Faithful Flatness

We shall say that an \( R \)-module \( F \) is \textit{faithfully flat} if it is flat if it is flat and for every nonzero \( R \)-module \( M \), \( F \otimes_R M \neq 0 \). An \( R \)-algebra \( S \) is \textit{faithfully flat} if it is faithfully flat when considered as an \( R \)-module. We shall see below that the completion of a local ring \( R \) is a faithfully flat \( R \)-algebra. Typically, \( W^{-1}R \) is flat but not faithfully flat: if \( W \) contains an element that is not already a unit, say \( f \), then \( W^{-1}R \otimes_R (R/fR) = 0 \). A nonzero free module over \( R \) is obviously faithfully flat.

\textbf{Proposition.} Let \( F \) be an \( R \)-module. The following conditions are equivalent:

1. \( F \) is flat and for every nonzero \( R \)-module \( M \), \( F \otimes_R M \neq 0 \) (i.e., \( M \) is faithfully flat).
2. \( F \) is flat and for every proper ideal \( I \) of \( R \), \( IF \neq F \).
3. \( F \) is flat and for every maximal ideal \( m \) of \( R \), \( mF \neq F \).
4. \( F \) is flat and for every \( R \)-linear map \( h : M \to N \), \( h \) is nonzero if and only if \( \text{id}_F \otimes h : F \otimes_R M \to F \otimes_R N \) is nonzero.
5. For every sequence of modules \( A \to B \to C \), the sequence is exact at \( B \) if and only if the sequence \( F \otimes_R A \to F \otimes_R B \to F \otimes_R C \) is exact at \( F \otimes_R B \).

\textit{Proof.} The conclusion in (2) is equivalent to \( F/IF = F \otimes_R (R/I) \neq 0 \). Therefore, (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3). Now assume (3) and let \( M \) be any nonzero module. Then \( M \) has a nonzero element \( u \). Let \( I = \text{Ann}_R u \), so that \( Ru \cong R/I \). Let \( m \) be a maximal ideal containing \( I \). Since \( IF \subseteq mF \neq F \), we have that \( F \otimes_R R/I \neq 0 \). Since \( R/I \cong Ru \hookrightarrow M \) and \( F \) is flat, we have that \( F/IF \hookrightarrow F \otimes_R M \), so that \( F \otimes_R M \neq 0 \). Thus, (3) \( \Rightarrow \) (1). This shows that (1), (2), and (3) are equivalent.

In (4), the “if” part is obvious. If we apply (4) to the map \( 0 \to M \), we see that (4) \( \Rightarrow \) (1). We need to show if (1) holds, the “only if” part of (4) holds. Suppose that \( M \to N \) factors \( M \to Q \hookrightarrow N \), where \( Q \) is the image of \( N \). The map is nonzero if and only if \( Q \neq 0 \). Then \( F \otimes_R M \to F \otimes_R N \) factors \( F \otimes_R M \to F \otimes_R Q \hookrightarrow F \otimes_R N \), where the map on the left is surjective by the right exactness of \( \otimes \), and the map on the right is injective because \( F \) is flat. By (1), we have that \( F \otimes_R Q \neq 0 \).

The fact that “only if” part of (5) holds implies that \( F \otimes_R - \) preserves short exact sequences, which is equivalent to the flatness of \( F \). Therefore, in the rest of the argument we may assume that \( F \) is flat.

To see that (5) is equivalent to the other conditions, let \( f, g \) respectively denote \( A \to B \) and \( B \to C \). Let \( D \) be the image of \( A \) in \( B \). Let \( E \) be the kernel of the map \( B \to C \), and let \( G \) be the image of \( B \in C \). This, we have \( A \to D, D \hookrightarrow B, E \hookrightarrow B, 0 \to E \to B \to G \to 0 \) is exact, and \( G \to C \). Because \( F \) is flat, all these conditions are preserved when apply \( F \otimes_R - \). This means that we may identify the image of \( F \otimes f \) with \( F \otimes_R D \), and the kernel of \( F \otimes g \) with \( F \otimes_R E \). The original sequence is exact at \( B \) if and only if \( D = E \). Obvious, this implies that we have exactness when we apply \( F \otimes_R - \): this only uses that \( F \) is flat. It remains to show that if the images of \( F \otimes D \) and \( F \otimes E \) are
equal in $F \otimes B$, then $D = E$. But if the images are equal, they will both be equal to the image of $F \otimes (D + E)$. Then $F \otimes ((D + E)/D) \cong (F \otimes (D + E))/\text{Im} (F \otimes D) = 0$, which shows that $(D + E)/D = 0$ by (1), and hence that $D + E = D$. But $D + E = E$ follows in exactly the same way. \qed

In the situation of the Corollary below, $m$ is the only maximal ideal of $R$, and $mS \neq S$ if and only if $m$ maps into $n$.

