map $\iota : \mathbb{Z} \to \mathbb{P}^n_K$. The sections of $\iota_*\mathcal{F}$ on an affine U are those of $\mathcal{F}$ on $U \cap Z$. The new sheaf, restricted to $\mathbb{P}^n - Z$, is zero. This is the global equivalent of viewing an (R/I)-module as an R-module by restriction of scalars. Note than an open affine cover of $\mathbb{P}^n_K$, intersected with Z, gives an open affine cover of Z (because a closed subscheme of an affine scheme is affine). The Cech complexes obtained from $\iota_*\mathcal{F}$ with respect to an open cover of $\mathbb{P}^n$ and $\mathcal{F}$ with respect to its intersection with Z are identical. Thus, $H^i(\mathbb{P}^n_K, \iota_*\mathcal{F}) \cong H^i(Z, \mathcal{F})$. The results of (17.4) above and (17.6) below are theorems of Serre [Se] that are easy corollaries of the local cohomology theory developed here.

(17.6) Theorem. *For any coherent sheaf* $\mathcal{F}$ *on the projection scheme* Z *over* K*:*

(a) $H^i(Z, \mathcal{F})$ *is a finite-dimensional vector space over* K.

(b) *If* $i \geq 1$ *then* $H^i(Z, \mathcal{F}(t)) = 0$ *if* $i > \dim Z$ *(or if* $i > \dim \operatorname{Supp} \mathcal{F}$*, where* $\operatorname{Supp} \mathcal{F}$ *is the support of* $\mathcal{F}$*, i.e.,* $\{z \in Z : \mathcal{F}_z \neq 0\}$*).*

Proof. By the remarks above, by considering $\iota_* \mathcal{F}$, we can reduce to the case where $\mathcal{F}$ is a sheaf on $X = \mathbb{P}^n$ itself (in part (c) we prove the version stated in terms of the support of $\mathcal{F}$).

For (b) note that we know that $H = \bigoplus_t H^i(X, \mathcal{F}(t))$ is a graded module over $R = K[x_0, \dots, X_n]$ for $i \geq 1$. The decreasing chain of submodules $H_{|\geq t}$ must therefore be eventually stable, which can only happen if all graded pieces are eventually zero. Moreover, $H^i(X, \mathcal{F}(t)) \cong (H_{|\geq t})/(H_{|\geq t+1})$ is a vector space with DCC, and therefore finite-dimensional for $i \geq 1$. It remains to prove part (a) when $i = 0$. One verifies it directly for the sheaves $\mathcal{O}_X(t)$ (cf. (17.4) above).

Suppose that $\mathcal{F} = {}^{\sim}M$ and that M has homogeneous generators $v_1, \dots, v_r$ of degrees $d_1, \dots, d_r$. Then there is a degree preserving map of the graded module $\bigoplus_j R(-d_j)$ onto M that sends the element 1 in $R(-d_j)$ (in which it has degree d_j) to v_j. This yields a surjection of sheaves $\bigoplus_j \mathcal{O}_X(-d_j) \twoheadrightarrow \mathcal{F}$ and, hence, a short exact sequence of sheaves:

$$0 \to \mathcal{G} \to \bigoplus_j \mathcal{O}_X(-d_j) \to \mathcal{F} \to 0$$

where $\mathcal{G}$ is simply the kernel of the map. The long exact sequence for cohomology yields:

$$\cdots \to \bigoplus_j H^0(X, \mathcal{O}_X(-d_j)) \to H^0(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to \cdots$$

Since we already know that the terms other than the middle term are finite-dimensional vector spaces, the result follows.

Part (c) can be proved as follows: suppose that $\mathcal{F} = {}^{\sim}M$, where M is graded. One shows that $\dim \operatorname{Supp} \mathcal{F} = \dim M - 1$. This implies that $H^{j+1}_m(M) = 0$ if $j > \dim \operatorname{Supp} \mathcal{F}$, for then $j + 1 > \dim M$, and the result now follows. (However, one can also prove part (c) by observing that when the support has dimension d it can be covered by $d+1$ open affinites.) $\square$

Note that the proof that the sections of a coherent sheaf on a projective scheme are a finite-dimensional vector space requires first establishing the result for higher cohomology. Serre's proof uses reverse induction on i, establishing the result for the highest cohomology first and working back down to the sections.

(17.7) Serre-Grothendieck duality: discussion. Let Z be a projective scheme over a field K and suppose that Z is *Cohen-Macaulay*, i.e., all its local rings are Cohen-Macaulay. (A stronger condition would be for it to have a homogenous coordinate ring that is Cohen-Macaulay.) Suppose that all components have the same dimension d. We can define a *canonical sheaf* (or *dualizing sheaf*) on Z, ω_Z as follows: embed Z in a projective space $X = \mathbb{P}^n_K$, and then let $\omega_Z \cong \mathcal{E}\mathrm{xt}^{n-d}_{\mathcal{O}_X}(\mathcal{O}_Z,\, \Omega^n_X)$. The sections on any open affine will give a canonical module for the ring of sections corresponding to that affine. Notice that if Z is smooth over K, our earlier results imply that ω_Z is canonically isomorphic with Ω^d_Z. The sheaf ω_Z is independent of the embedding of Z into a projective space: different choices of embeddings yield isomorphic sheaves.

