1. Let \( q = p^n = 4h + t_n \), where \( t_n \) is 1 or 3. Then \((z^3)^q = (z^3)^{4h+t_n} = (z^4)^h z^{3t_n} = \pm(w^4 + x^4 + y^4)^{3h}z^{3t_n}\). When we expand, at least one of the exponents on \( u \) or \( v \) or \( w \) is 4h. Hence, taking \( c = (uvw)^3 \) (not the most efficient choice), we have that \( c(z^3)^q \in (u, v, w)[q] \), since \( 4h + 3 \geq q \), and so \( z^3 \in (w, x, y)^* \). If \( q = p \) and \( t_1 = 1 \), \( z^{3p} = (w^4 + x^4 + y^4)^{3h}z^3 \), and 1, z, z^2, z^3 is a free basis for \( R \) over the polynomial ring \( K[w, x, y] \). Every term in the expansion is in \((w^q, x^q, y^q) \) except \((3h)w^h x^4 y^4 z^3 \), and so \( z^p \notin I[v] \). If \( q = p \) and \( t_1 = 3 \), \( z^{3p} = z^{12h+9} = (z^4)^{3h+2}z = \pm(w^4 + x^4 + y^4)^{3h+2}z \). When we expand, at least one of the exponents on \( w^4, x^4 \), or \( y^4 \) is at least \( h+1 \), and \( 4(h+1) \geq p \), so that every term is in \( I[v] \). Hence, \( z^p \in I[v] \) if and only if \( p \equiv 3 \) mod \( 4 \).

2. Since \( K[x_1, \ldots, x_n] \) is module-finite over \( R(d) \), we need to find \( \lim_{d \to \infty} S_{n, d, t, q}/d^n \), where \( S(n, d, t, q) \) is the number of choices \((a_1, \ldots, a_n) \in \mathbb{N}^n \) for the exponents on the variables such that \( d|\sum_{j=1}^n a_j \) and \( \sum_{j=1}^n [a_i/q] < dt \) (\([ \cdot ] \) denotes integer part). The number of choices for the \( b_i = [a_i/q] \) is the number of monomials of degree less than \( dt \) in \( x_1, \ldots, x_n \), which is \((d^{t-n}) \). Once the \( b_i \) are known, one must choose a remainder \( r_i \) in \( \{0, 1, \ldots, q-1\} \) for every \( i \), \( 1 \leq i \leq n \): moreover, the sum of the \( a_i \) must be divisible by \( d \). The first \( n - 1 \) of the \( r_i \) can be chosen in \( q^{n-1} \) ways. For the last, there is an additional restriction: the residue of \( n \) mod \( d \) must be \( r \) mod \( d \) where \( 0 \leq r < d-1 \) where \( d \) depends on the \( b_i \) and \( q \). This limits the choices to \( r, r + d, r + 2d, \ldots, r + \left[\frac{q-1-r}{d}\right]d \), i.e., there are \( \left[\frac{q-1-r}{d}\right] + 1 \) choices. This is \( \frac{q-1-r}{d} \geq \frac{q}{d} - 1 \), and at most \( \frac{q-1-r}{d} + 1 \leq \frac{q}{d} + 1 \), so the number of choices is \( \frac{q}{d} \) with an error of at most \( 1 \). If the average value of the error over the various choices is \( e_q \), we have the limit of \((d^{t-1+n})(q^{n-1})(q_d + e_q)/q^n = (d^{t-1+n})(\frac{1}{d} + e_d)\) as \( q \to \infty \). Since \( |e_q| \leq 1 \), the Hilbert-Kunz multiplicity is \((d^{t-1+n})/d\).

3. If \( f \in IS \cap R \) then for all \( q = p^n \), \( f^q \in I[q] \cap S \cap R \), and when we apply \( \theta \) we obtain that \( f^q/\theta(1) \in I[q] \). Since \( c = \theta(1) \neq 0 \), \( f \in I^* \).

4. If \( f \in I^* \) we have \( c \) nonzero such that \( cu^q \in I[q] \) for all \( q \gg 0 \), where \( q = p^n \). Hence, \( c(u^p)^q = cu^p \in I[q] \) for all \( q \gg 0 \), which shows that \( u^p \in (I[q])^* \). Hence, \((I^*)[p^n] \subseteq (I[p^n])^* \).

5. By 4., if \( f \in I^* \) then for all \( n \), \( f^{[p^n]} \in (I[p^n])^* \), and the defining property of \( c \) then implies that \( cf^{[p^n]} \in (I[p^n])^* \).

6. Let \( f \in I^* \) for \( I \subseteq R \), and \( c \in \tau(S) \cap R \). It suffices to show that \( cf \in I \). But since \( R \subseteq S \) are domains, \( f \in (IS)^* \), and so \( cf \in IS \). Since \( c, f \in R \), this implies \( cf \in IS \cap R = I \), as required.

EC 9. Suppose \( f \in I^* \) but that \( cf \notin I^F \). We can choose \( m \) maximal such that \( cf \notin I^F R_m \): we still have \( f \in (IR_m)^* \) in \( R_m \). Note that \( I^F R_m = (IR_m)^F \subseteq \) is clear. If \( r/w \in (IR_m)^F \), with \( w \in R - m \), we can choose \( q \) such that \( (r/w)^q \in (IR_m)^q = I[q] R_m \) and so \( (r/w)^q = j/v \) with \( j \in I[q] \) and \( v \in R - m \). \( n (rv)^q \in u^qv^{q-1}I[q] \) in \( I[q] m \) and so \( rv \in I[q] \), and so \( r/w = rv/wv \in I[q] R_m \), as required.) Choose a power \( Q \) of \( p \) with \( Q \geq N_m \). Since \( f \in (IR_m)^* \), \( f^Q \in ((IR_m)^{[Q]})^* \), by Problem 5. Since \( c^Q \in \tau(R_m) \), we then have that \( (cf)^Q = c^Q f^Q \in (IR_m)^{[Q]} \), and so \( cf \in (IR_m)^F = I^F R_m \), a contradiction.
EC 10. Suppose that $s = \deg(G) > \delta = \sum_{h=1}^{d} \deg(F_h)$ has maximum degree among the elements of $R$ that are nonzero mod $I = (F_1, \ldots, F_d)$ (there will exist such a form because the homogeneous maximal ideal is nilpotent mod $I$). Then for all $q = p^m$, we have $G^q \notin (F_1, \ldots, F_d)^{[q]} = I^{[q]}$, or else $G$ would be in $I^* = I$. We must have $G(F_1, \ldots, F_d) \subseteq I^{[q]}$ (or some $GF_j$ form of higher degree not in $I^{[q]}$). Then $G \in (F_1^q, \ldots, F_d^q) : R(F_1, \ldots, F_d) = I^{[q]} + F^{q-1}R$ where $F = F_1 \cdots F_d$ has degree $\delta$. Hence, we can replace $G$ by a homogeneous element of the form $F^{q-1}D$, where $D$ is a form of $R$ that is not in $I$, and so $\deg(D) \leq s$. Hence, $\deg(G) = (q - 1)\delta + \deg(D) \geq qm$. Since $\deg(D) \leq s$, $(q - 1)\delta + s \geq qs$, which shows that $(q - 1)\delta \geq (q - 1)s$ and so $\delta \geq s$, as required. □