1. (a) Every finitely generated submodule of C is killed by the product of elements killing its generators, and C is the direct limit of finitely generated submodules, and so it suffices to prove the result when C is killed by one element of R; this element kills all the Tors, which was shown in class.
(b) The long exact sequence for Tor yields

$$\cdots \to \mathrm{Tor}_1^R(M, C) \xrightarrow{\ \alpha\ } M \otimes_R A \xrightarrow{\ \beta\ } M \otimes_R B \to \cdots.$$

Since every element in the domain of α is a torsion element, this is also true for its image. Since $M \otimes_R A$ is a torsion-free, the image of α must be 0, and so β is injective. □

2. (a) The test ideal is evidently stable under all ring automorphisms, and so must be homogeneous with respect to the $\mathbb{N}^n$ grading inherited from the polynomial ring. This means that the test ideal is spanned over K by (and so generated by) monomials.
(b). If c is a test element, it follows that $I^* = \bigcup_q \{r \in R : r^q \in I^{[q]}\}$. Each ideal being intersected is easily seen to be monomial, and, hence, so is the intersection. If $I = (\nu_1, \ldots, \nu_k)$, a monomial μ is in I^* iff and only if $c\mu^q \in (\nu_1^q, \ldots, \nu_k^q)$ for all q, and this implies that for every q there exists i_q such that $c\mu^q \in \nu_{i_q}^q$. Since there are only finitely many choices for i_q, some ν_i recurs for infinitely many q. But then there are infinitely many values of q such that $c\mu^{\in}(\nu R)^q$, where $\nu = \nu_i$. By a class result, this implies that μ is in the integral closure of νR. Note that by a class theorem, integral closure and tight closure are the same for principal ideals. We have that μ satisfies an equation of the form $\mu^s + r_1\nu\mu^{s-1} + \cdots + r_s\nu^s = 0$, and we may take $s \geq 1$ as small as possiblre, We may replace each r_i by its monomial term g_i such that $g_i\nu^i\mu^{s-i}$ (which is a monomial) is a scalar times μ^s. Since s is as small as possible we may assume that $r_s \neq 0$ is a monomial term, or we get a lower degree equation by dividing by a power of μ. Thus, if μ is in the integral closure of νR, then $(\mu/\nu)^s$ is a scalar times r_s, and so is in the ring. On the other hand, if $(\mu/\nu)^s = r$, then $\mu^s - r\nu^s = 0$ shows that μ is integral over νR, and so in $(\nu R)^*$. □

3. We have that $\mathrm{syz}^1 K = \mathfrak{m}$. $\mathrm{syz}^1\mathfrak{m}$ is given by the relations on x, y, which minimal generate $\mathfrak{m}$. But polynomals (f, g) represent a relation if and only if $fx + gy = hx^2 + jxy$ for polynomials $h, j \in K[x, y]$. Since x divides the other three terms, it divides gy and so $g = xg_0$. Dividing by x in the polynomial ring yields that $f + g_0 y = hx + jy$, so that $f = hx + (j - gz - 0)y$. Thus, the condition $f \in \mathfrak{m}$ and $g \in xR$ are necessary aand sufficient for (f, g) to be a relation, and $xR \cong R/\mathrm{Ann}_R x = R/\mathfrak{m} \cong K$. That is $syz^1\mathfrak{m} \cong \mathfrak{m} \oplus K$. It is straightforward to check that $\mathrm{syz}^1(M \oplus N) \cong \mathrm{syz}^1 M \oplus \mathrm{syz}^1 N$. Hence, if $\mathrm{syz}^n K \cong K^{\oplus a_n} \oplus \mathfrak{m}^{\oplus b_n}$, we have $\mathrm{syz}^{n+1} K \cong \mathfrak{m}^{a_n} + (K \oplus \mathfrak{m})^{\oplus b_n} \cong K^{b_n} \oplus \mathfrak{m}^{a_n + b_n}$. We therefore have $a_0 = 1, b_0 = 0, a_1 = 0, b_1 = 1, a_{n+1} = b_n$ and $b_{n+1} = a_n + b_n = b_{n-1} + b_n$. Thus, b_n is the Fibonacci sequence $\{f_n\}_n$ and $a_n = b_{n-1}$. Thus $\mathrm{syz}^n(K) = K^{f_{n-1}} + \mathfrak{m}^{f_n}$.

Comment for the remaining problems. Note that if $N \subseteq M$ are R-modules, $q = p^e$, and $q' = p^{e'}$, then $\mathcal{F}^{e'}\big(\mathcal{F}^e(M)\big) \cong \mathcal{F}^{e+e'}(M)$. Also note that $N^{[q]} \subseteq F^e(M)$ and we have that $(N^{[q]})^{[q']} \subseteq F^{e'}\big(F^e(M)\big)$ may be identified with $N^{[qq']} \subseteq \mathcal{F}^{e+e'}(M)$.