Serre-Grothendieck duality asserts that for every coherent sheaf $\mathcal{F}$ on Z,

$$H^i(Z, \mathcal{F}) \cong \mathrm{Ext}^{d-i}_{\mathcal{O}_Z}(\mathcal{F}, \omega_Z)^*,$$

where $\_^*$ denotes $\mathrm{Hom}_K(\_\,,\, K)$. In fact, $\omega_X = \Omega^n_X$ is easily computed to be $\mathcal{O}_X(-n-1)$. Once can choose an isomorphism of $H^n(X, \Omega^n_X) \cong K$. This induces a map $H^d(Z, \omega_Z) \to K$ which, when composed with the Yoneda pairing, gives a pairing

$$H^i(Z, \mathcal{F}) \times \mathrm{Ext}^{d-i}_{\mathcal{O}Z}(\mathcal{F}, \omega_Z) \to H^d(Z, \omega_Z) \to K$$

which is *nonsingular*, i.e it induces an isomorphism

$$H^i(Z, \mathcal{F}) \cong \mathrm{Ext}^{d-i}_{\mathcal{O}_Z}(\mathcal{F}, \omega_Z)^*.$$

These isomorphisms are much like graded pieces of a local duality isomorphism. The case where $Z = \mathbb{P}^n$ is called *Serre duality*.

When $\mathcal{F}$ is a locally free sheaf (or *vector bundle*) $Ext^{d-i}_{\mathcal{O}_Z}(\mathcal{F}, \omega_Z)$ may be identified with $H^{d-i}(\mathcal{H}\mathrm{om}_{\mathcal{O}_Z}(\mathcal{F}, \omega_Z))$ and $\mathcal{H}\mathrm{om}_{\mathcal{O}_Z}(\mathcal{F}, \omega_Z)$ may be identified with $\mathcal{F}^\vee \otimes_{\mathcal{O}_Z} \omega_Z$ where $\mathcal{F}^\vee = \mathcal{H}\mathrm{om}_{\mathcal{O}_Z}(\mathcal{F}, \omega_Z)$. When Z is smooth, $\omega \cong \Omega^d_Z$, and we have that

$$H^i(Z, \mathcal{F}) \cong H^{d-i}(Z, \mathcal{F}^\vee \otimes_{\mathcal{O}_Z} \omega_Z \Omega^d_Z)^*.$$

In particular, if Z is a smooth projective curve, then for a line bundle $\mathcal{F}$ (locally free sheaf of rank one) we have $H^1(Z, \mathcal{F}) \cong H^0(\mathcal{F}^\vee \otimes_{\mathcal{O}_Z} \omega_Z \Omega_Z)^*$. Here, Ω_Z is itself a line bundle. This is "Roch's" part of the Riemann-Roch theorem. The rest of the theorem says that if $\chi(\mathcal{F}) = \dim_K H^0(Z, \mathcal{F}) - \dim_K H^1(Z, \mathcal{F})$ (the higher cohomology all vanishes) then

$\chi(\mathcal{F}) - \chi(\mathcal{O}_Z) = \deg \mathcal{F}$ (we will give the definition shortly). Here, $\chi(\mathcal{O}_Z) = 1 - g$ where $g = \dim_K H^1(Z, \mathcal{O}_Z)$ the *genus* of the curve Z.

(17.8) Discussion. We continue briefly our remarks on the Riemann-Roch theorem for smooth projective curves.

Let $\mathcal{F}$ be a line bundle (locally free sheaf of rank one) on a smooth, connected projective curve Z over an algebraically closed field K. On a sufficiently small nonempty open affine U in Z (but keep in mind that U will contain all but finitely many points of Z) $\mathcal{F}$ will have a section, σ. Think of two such sections as equivalent if they agree when restricted to some smaller nonempty open affine. These equivalence classes are called *meromorphic sections* of $\mathcal{F}$. The meromorphic sections $K(Z)$ of $\mathcal{O}_Z$ may be identified with the fraction field of any of the rings $\mathcal{O}_Z(U)$ for U a nonempty open affine in Z. It is easy to see that the meromorphic sections of $\mathcal{F}$ form a one-dimensional vector space over $K(Z)$, but there is no canonical generator.