**Corollary.** A flat homomorphism $h : (R, m) \to (S, n)$ of quasilocal rings is faithfully flat if and only if it is local, i.e., if and only if $m$ maps in to $n$. \qed

**Proposition.** If $M$ is flat (respectively, faithfully flat) over $R$ and $T$ is any $R$-algebra, $T \otimes_R M$ is flat (respectively, faithfully flat) over $T$.

**Proof.** If $f : A \to B$ is a map of $T$-modules, we may use the associativity of $\otimes$ to identify $A \otimes_T (T \otimes_R M) \to B \otimes_T (T \otimes_R M)$ with the map $A \otimes_R M \to B \otimes_R M$. Thus, if $f$ is injective, the flatness of $M$ over $R$ implies the new map is injective, while if $A$ is nonzero, so is $A \otimes_T (T \otimes_R M) \cong A \otimes_R M$. \qed

**Proposition.** If $S$ is a faithfully flat $R$-algebra and $I$ is an ideal of $R$, then the contraction of $IS$ to $R$ is $I$. Moreover, $R \to S$ is injective.

**Proof.** For any ideal $\mathfrak{A}$ of $R$, we have an injection $\mathfrak{A} \hookrightarrow R$, which yields an injection $\mathfrak{A} \otimes S \hookrightarrow S$ when we apply $\_ \otimes_R S$. The image of the injection is $\mathfrak{A}S$, so that $\mathfrak{A} \otimes_R S \cong \mathfrak{A}S$. If $\mathfrak{A}$ is the kernel of $R \to S$, we then have $\mathfrak{A} \otimes S \cong \mathfrak{A}S = 0$. Since $S$ is faithfully flat, this implies $\mathfrak{A} = 0$. This proves the second statement. But then for every $I$, the preceding result shows that $R/I \to S/IS$ is faithfully flat (take $M = S$, $T = R/I$), and so injective. The kernel is $J/I$, where $J$ is the contraction of $IS$ to $R$, and so $J = I$. \qed

**Corollary.** If $R \to S$ is faithfully flat, then $\text{Spec} (S) \to \text{Spec} (R)$ is surjective.

**Proof.** For every prime $P$ of $R$, the contraction of $PS$ to $R$ is $P$, which means that $PS$ is disjoint from the image $W$ of $(R - P)$ in $S$, which is a multiplicative system. Hence, there is a prime $Q$ of $S$ that contains $PR$ and is disjoint from $W$, and $Q$ must contract to $P$. \qed

**Proposition.** Let $(R, m) \to (S, n)$ be a flat local homomorphism of local rings. Then $\dim (S) = \dim (R) + \dim (S/mS)$. ($S/mS$ is called the closed fiber of $R \to S$.)

**Proof.** We use induction on $\dim (R)$. If $J = \text{Rad} (0)$ in $R$, $R/J \to S/JS$ is again flat and local, and, since both $J$ and $JS$ consist of nilpotents, the dimensions do not change (note that the closed fiber also has not changed.) Therefore, we may assume that $R$ is reduced. If $\dim (R) = 0$, then $R$ is a field, $m = 0$, and $S/mS \cong S$, so the result is clear. Otherwise, $m$ is not contained in the union of the minimal primes of $R$: choose $x \in m$ not in any minimal prime. Since $R$ is reduced, every associated prime of $(0)$ is minimal. Hence, $x$ is not a zerodivisor in $R$. Since $R \xrightarrow{x} R$ is injective, when we apply $S \otimes_R \_$ we obtain an injection $S \xrightarrow{x} S$. Thus, $\dim (R/xR) = \dim (R) - 1$, and $\dim (S/xS) = \dim (S) - 1$. But $R/xR \to S/xS$ is still flat local with the same closed fiber. By the induction hypothesis, $\dim (S) - 1 = \dim (R) - 1 + \dim (S/mS)$ and the result follows. \qed