Math 615, Winter 2022 **Problem Set #5**
Due: Tuesday, April 19, 2022

1. Let R be a Noetherian domain of prime characteristic $p > 0$ and $I \subseteq R$ and ideal. Let S be a module-finite extension domain of R. Prove that $(IS)^*_S \cap R = I^*$.

2. (a) Given a finitely generated $\mathbb{N}$-graded K-algebra R over a field K with $R_0 = K$, show that the $E_R(K)$ is isomorphic to $\mathrm{Hom}^*_K(R, K)$, which we define to be the set of linear functionals on R each of which vanishes on the graded components $[R]_n$ of R for all $n \gg 0$ (as a K-vector space, this is $\bigoplus_{n=0}^{\infty} \mathrm{Hom}_K([R]_n,\, K)$.)

(b) Let $K[w, x, y, z]$ and $S = K[r, s, t, u]$ be polynomial rings over a field K. Let $R := K[w, x, y, z]/(wz - xy)$. You may assume that $R \cong R' := K[rt, ru, st, su] \subseteq S$ under the K-algebra map such that w, x, y, z map to rt, ru, st, su, respectively. Think of the injective hull $E_S(K)$ of $K := S/(r, s, t, u)$ over S as the K-span of the strictly negative monomials in r, s, t, u. Show that $E_R(K)$ is isomorphic with the R'-submodule E_0 of $E_S(K)$ spanned over K by the strictly negative monomials μ^{-1}, where μ is a monomial of positive degree in R'.

3. Let $(R, \mathfrak{m}, K)$ be local and let M be any R module. Let $E := E_R(K)$ be the injective hull of the residue class field. Prove that:

(a) for every R-module $M \neq 0$, $\mathrm{Hom}_R(M, E) \neq 0$.

(b) for every R-module M, the map $M \to \mathrm{Hom}_R\big(\mathrm{Hom},_R(M,\, E),\, E\big)$ is injective.

4. Let $(R, \mathfrak{m}, K) \to (S, \mathfrak{n}, L)$ be a local homomorphism such that $\dim(S/\mathfrak{m}S) = 0$. Assume R is Cohen-Macaulay, and that $M \neq 0$ is finitely generated over S and R-flat.

(a) Prove that M is Cohen-Macaulay over S.

(b) Prove that the type of M is the product of the types of R and $M/\mathfrak{m}M$. [Suggestion: reduce to the case where $\dim(R) = 0$ and show that $\mathrm{Ann}_M\mathfrak{m} = (\mathrm{Ann}_R\mathfrak{m})M \cong (\mathrm{Ann}_R\mathfrak{m}) \otimes_R M$.]

5. Let R be essentially of finite type over a complete local ring $(A, \mathfrak{m})$ of characteristic $p > 0$. Let $K \subseteq A$ map onto $A/\mathfrak{m}$, and let Λ be a p-base for K. (You may assume that A is regular: see subsection 21.4 of the lecture notes.)

(a) Show that if R is Cohen-Macaulay then R^Γ is Cohen-Macaulay for all choices of $\Gamma \subseteq \Lambda$. (You may assume Γ is cofinite in Λ if you wish, but this does not matter.)

(b) Show that if R is a normal domain, then R^Γ is normal for all sufficiently small $\Gamma \ll \Lambda$. [You may use: if R is a domain and $V(J)$ is the singular locus, then R is normal if and only if $J = R$ or the depth of R on J is at least two.]

6. Let $(R, \mathfrak{m}, K)$ be a local Gorenstein ring, and let $E = E_R(K)$ be the injective hull. Show that if $E_1 \supset E_2 \supset \cdots \supset E_n$ is a strictly decreasing chain of submodules of E, then $\mathrm{Ann}_{\widehat{R}}E_n$ is a strictly increasing chain of ideals of $\widehat{R}$. Hence, E has DCC. (Since the injective hull of any local ring is the same when one completes and, in the complete case, the injective hull is a submodule of the injective hull of the residue class field of a regular local ring, the injective hull of the residue class field of every local ring has DCC.)

Extra Credit 9. Assume that if $M \neq 0$ is any finitely generated Cohen-Macaulay module over a regular local ring $(R, \mathfrak{m}, K)$ with $\dim(R) = n$ and $I = \mathrm{Ann}_R M$, then the projective dimension dimension of M is $h = \mathrm{height}(I)$ (which is the same as the depth of R on I). You may also assume that all minimal primes of I have the same height. Prove that $\_^{\vee} := \mathrm{Ext}^h_R(\_, R)$ is an exact contravariant functor from Cohen-Macaulay R-modules of dimension $d := n - h$ to Cohen-Macaulay modules of dimension d. Prove that $M^{\vee\vee} \cong M$, and that M and $M^{\vee}$ have the same annihilator. Prove that if x is a nonzerodivisor on M, then $(M/xM)^{\vee} \cong M^{\vee}/xM^{\vee}$. Also show that if $P \supseteq \mathrm{Ann}_R M$ is a prime ideal of R, then $(M_P)^{\vee} \cong (M^{\vee})_P$. The module $M^{\vee}$ is called the *Ext dual* of the Cohen-Macaulay module M.

Extra Credit 10. Keep the notation of the preceding problem.

(a) Show that $K^{\vee} \cong K$.

(b) Show that if M is killed by a power of $\mathfrak{m}$, then M and $M^{\vee}$ have the same length.

(c) Show that the type of any Cohen-Macaulay module M over R is the least number of generators of the Ext-dual. [Show that this reduces to the case where $\dim(M) = 0$.]