By a divisor on Z we mean a formal $\mathbb{Z}$-linear combination of points of Z, i.e., an element of the free abelian group whose generators are the points of Z. Given a nonzero element $f \in K(Z)$ we can form a divisor $\operatorname{div} f$ such that for $z \in Z$ the coefficient of z in $\operatorname{div} f$ is the order of f thought of as an element of the fraction field of $\mathcal{O}_{Z,z}$: this ring is a DVR. This is the *divisor of zeros and poles of* f. Given a divisor D we can form a line bundle $\mathcal{O}_Z(D)$ whose sections on U are those meromorphic functions f such that $\operatorname{div} f + D$ has nonnegative coefficients at all points of U. This bundle is a subsheaf of the constant sheaf on Z with coefficients in $K(Z)$. If g is a nonzero meromorphic function then replacing D by $D + \operatorname{div} g$ replaces $\mathcal{O}_Z(D)$ by the isomorphic bundle $g\mathcal{O}_Z(D)$.

On the other hand, given a line bundle $\mathcal{F}$ and a meromorphic section σ we can form a divisor D such that the coefficient of z is the order (in the fraction field of $\mathcal{O}_{Z,z}$) of the element $a \in K(Z)$ such that $a^{-1}\sigma$ generates $\mathcal{F}_z$ as an $\mathcal{O}_{Z,z}$-module. Each section of $\mathcal{F}$ on U can be written uniquely in the form $b\sigma$ where b is meromorphic, and the condition on a meromorphic function b for $b\sigma$ to be a section is that for all $z \in U$, $\operatorname{ord}_z b \geq \operatorname{ord}_z a^{-1}$, where $a^{-1}\sigma$ is the generator of $\mathcal{F}_z$. This says that b is a section of $\mathcal{O}_Z(D)$, and we see that $\mathcal{F} \cong \mathcal{O}_Z(D)$.

The upshot of this discussion is that there is a bijection between isomorphism classes of line bundles on Z and equivalence classes of divisors on Z, where two divisors are equivalent if they differ by the divisor of a nonzero meromorphic function.

The *degree* of a divisor is the sum of its coefficients. It is not hard to prove that the degree of the divisor of a meromorphic function is zero. Thus, we may define the degree of a line bundle to be the degree of any divisor that represents it.

Now, if D is any divisor and z is a point of Z, it is easy to see that there is an exact sequence of sheaves:

$$0 \to \mathcal{O}_Z(D) \to \mathcal{O}_Z(D+z) \to \mathcal{S}(z) \to 0$$

where the first nonzero map is an inclusion map. Moreover, $\mathcal{S}(z)$ is a sheaf supported only at z, and its sections on any neighborhood of z consist of a copy of K. It follows that $\chi(\mathcal{O}_Z(D+z)) = \chi(\mathcal{O}_Z(D)) + \chi(\mathcal{S}(z))$. But $H^0(Z, \mathcal{S}(z)) \cong K$ while the higher cohomology vanishes, since $\mathcal{S}(z)$ is supported at only one point. This shows that $\chi(\mathcal{S}(z)) = 1$. Thus, adding one point to a divisor increases χ of the corredsponding bundle by one, and it follows that subtracting a point decreases it by one. Starting with the divisor 0 (corresponding to $\mathcal{O}_Z$) we can add and subtract points one at a time until we obtain a given divisor D. It follows that

$$\chi(\mathcal{O}_Z(D)) = \chi(0_Z) + \deg D = 1 - g + \deg D.$$

This completes the proof of the Riemann-Roch theorem, given duality.

18. COHOMLOGICAL DIMENSION AND GENERATION OF IDEALS UP TO RADICALS

The following result is due to Peskine and Szpiro in their joint thesis [PS]:

(18.1) Theorem. *Let R be a regular domain of dimension n* of positive prime characteristic p*, and let I be an ideal of R of height h such that R/I is Cohen-Macaulay. Then $H^i_I(R) = 0$ for $i > h$.)*

Proof. The issue is local on R and we assume that R is local. The key point is that the application of Frobenius preserves the acyclicity of a finite free resolution of R/I over R, since $F^e : R \to R$ is flat when R is regular. It follows that $\mathrm{pd}_R R/I^{[q]} = \mathrm{pd}_R R/I$ for all $q = p^e$. Thus, each $R/I^{[q]}$ has the same depth as R/I, and so all of the modules $R/I^{[q]}$ are Cohen-Macaulay of projective dimension h. But we may then calculate $H^j_I(R) = \varinjlim_q \mathrm{Ext}^j_R(R/I^{[q]},\, R)$, and when $R/I^{[q]}$ is Cohen-Macaulay, there is a unique non-vanishing Ext, occuring in this case for $j = h$. The result is now immediate. □

This result is quite false in characteristic zero!

