Exact Sequences with a flat cokernel and a sketch of properties of Tor

We give two arguments to prove the following fact:

Lemma. *Let* $0 \to N \to M \to F \to 0$ *be an exact sequence of* R*-modules such that* F *is* R*-flat. Then for every* R*-module* A*,* $0 \to A \otimes_R N \to A \otimes_R M \to A \otimes_R F \to 0$ *is exact.*

The first proof is completely elementary. The second uses properties of Tor, and is quite straightforward if one is familiar with Tor. Prior to giving the argument, we give a brief treatment of properties of Tor.

Elementary proof. Map a free R-module G onto A, and let B denote the kernel, so that $0 \to B \to G \to A \to 0$ is exact. Think of this exact sequence as written as a column. We may tensor the row and column together to obtain a double array (all tensor products are over R) in which the squares commute:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & B \otimes N & \longrightarrow & B \otimes M & \longrightarrow & B \otimes F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \alpha & & \\
0 & \longrightarrow & G \otimes N & \xrightarrow{g} & G \otimes M & \longrightarrow & G \otimes F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A \otimes N & \xrightarrow{f} & A \otimes M & \longrightarrow & A \otimes F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By the right exactness of tensor product, the rows are exact except at the left and the columns are exact except at the top. However, because G is free, the middle row is exact, and because F is flat, the rightmost column is exact. We want to show that f is injective. Suppose that $u \in \mathrm{Ker}\,(f)$. Pick $v \in G \otimes N$ that maps to u. Let w be ithe image of v in $G \otimes M$. Then w maps to 0 in $A \otimes M$, since the image of w is the image of v, and if we use the maps forming the other two sides of the square, we get $v \mapsto u \mapsto f(u) = 0$. Hence, w is the image of some $x \in B \otimes M$. Let y be the image of w in $B \otimes F$. We claim that $y = 0$. Since α is injective, it suffices to show that the image z of y in $G \otimes F$ is 0. If we use the other two sides of the square, we see that this is the same as the image of w. Since w is the image of v, this is 0. Since $y = 0$, x is the mage of an element x' in $B \otimes N$. Let v' be the image of x in $G \otimes N$. Then v and v' both map to w in $G \otimes M$, so that $v - v' \mapsto 0$. Since g is injective, we have that $v = v'$. But then u is the image of v', and so it is the image of x'. Hence, $u = 0$. $\square$

For those familiar with Tor, we give a different argument. First note that when C, D are R-modules the R-modules $\mathrm{Tor}_i^R(C,\,D)$ are defined for all integers i. The superscript R is frequently omitted when the base ring is understood from context. For negative i, they vanish, while $\mathrm{Tor}_0(C,\,D) = C \otimes D$. There is a canonical isomorphism $\mathrm{Tor}_i(C,\,D) \cong \mathrm{Tor}_i(D,\,C)$ that generalizes the symmetry of the tensor product. When D is flat and, hence, when C is flat, $\mathrm{Tor}_i(C,\,D) = 0$ for all $i \geq 1$. Moreover, if $0 \to C' \to C \to C'' \to 0$ is exact, there is a (functorial) long exact sequence:

$$\cdots \to \mathrm{Tor}_i(C',\,D) \to \mathrm{Tor}_i(C,\,D) \to \mathrm{Tor}_i(C'',\,D) \to \mathrm{Tor}_{i-1}(C',\,D) \to \cdots$$

$$\to \mathrm{Tor}_1(C',\,D) \to \mathrm{Tor}_1(C,\,D) \to \mathrm{Tor}_1(C'',\,D) \to C' \otimes D \to C \otimes D \to C'' \otimes D \to 0.$$

We only need the rightmost four terms of the long exact sequence, which we may utilize in the case where $C'' = F$, $C' = N$, $C = M$, and $D = A$. The conclusion we want is immediate. $\square$

We make the following further comments about Tor. If C is held fixed, $\mathrm{Tor}_i(C,\,\_)$ is a covariant functor form R-modules to R-modules. The same holds for $\mathrm{Tor}_i(\_,\,D)$ when D is held fixed. If R is Noetherian and C, D are Noetherian, so are all the modules $\mathrm{Tor}_i(C,\,D)$.

If $r \in R$, the map given by multiplication by r on C induces the map given by multiplication by r on $\mathrm{Tor}_i(C,\,D)$, and the same applies for the map given by multiplciation by r on D. It follows that $\mathrm{Tor}_i(C,\,D)$ is killed by the sum of the annihilators of the modules C and D.

One way of defining Tor is as follows. If A is an R-module, one can construct a left free resolution of A as follows. Map a free R-module onto A, call it P_0. Then map a free R-module P_1 onto $\mathrm{Ker}\,(P_0 \twoheadrightarrow A)$. This yields a sequence $P_1 \to P_0 \to A \to 0$ exact except at P_1. Recursively, if one has an exact sequence $P_i \to P_{i-1} \to \cdots P_1 \to P_0 \to A \to 0$, one may extend the resolution further by mapping a free module P_{i+1} onto $\mathrm{Ker}\,(P_i \to P_{i-1})$. This yields a (very possibly infinite) free resolution $\cdots \to P_n \to P_{n-1} \cdots \to P_1 \to P_0 \to A \to 0$ which is exact. Let $P_\bullet$ be the sequence obtained by replacing A by 0. This sequence is exact except at P_0. It is still a complex, i.e., the composition of any two consecutive maps is 0. One gets a new complex $P_\bullet \otimes_R D$ by applying $\_ \otimes_R D$, namely $\cdots \to P_n \otimes_R D \to P_{n-1} \otimes_R \cdots \to P_1 \otimes_R \to P_0 \otimes_R \to 0$.

Given any complex $G_\bullet$, say $\cdots G_n \to G_{n-1} \cdots$ we may define its homology $H_n(G_\bullet)$ at the n th spot as Z_n/B_n, where $Z_n = \ker(G_n \to G_{n-1})$ and $B_n = \mathrm{Im}\,(G_{n+1} \to G_n$. Then $\mathrm{Tor}^n(C,\,D) = H_n(P_\bullet \otimes_R D)$. This turns out to be independent of the free resolution up to canonical isomorphism. In fact, one may use any projective resolution or even a resolution by flat modules to calculuate Tor.