Test elements using the Lipman-Sathaye theorem

**Theorem.** Let $R \subseteq S$ be Noetherian domains of positive characteristic $p$ with fraction fields $\mathcal{K} \subseteq \mathcal{L}$ such that $R$ is regular, $S$ is module-finite over $R$, and $\mathcal{L}$ is separable over $\mathcal{K}$. Then $R^{1/q} \otimes_R S \cong R^{1/q}[S]$ is faithfully flat over $S$, where we think of $R^{1/q}[S] \subseteq S^{1/q}$. Let $c$ be an element of $R$ such that $cS^{1/q} \subseteq R^{1/q}[S]$ for all $q = p^n$. Elements of $\mathcal{J}_{S/R}$ will have this property. Then for every ideal $I$ of $S$, $cI^n \subseteq I$. Hence, every nonzero element of $\mathcal{J}_{S/R}$ is a test element for tight closure.

**Proof.** Since $R^{1/q}$ is faithfully flat over $R$, $B = R^{1/q} \otimes_R S$ is faithfully flat over $S$. Hence, elements of $S - \{0\}$ and of $R - \{0\}$ are nonzerodivisors on $B$, and if tensor with $\mathcal{K} = (R)$ we obtain that $R^{1/q} \otimes_R S \subseteq \mathcal{K} \otimes_R (R^{1/q} \otimes_R S) \cong (\mathcal{K} \otimes_R \mathcal{K}) \otimes_R (R^{1/q} \otimes_R S) \cong (by\ the\ associativity\ of\ tensor) (\mathcal{K} \otimes_R (R^{1/q}) \otimes_R (\mathcal{K} \otimes_R S) \cong \mathcal{K}^{1/q} \otimes_R \mathcal{L}$, where $\mathcal{L} = (L)$. (With an integral extension domain $D$ of a domain $R$, inverting every element of $R - \{0\}$ inverts every element of $D - \{0\}$, since each nonzero element of $D$ has a nonzero multiple in $R$.) The tensor product of two modules over $W^{-1}R$ is the same whether the base is taken to be $R$ or $W^{-1}R$, so the last term becomes $\mathcal{K}^{1/q} \otimes_K \mathcal{L}$. Because $L$ is separable over $\mathcal{K}$, $\mathcal{L} \cong \mathcal{K}[X]/(F)$ where $F$ is a separable monic polynomial in $X$, and the tensor product is $\mathcal{K}^{1/q}[X]/(F)$, which is reduced. Thus, $\mathcal{K}^{1/q} \otimes_L$ is reduced. Every element $u$ has the form $\sum_{j=1}^t \alpha_j^{1/q} \otimes \lambda_j$ with the $\alpha_j \in \mathcal{K}$ and the $\lambda_j \in \mathcal{L}$. Then $u^n \sum_{j=1}^t t \alpha \otimes \lambda^q = 1 \otimes \sum_{j=1}^t \alpha_j \otimes \lambda^q \in \mathcal{L}$, and so is a unit if it is not zero. Since $\mathcal{K}^{1/q} \otimes_K \mathcal{L}$ is reduced, every nonzero element $u$ has a nonzero power in $\mathcal{L}$ that is a unit, and so every nonzero element is a unit. Thus, $\mathcal{K}^{1/q} \otimes_K \mathcal{L}$ is a field, and the map $\mathcal{K}^{1/q} \otimes_K \mathcal{L} \to \mathcal{L}^{1/q}$ induced by $\mathcal{K}^{1/q} \subseteq \mathcal{L}^{1/q}$ and $\mathcal{L} \subseteq \mathcal{L}^{1/q}$ is injective, with image $\mathcal{F} = \mathcal{K}^{1/q}[L]$. Thus, $R^{1/q} \otimes_R S \subseteq S^{1/q}$ is injective, with image $B = R^{1/q}[S]$. We claim that $\mathcal{F} = \mathcal{L}^{1/q}$. To see this note that $[\mathcal{F} : \mathcal{K}^{1/q}] = [L : K] = [L^{1/q} : \mathcal{K}^{1/q}]$. Hence, the fraction field of $R^{1/q}[S]$ is the same as the fraction field of $S^{1/q}$, namely, $\mathcal{L}^{1/q}$. Hence, $S^{1/q}$ is contained in the integral closure of $R^{1/q}[S]$. By the Lipman-Sathaye theorem, $\mathcal{J}_{B/R^{1/q}}S^{1/q} \subseteq B$. But if $S = T/(F_1, \ldots, F_m)$ is a presentation over $R$, where $T$ is a polynomial ring over $R$, then $B = R^{1/q} \otimes T/(F_1, \ldots, F_m)$ is a presentation over $R^{1/q}$. It follows that $\mathcal{J}_{B/R^{1/q}} = \mathcal{J}_{S/R} B$, and so $\mathcal{J}_{S/R}S^{1/q} \subseteq B$.

Now suppose that $c$ is any element in $S$ such that $cS^{1/q} \subseteq R^{1/q}[S]$ for all $q = p^n$. If $s \in I^s$, there exists $d \in S - \{0\}$ such that $ds^q \in I^q$ for all $q \gg 0$. Every nonzero element of $S$ has a multiple in $R$. Hence, we may replace $d$ by a nonzero multiple in $R$ and henceforward assume that $d \in R - \{0\}$. Taking $q$th roots yields $d^{1/q} \in IS^{1/q}$ and hence $cd^{1/q} \in (cS^{1/q}) \subseteq IB$, where $B = R^{1/q}[S]$. Then $d^{1/q} \in IB :_B cs = IB :_B csB = (I :_S cs)B$ since $d^{1/q} \in B$ and $B$ is $S$-flat, and this holds for all $q \gg 0$. But then $d \in (I :_S cs)^q$ for all $q \gg 0$. If $I :_S cs$ is a proper ideal of $S$, this is impossible: if we local at maximal ideal containing $I :_S cs$, $d \neq 0$ will be in every power of the maximal ideal of the local ring obtained. Hence, $I :_S cs = S$, i.e., $cs \in I$. \[\square\]