We shall now focus on giving counterexamples to this theorem in characteristic zero by studying the case where R is a polynomial ring in $(n+1)n$ indeterminates x_{ij} over a field of characteristic zero K, and I is the ideal generated by the size n minors Δ_j of the matrix $X = (x_{ij})$, where Δ_j is the determinant of the $n \times n$ matrix obtained by deleting the j th column. It is not difficult to show that one has a free resolution

$$0 \to R^n \xrightarrow{X} R^{n+1} \xrightarrow{\Delta} R \to R/I \to 0$$

where Δ is the $1 \times (n+1)$ matix whose j th entry is $(-1)^{j-1}\Delta_j$. This shows that $depth_m R/I = (n+1)n - 2$. Since $\mathrm{ht}\, I \geq 2$, clearly, $\dim R/I \leq (n+1)n - 2$. Thus, we must have equality, and R/I is Cohen-Macaulay (this is valid in all characteristics). Moreover, I has height 2. In characteristic p, we then have $H^j_I(R) = 0$ for $j \geq 3$ by the result of Peskine-Szpiro.

We shall prove, however, in two quite differnt ways, that, in equal characteristic zero, $H^{n+1}_I(R) \neq 0$. This shows that, in equal characteristic zero, I requires $n+1$ generators up to radicals! (The same is true in characteristic p, but requires local étale cohomology for the proof.) This appears to be just as difficult when $n = 2$ as in the general case.

The key point in the first proof is that, in characteristic zero, the K-homomorphism $A = K[\Delta_1, \ldots, \Delta_{n+1}] \subseteq R$ splits over A: A is a direct summand of R as an A-module. Moreover, A is a polynomial ring in the Δ's. Let Q be the ideal of A generated by the Δ's. Assuming the splitting, we get an injection of $H^{n+1}_Q(A) \to H^{n+1}_Q(R) = H^{n+1}_{QR}(R) = H^{n+1}_I(R)$, and we know that $H^{n+1}_Q(A)$ is not zero, since Q is a maximal ideal of A of height $n+1$. This shows that $H^{n+1}_I(R) \neq 0$.

One has the splitting because $G = SL(n, K)$ is a reductive linear algebraic group, and is we let G act linearly on R by letting $\alpha \in G$ send the entries of the matrix X to the entries of the matrix αX, A is the fixed ring of the action one then obtains on R. These results may be found in H. Weyl's book [W].

The second proof that $H^{n+1}_I(R)$ is not zero in equal characteristic zero is by topological methods that are quite instructive. The idea is to relate the vanishing of local cohomology in the algebraic sense to the vanishing of singular cohomology in a purely topological sense: the transition is made by studying the cohomology of sheaves of differential forms on suitable varieties. There is a lot of machinery underlying this argument (e.g., algebraic DeRham cohomology, whose definition requires hypercohomology, spectral sequences, etc.). I will sketch the argument, giving the definitions, but omitting one key proof (the theorem of Grothendieck that the algebraic DeRham cohomology of a smooth variety of finite type over $\mathbb{C}$ is the same as the singular cohomology).

Let X denote a smooth scheme of dimension d of finite type over a field K. The universal differentials from the rings of sections of $\mathcal{O}_X$ on open affines to the corresponding modules of sections of Ω^1_X (the subscript $_{/K}$ is omitted throughout here) yield a K-linear map $\mathcal{O}_X \to \Omega^1_X$, and this extends to give a complex $\Omega^\bullet_X$ of D-linear maps:

$$0 \to 0_X \to \Omega^1_x \to \Omega^2_X \to \cdots \to \Omega^n_X \to 0$$

called the *algebraic DeRham complex*. An analogous construction on a C^∞ manifold using $\mathbb{R}$-valued differntial forms yields an exact sequence of sheaves: when one takes global sections, the cohomology gives the singular chomology of X. Here, we do not get an exact sequence of sheaves (i.e., even locally, a "closed" differential form is not necessarily "exact" in the algebraic category). Instead of taking ordinary cohomology, we take *hypercohomology*: Briefly, we write down an injective resolution of the complex — this yields a double complex of injective sheaves in which the j th column (which begins with Ω^j_X) is an injective resolution of Ω^j_X, while each row is a complex of injective modules. Moreover, this can be set up (and is required to be set up) so that if one takes the cohomology of all the horizontal rows, each at the j th spot, together, as the row varies one gets an injective resolution of the cohomology of $\Omega^\bullet_X$ at the j th spot. Consider the double complex of injectives: from it one forms a total complex. Now take global sections and calculate the cohomology. The result is called the *hypercohomology* of the complex $\Omega^\bullet_X$, and we denote it $\mathcal{H}^\bullet(X, \Omega^\bullet_X)$.

We now define the algebraic DeRham cohomology $H^i_{\mathrm{DR}}(X)$ by the formula $H_DR^i(X) \cong \mathcal{H}^i(X, \Omega^\bullet_X)$. This may seem cumbersome, but it gives the right answer: a theorem of Grothendieck asserts that when X is a smooth variety of finite type over $\mathbb{C}$, the field of complex numbers, then $H^i_{\mathrm{DR}}(X) \cong H^i(X^h; \mathbb{C})$, where X^h is the underlying (Hausdorff) topological space of X in the usual topology (so that it is a real $2d$-manifold) and H^i denotes singular cohomology.

