Math 615, Winter 2022 **Problem Set #1: Solutions**

1. Let $p = 3h+2$, $h \in \mathbb{N}$. Then $z^{2p} = z^{6h+4} = (z^3)^{2h+1}z = -(x^3+y^3)^{2h+1}z$. Each term in the binomial theorem expansion of $(x^3+y^3)^{2h+1}$ is a multiple of $(x^3)^{a_i}(y^3)^{b_i}$ where $a_i, b_i \in \mathbb{N}$ and $a_i+b_i = 2h+1$. If $3a_i < p$ and $3b_i < p$, then $3a_i+3b_i \le p-1+p-1 = 6h+2$, But $3(a_i+b_i) = 6h+3$, a contradiction. It follows that each term is in (x^p, y^p), and, hence, so is the sum.

2. Following the suggestion, if $u \in M \setminus \{0\}$ is killed by power of $\mathfrak{m}$, choose t is large as possible such that $\mathfrak{m}^t u \neq 0$. Any nonzero element of $\mathfrak{m}^t u$ is killed by $\mathfrak{m}$. Take $M = R/I$. If $u \in I^* \setminus I$, choose a multiple v of u that reprsents an element of the socle in R/I but is not 0. Then $v \in I^* \setminus I$.

3. As indicated, it suffices to show that when $\dim(R) = 1$ the socle V mod xR is isomorphic to the socle W mod xyR when x, y are parameters (equivalently, nonzerodivisors). Note: When y is a nonzerodivisor and $yA \subseteq yB$ then $A \subseteq B$. We show multiplication by y gives the needed isomorphsism. If $v \in R$ represents an element of V then $\mathfrak{m}v \subseteq xR$ implies that $\mathfrak{m}yv \subseteq xyR$. The map is injective because if $yv - yv' = xyr$ then $y(v - v' - xr) = 0$ and $v \equiv v'$ in R/xR. Now suppose $w\mathfrak{m} \in xyR$. Then $x \in \mathfrak{m}$, and $wx = xyr$, so that $x(w - yr) = 0$ and $w = yr$. To show the map surjective, it suffices to show $r\mathfrak{m} \subseteq xR$. But $w\mathfrak{m} \subseteq xyR$ so that $yr\mathfrak{m} \subseteq xyR$, and, hence, $r\mathfrak{m} \subseteq xR$. $\square$

4. This is immediate from **2.**

5. If $u \in (N : f)^*_M$ we have $c \in R^\circ$ such that $cu^q = \sum_{j=1}^{n_q} r_{qj} u_{qj}^q$ for all $q \ge q_0$, where every $u_{qi} \in N :_R f$ and so satisfies $fu_{qi} \in N$. If we multiply by f^q we have $c(fu)^q = \sum_{j=1}^{n_q} r_{qi}(fu_{qj})^q \in N^{[q]}$ for $q \ge q_0$. Since $N = N^*$, $fu \in N$ and $u \in N :_R f$, as required. $\square$

6. If $f \in J$ then since $JM = IM$ we can write $ju = \sum_{i=1}^n g_i m_i$ with all $g_i \in I$ and all $m_i \in M$. Applying the e-fold iterated composition F^e, we obtain $j^q u = \sum_{i=1}^n g_i^q F^e(m_i)$, since $F^e(u) = u$. Applying the R-linear map θ now yields $cj^q = j^q\theta(u) = \sum_{i=1}^n g_i^q \theta\big(F^e(m_i)\big) \in I^{[q]}$ for all q, and so $j \in I^*$. $\square$

EC1. and 2. The identification is immediate from the adjointness. Note also that the value of $^\vee$ on $K = A/\mathfrak{m}$ is K (since it is the same as $\mathrm{Hom}_K(\_\,, K)$ and that the value on $(A/\mathfrak{m})^t$ is an isomorphic module. For **EC2.**, let $a_1, \ldots, a_n$ generate $\mathfrak{m}$. Let $\alpha : M \to M^n$ by $m \mapsto (a_1 m, \ldots, a_n m)$. We have

$$0 \to \mathrm{Ann}_M \mathfrak{m} \to M \xrightarrow{\ \alpha\ } M^n$$

is exact. When we apply $\_^\vee$ we get an exact sequence

$$(M^\vee)^n \xrightarrow{\ \alpha^\vee\ } M^\vee \to (\mathrm{Ann}_M \mathfrak{m})^\vee \to 0,$$

where the dual map $\alpha^\vee$ takes $(m_1, \ldots, m_n) \mapsto \sum_{i=1}^n a_i m_i$. Thus, cokernel $\alpha^\vee$ is $M^\vee/\mathfrak{m}M^\vee$, and so $(\mathrm{Ann}_M \mathfrak{m})^\vee \cong M^\vee/\mathfrak{m}M^\vee$. The two modules are vector spaces of the same dimension. Since $M^{\vee\vee} \cong M$, one may interchange the roles of M and $M^\vee$. Finally, A and $A^\vee$ alwaays have have the same length. We have that A is Gorenstein if and only if it is type one if and only if $A^\vee$ is cyclic, and has the form A/J. But A and A/J caannot have the same length unless $J = (0)$. Thus, $A^\vee$ is cyclic if and only if $A^\vee \cong A$. $\square$

For **EC1** let the type be t. By **EC2**, $M^\vee$ has t generators and we may map $A^t \twoheadrightarrow M^\vee$. Applying the exact functor $^\vee$ again, we obtain an injection $M^{\vee\vee} \hookrightarrow \mathrm{Hom}_A(A^t, E)$. But $M \cong M^{\vee\vee}$ and $\mathrm{Hom}_A(A^t, E) \cong E^t$. $\square$