Math 615, Winter 2012
Problem Set #5
Due: Monday, April 16.

1. Let $M$ be a Cohen-Macaulay module of dimension $d$ over a local ring $(R, m, K)$. Let $x = x_1, \ldots, x_d$ be a system of parameters (i.e., a maximum regular sequence) on $M$, and let $x = x_1$, so that $x$ is a nonzerodivisor on $M$.
   (a) Prove that $\text{Ext}_R^d(K, M) \cong \text{Ext}_R^{d-1}(K, M/xM)$.
   (b) Prove that $\text{Ext}_R^d(K, M) \cong \text{Hom}_R(K, M/(x)M)$. Hence, the $K$-vector space dimension of $\text{Hom}_R(K, M/(x)M) \cong \text{Ann}_{M/(x)M}m$ is independent of the choice of system of parameters $x_1, \ldots, x_d$.
   The positive integer $\dim_K \text{Ext}_R^d(K, M)$ is called the type of $M$. Also show that the type of $M$ is the same as the type of $\hat{M}$ over $\hat{R}$.

2. A local ring $(R, m, K)$ that has type 1 as a module over itself is called Gorenstein. Prove if $R$ is regular, then $R$ is Gorenstein, and that if $R$ is Gorenstein, so is $R/(f_1, \ldots, f_k)R$ whenever $f_1, \ldots, f_k$ is part of a system of parameters for $R$.

3. Let $M$ be a Cohen-Macaulay module of dimension $d$ over a regular local ring $(R, m, K)$ of dimension $n$. Show that the type of $M$ is the same as the least number of generators of its Ext dual $\text{Ext}_R^{n-d}(M, R)$. (It may be helpful to reduce to the case where Krull dim $M = 0$.)

4. Let $X = (x_{ij})$ denote a $3 \times 2$ matrix of indeterminates over a field $K$, and let $R$ be the polynomial ring in the six variables $x_{ij}$ over the field $K$. Let $m$ denote the ideal generated by the variables. Let $\Delta_1$, $-\Delta_2$ and $\Delta_3$ be the determinants of the $2 \times 2$ matrices obtained by omitting the first, second and third rows of the matrix, respectively, and let $Y$ be the $1 \times 3$ matrix $(\Delta_1 \Delta_2 \Delta_3)$, so that $YX = (0)$. Let $P = (\Delta_1, \Delta_2, \Delta_3)\hat{R}$. You may assume that the complex (1) $0 \rightarrow R^2 \xrightarrow{X} R^3 \xrightarrow{Y} R \rightarrow R/P \rightarrow 0$ is exact, and so gives a free resolution of $R/P$. Let $Q$ be the ideal generated by $x_{12}$, $x_{11} - x_{22}$, $x_{21} - x_{32}$, and $x_{31}$ in $R$. Show that the images of these elements form a homogeneous system of parameters for $R/P$, determine the type of $R_m/P R_m$ in two different ways, and determine the intersection multiplicity of $Z = V(P)$ and $L = V(Q)$ at the origin.

5. Let $R = K[[x, y]]/(xy)$, where $K$ is a field. Determine the minimal first modules of syzygies of $R/xR$ and $R/yR$. Describe a minimal free resolution of $R/xR$ over $R$ and determine all the Betti numbers of $R/xR$ over $R$. Find $\text{Tor}_i^R(R/xR, R/yR)$ for all $i \geq 0$.

6. Let $(R, m, K) \rightarrow (S, n, L)$ be a flat local extension such that $\dim S/mS = 0$. Let $M$ be a Cohen-Macaulay module over $R$. Show $S \otimes_R M$ is Cohen-Macaulay and its type is the product of the type of $M$ and the type of $S/mS$.

EXTRA CREDIT 9. Prove that the complex described in #4. is exact.

EXTRA CREDIT 10. Use the resolution in #4. to calculate the Hilbert function of $R/P$. Show that $R/P$ maps as $K$-algebra onto the Segre product $T$ of the polynomial rings $K[x, y, z]$ and $K[s, t]$ so as to preserve degree. The Hilbert function of $T$ was calculated in an earlier exercise. Conclude from the fact that $R/P$ and $T$ have the same Hilbert function that the map $R/P \rightarrow T$ is an isomorphism.