There is a spectral sequence for hypercohomology: it is simply one of the spectral sequences asociated with the double complex utilized in the definition, and in this instance it yields a spectral sequence $H^q(X, \Omega^P_X) \Rightarrow H^n_{DR}(X)$. This means that there is a complex in which the n th term is $\bigoplus_{p+q=n} H^q(X, \Omega^P_X)$ with the following propety: one can take its cohomology, get a new complex, take the cohomology again, get a new complex, etc., continuing in this way, until, at any given spot, one has, eventually, as the "stable" answer, an associated graded module of $H^n_{\mathrm{DR}}(X)$.) This may seem like weak information, but it suffices to deduce, for example, that if $H^q(X, \Omega^P_X)$ vanishes whenever $p + q = n$, then $H^n_{\mathrm{DR}}(x) = 0$.

We refer to [Ha2], Ch III, §7, for more information. Note that we have at once:

(18.2) Proposition. *Let X be a smooth variety of finite type over the complex numbers $\mathbb{C}$ of dimension d. If $H^i(X, \mathcal{F}) = 0$ for all coherent sheaves $\mathcal{F}$ on X when $i > r$, then $H^j(X; \mathbb{C}) = 0$ for $j > d + r$.*

Proof. $H^j(X; \mathbb{C}) \cong H^j_{DR}(X)$, and so it suffices to show that $H^q(x, \Omega^p) = 0$ for $q + p \geq j$. But either $p > d$ (in which case $\Omega^p = 0$) or $q > r$, which forces $H^q(X, \Omega^p)$ to vanish. $\square$

(18.3) A topological proof that $H^{n+1}_I(R) \neq 0$. We again consider an $n+1$ by n matrix of indeterminates. It is not hard to deduce the case of an arbitrary field K of characteristic 0 from the case where $K = \mathbb{C}$: henceforth we assume that $K = \mathbb{C}$. Let I be the ideal genrated by the size n minors of the matrix in $R = \mathbb{C}[X_{ij}]$. If $H^i_I(R) = 0$ for $i = n + 1$ then it is zero for all $i \geq n + 1$. It then follows that $H^i(X, \mathcal{F}) = 0$ for $i \geq n$ for every quasichoherent sheaf $\mathcal{F}$ on the quasiffine scheme $X = \operatorname{Spec} R - V(I)$, since this cohomology agrees with the local cohomology $H^{i+1}_I(M)$ for some R-module M. It now follows from the proposition that if (by slight abuse of notation) X is the manifold $\mathbb{C}^{(n+1)n} - V(I)$ consisting of all $n + 1$ by n matrices of rank n, then $H^{(n+1)n+n}(X; \mathbb{C}) = H^{(n+2)n}(X; \mathbb{C}) = 0$.

We can now complete the argument by purely topological methods: the point is that X is homotopic to a compact orientable manifold of real dimension $(n + 2)n$, whose highest cohomology can therefore not be zero.

The idea of the proof of this homotopy is to adjust, continuously, the length of the first column of the matrix until it has length one: then to change the second column by subtracting off a multiple of the first (the multiplier varies continuously) until it is orthogonal to the first, then to change the length of the second column until it is one, etc. This idea shows that C is homotopic to the space $Y = Y(n)$ of $n + 1$ by n matrices such that each column is a unit vector and the columns are mutually orthogonal. Y is a compact manifold. The first column varies in a sphere of dimension $2n + 1$. For a given first column the second column varies in a sphere of dimension $2n - 1$, and so forth. If all columns but the last are held fixed, the last column varies in a 3-sphere. Thus, the dimension of Y is $(2n + 1) + (2n - 1) + \cdots + 3 = (n + 2)n$. The projection map from $Y(n)$ to S^{2n+1} which takes each matrix to its first column makes Y a bundle over S^{2n+1}.

Since S^{2n+1} is not only orientable but simply connected, and since, inductively, each fiber (which may be thought of as a $Y(n-1)$) is orientable, the total space Y is orientable. $\square$

19. THE LOCAL HARTSHORNE-LICHTENBAUM VANISHING THEOREM

Part (a) of the theorem below is a rather simple statement about cohomological dimension in complete local domains. Part (b) is a somewhat more technical elobaoration that gives a best possible result along these lines. The proof of part (b) reduces very quickly to establishing part (a).

