Module-finite Extensions of Complete Local Rings

Theorem. A module-finite extension $S$ of a complete local ring $(R, m, K)$ is a finite product of complete local rings: moreover, there is a bijection between the factors and the maximal ideals of $S$. Therefore, if $S$ is a domain (then $R$ will automatically be a domain as well), $S$ is complete and local.

Proof. Let $Q$ be any maximal ideal of $S$, and suppose it contracts to $P$ in $R$. Then $R/P \hookrightarrow S/Q$ is module-finite, and so $R/P$ has Krull dimension 0. Therefore, $Q$ lies over $m$. The primes of $S$ lying over $m$ correspond to the primes of $S/mS$, which is module-finite over $R/m = K$ and so zero-dimensional. Therefore, $S$ has only finitely many maximal ideals, all of which lie over $m$. Let $Q_1, \ldots, Q_n$ be the maximal ideals of $S$. Let $J = \bigcap_{i=1}^n Q_n$. Then $\text{Rad}(mS) = J$, and $J$ has a power in $mS$, say $J^k$. Since $S$ is module-finite over $R$ and $R$ is complete, $S$ is $m$-adically complete. Since $J^k \subseteq mS \subseteq J$, $S$ is $J$-adically complete. Note that $J^t = (Q_1 \cap \cdots \cap Q_n)^t = (Q_1 \cdots Q_n)^t$ (since the $Q_i$ are pairwise comaximal) = $Q_1^t \cap \cdots \cap Q_n^t = Q_1^t \cap \cdots \cap Q_n^t$ (since the $Q_i^t$ are pairwise comaximal). Hence,

$$S \cong \lim_{\leftarrow} S/J^t \cong \lim_{\leftarrow} S/(Q_1^t \cap \cdots \cap Q_n^t) \cong \lim_{\leftarrow} ((S/Q_1^t) \times \cdots \times (S/Q_n^t))$$

by the Chinese Remainder theorem. Since limit (or inverse limit) commutes with finite products, this is

$$\lim_{\leftarrow} ((S/Q_1^t) \times \cdots \times (S/Q_n^t)).$$

When $Q$ is maximal in $S$, $S/Q^t$ is already a local ring (the only prime that contains $Q^t$ is $Q$) and so $S/Q^t \cong S_Q/(QS_Q)^t$. Hence, $\lim_{\leftarrow} S/Q_i^t$ may be identified with $(Q_i S_{Q_i})-$adic completion of $S_{Q_i}$. Thus, $S$ is isomorphic with a product of complete local rings, one for each of its maximal ideals. Evidently, $S$ cannot be a domain unless there is only one factor, in which case $S$ is local. $\Box$