4. If $cu^Q \in N^{[Q]}$ for all $Q \geq Q_0$, then $c^q u^{qQ} \in (N^{[Q]})^{[q]} = N^{[qQ]} = (N^{[q]})^{[Q]}$ for all $Q \geq Q_0$, and so using c^q for the test shows that $u^q \in (N^{[q]})^*$ □

5. (a) If $u^q \in I^{[q]}$, then by taking Q/q powers one has $1 \cdot u^Q \in I^{[Q]}$ for all powers Q of p greater than or equal to q. $\square$
(b) When R is reduced, the ring map $R^p \subseteq R$ is isomorphic with $R^{pq} \subseteq R^q$ for every $q = p^e$. The composition $R^{p^e} \subseteq R^{p^{e-1}} \subseteq \cdots \subseteq R^{p^2} \subseteq R^p \subseteq R$ shows that for all $q \to \mathfrak{p}^e$, every ideal of R^q is contracted from R . I.e., ideals of R^q are contracted from R for all $q = p^e$ iff ideals of R^p are contracted from R. If I is an ideal of R, this says that if we extend the corresponding ideal $I_q = \{i^q : i \in I\} \subseteq R^q$ to R and contract, we get I_q. Since the extension of I_q to R is precisely $I^{[q]}$, this says exactly that $r^q \in I^{[q]}$ implies $r^q \in I_q$ and so $r \in I$. $\square$

6. Suppose $c^n \in \tau(R)$. Then so is c^q for $q = p^e \geq n$. If $N \subseteq M$ are finitely generated and $u \in N^*_M$ then $c^q u^Q \in N^{[Q]}$ for all Q. For any $q' = p^{e'}$ let $Q = qq'$. Then we have $c^q u^{qq'} \in N^{[qq']}$ and so which yields $(*)$ $\quad (cu^{q'})^q \in N^{[qq']} \cong (N^{[q']})^{[q]}$. The fact that F is split implies that F^e is split, i.e. the map $R \to S = R$ given by F^e is split. Let θ be the R-linear splitting. Let $v = cu^{q'}$. Let $W = N^{[q']}$. Then what we know from $(*)$ is that $1 \otimes v$ is in the image $S \otimes_R W$, i.e., $1 \otimes v = \sum_{i=1}^h s_i \otimes w_i$. Then θ gives a map $S \otimes_R M \to R \otimes_R M \cong M$ whose restriction to $1 \otimes_R M$ is the identity, and this yields $v = \sum_{i=1} \theta(s_i) w_i \in W$. This says that $cu^{q'} \in N^{[q']}$ for all q', N, M, and $u \in N^*_M$. Thus, $c^n \in \tau(R)$ implies that $c \in \tau(R)$. $\square$

EC3. The length needed is the number of monomials in R of even degree not in the ideal $(x_1^{2q}, x_1^q x_2^q, x_2^{2q})$. (We can work in the polynomial ring, because killing a power of each variable automatically gives a local ring.) These are the terms $x^a y^b$ such that $a < 2q$, $y < 2q$, at least one of a, b is $< q$, and $a + b$ is even. The number we want is the number of choices of (a, b) with even sum such that either $a < q$ and $b < 2q$ or $q \leq a < 2q$ and $0 \leq b < q$. (This is the number of lattice points in a $2q$ by $2q$ square with the upper right q by q corner cut out, but with the restriction that the sum of the coordinates is even.) Without the parity restriction on the sum, this would be $2q^2 + q^2 = 3q^2$. In this counting problem, once a is fixed, approximately half the choices for b are eliminated (one gets every other lattice point on a certain line segment) with an error of at most 1. Hence, the error is bounded by a constant times q, say Cq. Thus, the number of choices is $3q^2/2$ with an error bounded by Cq. Once we divide by q^2, the error term will approach 0 as $q \to \infty$. Hence, the limit is $3/2$.

EC4. For the first statement consider a filtration of M with $L = \ell_R(M)$ factors isomorphic with $K = R/\mathfrak{m}$. When we apply $S \otimes_R \_$ with S flat we get a filtration with L factors, each of which is ismorphic with $S \otimes_R K \cong S/\mathfrak{m}S$. The statement in the problem follows. If R is regular, F^e is flat, and and the extension of $\mathfrak{m}$ is $\mathfrak{m}^{[q]}$, so that the term $S/\mathfrak{m}S$ in the first part corresponds to $\overline{R} = R/\mathfrak{m}^{[q]}$. Thus, it suffices to show that the length of $\overline{R}$ is q^d. Totally order the d-tuples of elements of $\mathbb{N}$ in $[0, q-1]^d$ so that $(a_1, \ldots, a_d) < (b_1, \ldots, b_d)$ iff for the least i such that $a_i \neq b_i$ we have $a_i > b_i$. Use this ordering of exponent vectors to totally order the corresponding monomials $x_1^{a_1} \cdots x_d^{a_d}$. Call the monomials $\mu_1 < \cdots < \mu_i < \cdots < \mu_d$ where $\mu_1 = x_1^{q-1} \cdots x_d^{q-1}$ and $\mu_{q_d} = 1$. Then

$$0 \subset (\mu_1)\overline{R} \subset \cdots \subset (\mu_1, \ldots, \mu_i)\overline{R} \subset \cdots \subset (\mu_1, \ldots, \mu_{q^d})\overline{R} = \overline{R}$$

filters $\overline{R}$ so that each factor is cyclic, nonzero, and killed by $\mathfrak{m}$ (increasing one exponent by one produces either an exponent of q or a monomial that is smaller in the total ordering), i.e., each factor is $\cong K$. This shows that the length of $\overline{R}$ is q^d. $\square$