(19.1) Theorem (the local Harshorne-Lichtenbaum vanishing theorem). *Let (R, M, K) be a local ring of dimension d and I a proper ideal of R.*

(a) *If R is a complete local domain and I is not primary to m then $H^d_I(R) = 0$ (i.e. the cohomological dimension of the pair (R, I) is at most $d-1$) ($\Rightarrow H^d_I(M) = 0$ for every R-module M).*

(b) $H^d_I(R) = 0$ *if and only if for every minimal prime P of the completion $\widehat{R}$ of R such that* $\dim \widehat{R}/P = d$, *$I\widehat{R} + P$ is not primary to the maximal ideal of $\widehat{R}$.*

Proof that (a) $\Rightarrow$ (b). Assume part (a). Since $\widehat{R}$ is faithfully flat over R and $H^d_{I\widehat{R}}(\widehat{R}) \cong H^d_I(\widehat{R}) \cong \widehat{R} \otimes_R H^d_I(R)$, we might as well assume that R is complete. Suppose that for some minimal prime P of R with $\dim R/P = d$ we have that $P + I$ is primary to m. The long exact sequence for local cohomology yields a surjection $H^d_I(R) \twoheadrightarrow H^d_I(R/P)$ (since H^{d+1}_I vanishes), and since I expands to an ideal J primary to the maximal ideal in R/P and $H^d_I(R/P) \cong H^d_J(R/P) \neq 0$, we see that "only if" part holds. Now suppose that for every minimal prime P of R with $\dim R/P = d$ we have that $I + P$ is not primary to m. For minimal primes P with $\dim R/P < d$ we evidently have $H^d_I(R/P) = 0$, while for the other minimal primes P we have that $H^d_I(R/P) = H^d_J(R/P)$ with $J = I(R/P)$, and J is not primary to m, so that it follows from part (a) that $H^d_J(R/P) = 0$. Since $H^d_I(R/P) = 0$ for every minimal prime P of R and since every finitely generated R-module has a finite filtration by modules each of which is killed by some minimal prime P, it follows that $H^d_I(M) = 0$ for every finitely generated (and, hence, every) R-module M. $\square$

Thus, part (a) is really the heart of the theorem. We defer its proof, however, until we have established several lemmas.

(19.2) Discussion. Let R be a Noetherian ring. If $I \subseteq J \subseteq R$ then there is a surjection $R/I^t \twoheadrightarrow R/J^t$ for all t, and these maps yield maps $\mathrm{Ext}^i_R(R/J^t,\, M) \to \mathrm{Ext}^i_R(R/I^t, M)$ for all i, t and R-modules M. Thus, there is a natural induced map $H^i_J(M) \to H^i_I(M)$. E.g., when $i = 0$, sections supported on $V(J)$ are clearly supported on $V(I)$.

In the case where $J = I + xR$ we have for H^0 an obvious exact sequence $0 \to H^0_J(M) \to H^0_I(M) \to H^0_I(M_x)$ which is readily seen to be exact when M is injective. (This comes down to the case where $M = E(R/P)$, and we may assume that R is local and M is the injective hull of the residue field. The map $M \to M_x$ is then either an isomorphism or else zero. The quasicoherent sheaves corresponding to injective modules over Noetherian rings are flasque.) Thus, if we apply the three indicated functors $H^0_J(\_), H^0_I(\_), and H^0_I(\_{}_x)$ to an injective resolution of M we get a short exact sequence of complexes leading to a long exact sequence of cohomology. To wit:

(19.3) Proposition. *Let $J = I + xR$, where $I \subseteq J \subseteq R$ are ideals of the Noetherian ring R and $x \in R$. Let M be an R-module. Then there is a long exact sequence of cohomology:*

$$\cdots \to H^{i-1}_I(M_x) \to H^i_J(M) \to H^i_I(M) \to H^i_I(M_x) \to H^i_J + 1(M) \to \cdots$$

that is functorial in M. $\square$

(19.4) Remark. An alternative proof can be based on the fact that if $\underline{z} = z_1, \ldots, z_n$ generate I then $K^{\bullet-1}(\underline{z}^\infty; M)$ is a subcomplex of $K^\bullet(\underline{z}^\infty, x^\infty; M)$, and the quotient complex is easily identified with the complex $K^\bullet(\underline{z}^\infty; M)$.

(19.5) Discussion. We now claim that part (a) of Theorem (19.1) reduces to the case where I is a prime ideal of R such that $\dim R/I = 1$. For suppose that there is a counterexample and choose one with I maximal. If I is not a prime ideal of R such that $\dim R/I = 1$, then there exists an element $x \in m - I$ such that $J = I + xR$ is not m-primary. Then we get:

$$\cdots \to H^d_J(R) \to H^d_I(R) \to H^d_I(R_x) \to \cdots$$

and $\dim R_x < d$, so that $H^d_I(R_x) = 0$ Thus, the map $H^d_J(R) \to H^d_I(R)$ is onto, and so J is also a counterexample, contradicting the maximality of I. Thus, to complete the proof of Theorem (19.1), it suffices to prove:

(19.6) Theorem. *Let (R, m, K) be a complete local domain with* $\dim R = d$ *and let P be a prime with* $\dim R/P = 1$. *Then $H^d_P(R) = 0$.*

The rest of the proof consists of two reductions: first, we reduce to the case where R is Gorenstein. Second, we show that in the Gorenstein case one can compute local cohomology using symbolic powers instead of ordinary powers. The particular reduction to the Gorenstein case that we are going to use is due to M. Brodmann and Craig Huneke, independently.

