1. Each of $M$, $N$ is the direct sum of a free module and a torsion module: say $M = F \oplus A$ and $N = G \oplus B$. Since Tor distributes over $\oplus$ and higher Tors with free modules are 0, $\text{Tor}_1^R(M, N) \cong \text{Tor}_1^R(A, B)$. Thus, it suffices to show that $\text{Tor}_1^R(A, B) \cong A \otimes_R B$. $A$ and $B$ are finite direct sums of cyclic torsion modules. Since both $\text{Tor}_1^R(\_ , \_)$ and $\otimes_R$ distribute over $\oplus$, we may assume that $A = R/aR$, $B = R/bR$, $a, b \neq 0$. Then $A \otimes_R B = R/dR$ where $d = \text{GCD}(a, b)$. Use the resolution $0 \to R \to \to R \to 0$ for $R/aR$ to see that $\text{Tor}_1^R(R/aR, R/bR) = \text{Ker}(R/bR \twoheadrightarrow R/bR)$. If $a = a'd$, $b = b'd$ where $\text{GCD}(a', b') = 1$, the kernel is $b'R/bR$ (since $\text{GCD}(a', b') = 1)$ $\cong b'R/b'dR \cong R/dR = R/(a, b)R$.

2. If $M$ is any $R$-module and $f$ is a nonzerodivisor in $R$, $0 \to R \xrightarrow{f} R \to 0$ resolves $R/fR$, and so $\text{Ext}_1^R(R/fR, M)$, computed by applying $\text{Hom}_R(\_ , M)$ is $\text{Coker}(M \xrightarrow{f} M) \cong M/fM$. Hence, $\text{Ext}_1^R(R/fR, R) \cong \text{Hom}_R(R/fR, R)$ and $\text{Ext}_1^R(R/fR, R/gR) \cong \text{Hom}_R(R/(f, g)R)$ in all cases. Now suppose that $M \cong F \oplus A$ and $N \cong G \oplus B$ over the PID $R$ as in Problem #1. Then $\text{Ext}_1^R(F, \_)$ vanishes, and so $\text{Ext}_1^R(M, N) \cong \text{Ext}_1^R(A, N)$. For each cyclic summand $R/fR$ of $A$, $\text{Ext}_1^R(R/fR, N) \cong N/fN \cong (R/fR) \otimes_R N$. Hence, $\text{Ext}_1^R(M, N) \cong A \otimes_R N$.

3. Every prime in $R$ contains one of the $X_i$, and so is in the image of some $G_0(R/X_iR) \to G_0(R)$. But $G_0(R/X_iR) \cong \mathbb{Z}$, where $[R/X_iR]$ generates. It follows that the $n$ elements $[R/X_iR]$ span $G_0(R)$. Now suppose that $\sum_{i=1}^n a_i [R/X_iR] = 0$, where the $a_i \in \mathbb{Z}$. To complete the proof, it suffices to show that all the $a_i$ are 0. Let $P_i = X_iR$. The map $M \mapsto \text{length}_R(M/P_i)$ is additive on short exact sequences and so induces a map $G_0(R) \to \mathbb{Z}$ that sends $[R/P_i] \mapsto 0$ if $i \neq j$ and $[R/P_i] \mapsto 1$. Hence, its value on $\sum_{i=1}^n a_i [R/X_iR]$ is $a_i$. Consequently, if the sum is 0, all the $a_i$ are 0. □

4. Necessity is clear. We use Noetherian induction on $R$: we may assume the result for all $R/I$, $I \neq 0$. Let $P_1, \ldots, P_k$ be the minimal primes of $R$. Then $\text{Spec}(R) = \bigcap_i V(P_i)$, and so it suffices to show that $U_i = U \cap V(P_i)$ is open in $V(P_i)$ for all $i$. $U_i$ satisfies the same conditions within $V(P_i) = \text{Spec}(R/P_i)$. So we have reduced to the case of one minimal prime, $P$. If $P \notin U$, $U$ must be empty. Otherwise, choose $f \notin P$ such that the open set $D_f \subseteq U$. It suffices to show that $U \cap V(f)$ is open in $V(f)$. But we may identify $\text{Spec}(R/fR)$ with $V(f)$, and then $U \cap V(f)$ satisfies the same conditions. By the hypothesis of Noetherian induction, $U \cap V(f)$ is open in $V(f)$. □

5. Note that dim $M = d$ implies that $I = \text{Ann}_R M$ has height $n-d$. Let $x_1, \ldots, x_h = x$ be a maximal regular sequence on $R$ in $I$, and let $P \supseteq I$ be an associated prime of $(x)$. Then $x$ is a maximal regular sequence in $R_P$, which is Cohen-Macaulay, so $P$ has height $h$ $\geq$ height $I$. Let $Q$ be a minimal prime of $I$ of height $n-d$. Any regular sequence in $I$ is part of a system of parameters for $R_Q$, so $h \leq n-d$. Thus, $h = n-d$, and depth$_I R = n-d$. By a class theorem, the first nonvanishing $\text{Ext}_j^R(M, R)$ occurs with $j = n-d$. Since $M$ is Cohen-Macaulay, by the Auslander-Buchsbaum theorem $\text{pd}_R M = \text{depth}(R) - \text{depth}(M) = n - \dim(M) = n-d$. Hence, there is a unique nonzero $\text{Ext}_j^R(M, R)$ for $j = n-d$. So $\text{Ext}_j^R(M, R) = H^j(\text{Hom}_R(G, R)) = \text{Hom}_R(G, R)$, numbered backwards, is a free resolution of $\text{Ext}_j^R(M, R) = M^\vee$. The matrices for the dual bases in the dual complex are the transposes of those in the original complex, and so have entries in $m$. Hence, this resolution
of $M^\vee$ is minimal. If we use the dual complex to calculate $(M^\vee)^\vee = \Ext_R^{n-d}(M^\vee, R)$ we get the original complex back, and so $M^{\vee\vee} \cong M$. Clearly, $\Ann_R M$ kills $M^\vee$, and $\Ann_R M^\vee$ kills $(M^\vee)^\vee$. Hence, $M$ and $M^\vee$ have the same annihilator and the same dimension. Also $\pd(M^\vee) = n - d$ implies that depth $(M^\vee) = d$. Thus, $M^\vee$ is Cohen-Macaulay, and $M^{\vee\vee} \cong M$. Exactness is immediate from the long exact sequence for Ext (only the terms in degree $n - d$ are nonzero). Note also that if $x$ is a nonzerodivisor on $M$, the long exact sequence for Ext gives that $x$ is not a zerodivisor on $M^\vee$ and an isomorphism $(M/xM)^\vee \cong M^\vee/xM^\vee$, which yields a different proof, by induction on $\dim(M)$, that $M^\vee$ is Cohen-Macaulay.