We first note:

(19.7) Lemma. *Let R be a normal domain and let S be a domain generated over R by one integral element, s. Then the minimal monic polynomial f of s over the fraction field of R has coefficients in R, and $S \cong R[x]/fR[x]$.*

Proof. Let T be an integral closure of S in an algebraically closed field L containing the fraction field of S. Then f splits over L. Let G be a monic polynomial over R satisfied by s. Then f divides G (working over the fraction field of R),which shows that all roots of f are roots of G and, hence, integral over R. The coefficients of f are elementary symmetric functions of its roots, and, hence, are both integral over R and in the fraction field of R. Since R is normal, the coefficients of f are in R. Thus $f \in R[x]$.

Let $I = \mathrm{Ker}\,(R[x] \to R[s])$ for the R-algebra map sending x to s. Then $R[s] \cong R[x]/I$. Evidently, $f \in I$. Now suppose that $F \in I$. Carry out the division algorithm for dividing F by f over R: recall that f is monic here. The result is the same as if we were working over the fraction field of R. Since $F(s) = 0$, if $f \mid F$ in $R[x]$. Thus, $I = fR[x]$ $\square$

(19.8) Proof that (19.6) for Gorenstein domains implies (19.6) for domains in general. Let P be a prime ideal of the complete local domain (R, m, K) such that $\dim R/P = 1$ Let $\dim R = d$. Choose a system of parameters $x_1, \ldots, x_d$ for R such that $x_1, \ldots, x_{d-1} \in P$. In the mixed characteristic p case it is possible to do this so that one of the x_i is equal to p (where $i < d$ if $p \in P$ and $i = d$ if $p \notin P$), by standard prime avoidance arguments. Let V be a coefficient ring for R (which will be a field in the equicharacteristic case and will be a DVR (V, pV) in the mixed characteristic case) and form a regular ring by adjoining the power series in the parameters (other than p, in mixed characteristic) to V. In this way we obtain a regular ring $A \subseteq R$ such $P \bigcap A$ has height $d-1$ (it will contain x_i for $i \leq d-1$). Let $P = Q_1, \ldots, Q_r$ denote the prime ideals of R lying over $(x_1, \ldots, x_{d-1})A$. Then we can choose θ in P and not in any other of the Q_j. Let $B = A[\theta]$. Then B is a complete local domain, and by (19.7) is $\cong A[x]/fA[x] \cong A[[x]]/fA[[x]]$, since θ is in the maximal ideal, and, hence, B is Gorenstein. (The quotient ring of a regular ring by an ideal generated by an R-sequence is Gorenstein. In this case, the R-sequence has length

one.) It is then clear that P is the only prime of B lying over $\mathfrak{p} = P \cap B$, since any other must lie over $(x_1, \dots, x_{d-1})A$, and so must be one of the $Q's$, and the others are excluded since they do not contain θ. This implies that P is the radical of the ideal $\mathfrak{p}R$. Thus, $H^d_P(M) = H^d_{\mathfrak{p}}(M)$ (with M viewed as a B-module on the right), and so it suffices to prove the theorem for B and $\mathfrak{p}$. $\square$

We next observe the following very useful fact, due to Chevalley:

(19.9) Theorem (Chevalley's theorem). *Let M be a finitely generated module over a complete local ring (R, m, K) and let $\{M_t\}_t$ denote a nonincreasing sequence of submodules. Then $\bigcap_t M_t = 0$ if and only if for every integer $N > 0$ there exists t such that $M_t \subseteq m^N M$.*

Proof. The "if" part is clear. Suppose that the intersection is 0. Let V_{tN} denote the image of M_t in $M/m^N M$. Then the V_{tN} do not increase as t increases, and so are stable for all large t. Call the stable image V_N. Then the maps $M/m^{N+1}M \to M/m^N M$ induce surjections $V_{j+1} \twoheadrightarrow V_j$. The inverse limit of the V_N may be identified with a submodule of the inverse limit of the $M/m^N M$, i.e. with a submodule of M, and any element of the inverse limit is in $\bigcap_{t,N}(M_t + m^N M) = \bigcap_t \big(\bigcap_N (M_t + m^N M)\big) = \bigcap_t M_t$. If any V_N is not zero, then since the maps $V_{j+1} \twoheadrightarrow V_j$ are surjective, the inverse limit of the V_j is not zero. But V_N is zero if and only if $M_t \subseteq m^N M$ for all $t \gg 0$. $\square$

Thus, in a complete finitely generated module, a nonincreasing sequence of submodules has intersection 0 if and ony if the terms are eventually contained in arbitrarily high powers of the maximal ideal times the module. From this we can deduce:

(19.10) Proposition. *Let (R, m, K) be a complete local domain and let P be a prime ideal such that $\dim R/P = 1$. Then the powers P^t of P and the symbolic powers $P^(t)$ of P are cofinal. In other words, for every integer $t > 0$ there exists N such that $P^{(N)} \subseteq P^t$ (of course, we always have $P^t \subseteq P^{(t)}$).*

Proof. First note that $\bigcap_t P^{(t)} \subseteq \bigcap_t P^{(t)} R_P = \bigcap_t (PR_P)^t = (0)$. Since R is comnplete, Chevalley's theorem implies that $P^(N)$ is contained in arbitrarily high powers of m for large enough N.

Now fix t. Then P^t can have only P, m as associated primes, since $V(P) = \{P, m\}$, and so the primary decompositon tells us that $P^t = P^{(t)} \cap J$ where J is either primary to m or else R. For $N \geq t$ we have that $P^{(N)} \subseteq P^{(t)}$, while Chevalley's theorem implies that $P^{(N)} \subseteq J$ for all $N \gg 0$. Thus $P^{(N)} \subseteq P^t$ for all $N \gg 0$ $\square$

(19.11) Lemma. *Let M be a finitely generated Cohen-Macaulay module of dimension d over a Gorenstein local ring (R, m, K) of dimension n. Then* $\operatorname{Ext}^j_R(M, R) = 0$ *except for* $j = n - d$.

Proof. This is immediate from local duality, since $H^i_m(M)$ vanishes except when $i = d$.

One can also proceed as follows: the case where $M = K$ is immediate from the basic properties of Gorenstein rings, and the case where $\dim M = 0$ follows by induction on the length of M. If $d > 0$ proceed by induction on d. Pick a nonzerodivisor x in m on M, and apply the long exact sequence for $\operatorname{Ext}^\bullet_R(\_\,, R)$ to $0 \to M \xrightarrow{x} M \to M/xM \to 0$. It follows from Nakayama's lemma that whenever $\operatorname{Ext}^{j+1}_R(M/xM,\, R) = 0$ then $\operatorname{Ext}^j_R(M,\, R) = 0$, and the result is then immediate from the induction hypothesis . $\square$

(19.12) Proof of Theorem (19.6). We have seen that it suffices to prove that if R is a complete local Gorenstein domain of dimension d and P is a prime such that $\dim R/P = 1$, then $H^d_P(R) = 0$. But because the symbolic powers of P are cofinal with the ordinary powers, we may compute the local cohomology as $\varinjlim_t \operatorname{Ext}^d_r(R/P^{(t)},\, R)$. Because $R/P^{(t)}$ is a local ring of dimesion 1 and the maximal ideal is not an associated prime of (0), $R/P^{(t)}$ is Cohen-Macaulay. It follows from the preceding proposition that $\operatorname{Ext}^j_R(R/P^{(t)},\, R)$ is zero except when $j = d - 1$. $\square$

We have now completed the proof of Theorem (19.1) as well.

(19.13) Exercise. Let (R, m, K) be a Cohen-Macaulay local ring with canonical module ω. Let $\dim R = n$. If M is a Cohen-Macaulay R-module of dimension d let $M^* = \operatorname{Ext}^{n-d}_R(M, \omega)$.

(a) Show that * is a contravariant functor from Cohen-Macaulay modules of dimension d to Cohen-Macaulay modules of dimension d. Show also that if $0 \to N \to M \to Q \to 0$ is a short exact sequence of Cohen-Macaulay modules of dimension d then the sequence $0 \to Q^* \to M^* \to N^* \to 0$ is exact, while if N and M have dimension d and Q has dimension $d-1$ then there is a short exact sequence $0 \to M^* \to N^* \to Q^* \to 0$. Show that if x is not a zerodivisor on the Cohen-Macaulay module M then $(M/xM)^* \cong M^*/xM^*$. Note that $R^* \cong \omega$ and $\omega^* \cong R$.

(b) Show that if $S \to R$ is a local homorphism, where S is Cohen-Macaulay with canonical module ω_S, and R is module-finite over the image of S, then * calculated over S, when restricted to Cohen-Macaulay R-modules, is a functor isomorphic with * calculated over R. (Identify $\omega \cong \operatorname{Ext}^h_S(R, \omega_S)$ where $h = \dim S - \dim R$.)

(c) Show that the calculation of * is compatible with completion.

(d) Show that for every Cohen-Macaulay R-module $M, M^{**} \cong M$. (One may reduce to the complete case, and then to the case where the ring is regular by mapping a regular ring onto R.)

(e) Show that $\mathrm{Ann}_R M = \mathrm{Ann}_R M^*$.

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