6. Given the short exact sequence $0 \to A \to M \to B \to 0$, denoted $\mathcal{S}$, we get a map of $G_\bullet$ to the complex $0 \to A \to M \to 0$ that lifts id$_B$. The map $h : G_1 \to A$ must kill $\Im G_2$ and so gives a map $f : B_1 \to A$. Every such $f$ in fact arises from a unique $h : G_1 \to A$ that kills $\Im G_2$. The maps $A \to M$ and $G_0 \to M$ give a map $A \oplus G_0 \to M$ and $a \oplus b$ maps to 0 iff $b \in B_1$ and $a + f(b_1) = 0$. Thus $0 \to A \to M \to B \to 0$ arises, up to isomorphism, from $f : B_1 \to A$ by the construction given: with $N_f = \{(f(b_1) + - b_1 : b_1 \in B_1\}$, the constructed module in the middle is $M_f = (A \oplus G_0)/N_f$. To get the bijection with $\Ext^1_R(A, B)$ it suffices to show that $f, f' : B_1 \to A$ give isomorphic extensions iff $f - f'$ extends to $G_0$. For “if”, note that if $\phi$ extends $f' - f$ to $G_0$ the map $A \oplus G_0 \to A \oplus G_0$ given by $a \oplus u \mapsto (a + \phi(u)) \oplus u$ induces the isomorphism. For “only if”, we may use the isomorphism of sequences to get a map $\theta : A \oplus G_0 \to A \oplus G_0$ that is the identity on $A$ and induces the identity map on $B$, which means we may take the map on $G_0$ to have the form $\phi \oplus \text{id}_{G_0}$, where $\phi : G_0 \to A$. For $\theta$ to take $Z_f$ to $Z_{f'}$, as needed, we must have that $\phi$ extends $f' - f$ to all of $G_0$. We have at once from the construction of the connecting homomorphism (using the snake lemma on the short exact sequence of complexes obtained by taking $\Hom_R(G_\bullet, \mathcal{S})$ that id$_B$ maps to $[h] \in \Ext^1(B, A)$: id$_B$ is induced by the map $G_0 \to M$, and composing with $G_1 \to G_0$ gives $h$, which actually takes values in $A$.

EXTRA CREDIT 7. Since $G_0(R/xR) \to G_0(R) \to G_0(R_x) \to 0$ is exact, $G_0(R)$ will be generated by $\Im G_0(R/xR)$ and lifts of generators of $G_0(R_x)$. Since $R_x \cong K[x, y, z]_x$ (we may solve for $u$) and $G_0(K[x, y, z]) \cong \mathbb{Z}$, generated by the class of the ring, we have that $G_0(R)$ is generated by $[R]$ (lifting $[R_x]$) and $\Im G_0(R/xR)$. Now, $R/xR \cong (K[y, z]/(yz))[u]$, polynomial over $K[y, z]/(yz)$. Hence, $G_0(R/xR) \cong G_0(K[yz]/(yz)) \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by $[R/(x, z)]$ and $[R/(x)]$ by Problem #3. But $[R/xR] = 0$ in $G_0(R)$, and $0 \to R/(x, y) \xrightarrow{\sim} R/xR \to R/(x, z)$ is exact, so that $[R/(x, z)] = -[R/(x, y)]$ in $G_0(R)$. Hence, $[R]$ and $[R/(x, y)]$ generate $G_0(R)$. Since $R$ is a domain, $\mathbb{Z} \cdot [R]$ splits off. We need only show that $[R/(x, y)]$ is not torsion in $\mathcal{G}_0(R)$. But $P = (x, y)R$ has infinite order in the divisor class group of $R$: note that $R \cong K[as, at, bs, bt]$ with as, at, bs, bt corresponding to $x, y, u, v$ respectively, that $(x, y)^k$ is the contraction of the primary ideal $a^k$ in $K[a, b, s, t]$, hence primary, and so $P^{(k)} = P^k$, which needs $k + 1$ minimal generators. □

EXTRA CREDIT 8. Finite length case. We have $0 \to \Hom_R(C, A) \to \Hom_R(C, B) \to \Hom_R(C, C) \to D \to 0$. Because $B \cong A \oplus C$ in some way, length $\Hom_R(C, B)$ is the sum of the lengths of $\Hom_R(C, A)$ and $\Hom_R(C, C)$. Hence, length $D = 0$, and so $D = 0$. But then id$_C$ is the image of a map $C \to B$, which means that the sequence splits. We leave the general case as a continuing problem. □