Commutative Algebra Seminar: Lecture of February 23, 2006

We shall say that a complex $G' \xrightarrow{f} G \xrightarrow{g} G''$ is *split exact* at G if there are maps $\beta : G'' \to G$ and $\alpha : G \to G'$ such that $f \circ \alpha + \beta \circ g = \mathrm{id}_G$.

A finite exact sequence of projective modules

$$0 \to P_n \to \cdots \to P_0 \to 0$$

is split exact exact at each spot. To see why, we first note that P_i is the direct sum the image B_i of P_{i+1} and a module C_i (once we know this, we also have that C_i is mapped isomorphically onto B_{i-1}, and it follows as well that B_i and C_i are both projective). We can prove this assertion by induction on the length of the complex. The case where $n \leq 1$ is obvious. Suppose that $n \geq 2$. The surjection $P_1 \twoheadrightarrow P_0$ is split, since P_0 is projective, and so we can write

$$P_1 = B_1 \oplus C_1,$$

where B_1 is the image of $P_2 \to P_1$, and C_1 maps isomorphically onto $P_0 = B_0$. We may take $C_0 = 0$. Then B_1 and C_1 are projective, and our claim follows from the induction hypothesis and the fact that

$$0 \to P_n \to \cdots \to P_2 \to B_1 \to 0$$

is a shorter exact sequence of projective modules.

Let d_i be the map $P_i \to P_{i-1}$ for each i. Then $d_i : B_i \oplus C_i \to B_{i-1} \oplus C_{i-1}$ is the direct sum of the map $B_i \to 0$ and an isomorphism $\gamma_i : C_i \to B_{i-1}$ composed with the injection $B_{i-1} \hookrightarrow B_{i-1} \oplus C_{i-1}$. We take $\alpha_i : P_i \to P_{i+1}$ to be the map that is 0 on B_i and is the composition of γ^{-1} with the inclusion $C_{i+1} \hookrightarrow B_{i+1} \oplus C_{i+1}$ on B_{i-1}. It is perfectly straightforward to verify that $\mathrm{id}_{G_i} = d_{i+1} \circ \alpha_i + \alpha_{i-1} \circ d_i$, since the first summand is 0 on C_i and the identity on B_i while the second summand is 0 on B_i and the identity on C_i.

We next want to discuss the situation when the modules in the sequence $G' \xrightarrow{f} G \xrightarrow{g} G''$ are Noetherian and the sequence becomes split exact at G_u after one localizes at an element u of the base ring A. We write $\widetilde{h}$ for the map of localizations over A_u induiced by an A-linear map h. Then there are linear maps $\alpha_0 : G''_u \to G_u$ and $\beta_0 : G_u \to G'_u$ such that $\widetilde{f} \circ \alpha_0 + \beta_0 \circ \widetilde{g} = \mathrm{id}_{G_u}$. Since $\mathrm{Hom}_A(G, G'')_u \cong \mathrm{Hom}_{A_u}(G_u, G''_u)$ and $\mathrm{Hom}_A(G', G)_u \cong \mathrm{Hom}_{A_u}(G'_u, G_u)$ for $s_1 \gg 0$ we can write $\alpha_0 = u^{-s_1}\alpha_1$ and $\beta_0 = u^{-s_1}\beta_1$, where $\alpha_1 \in \mathrm{Hom}_A(G, G'')$ and $\beta_1 \in \mathrm{Hom}_{A_u}(G_u, G''_u)$. It follows that $f \circ \alpha_1 + \beta_1 \circ g - u^{s_1}\mathrm{id}_G \in \mathrm{Hom}_A(G, G)$ vanishes upon localization at u, and so is killed by u^t for some $t \gg 0$. Then with $\alpha = u^t\alpha_1$, $\beta = u^t\beta_1$, and $s = s_1 + t$, we have that

$$(\#_v) \qquad f \circ \alpha + \beta \circ g = v\,\mathrm{id}_G$$

with $v = u^s$.

We want to consider the condition given by $(\#_v)$. Of course, if $v = 1$, this is simply the condition that the sequence be split exact at G. Condition $(\#_v)$ implies that v kills the homology at G: if z is a cycle, i.e., if $g(v) = 0$, we have that

$$vg = v\mathrm{id}_g(v) = f\big(\alpha(v)\big) + \beta\big(g(v)\big) = f\big(\alpha(v)\big)$$

is a boundary. If one is in positive prime characteristic p and applies the iterated Frobenius functor F^e, one has

$$F^e(f) \circ F^e(\alpha) + F^e(\beta) \circ F^e(g) = v^{p^e} \mathrm{id}_{F^e(G)},$$

so that condition $(\#_{v^{p^e}})$ holds at $F^e(G)$ for $F^e(G') \to F^e(G) \to F^(G'')$. If $\_^\vee$ denotes the functor $\mathrm{Hom}_A(\_,\, A)$, we have that

$$g^\vee \circ \beta^\vee + \alpha^\vee f^\vee = v\,\mathrm{id}_{G^\vee},$$

which shows that $(\#_v)$ holds for ${G''}^\vee \to G^\vee \to {G'}^\vee \to 0$.

With these preliminary remarks, we can now prove:

Lemma. *Let A be a local ring of characeristic $p > 0$ and let M be an A-module of dimension k. Let $G' \xrightarrow{f} G \xrightarrow{g} G''$ be a complex of finitely generated free A-modules which is split exact at G if we localize at any element in the maximal ideal of A. For every e, let $H^{(e)}(M)$ denote the homology of the complex*

$$\mathrm{Hom}_A\big(F^e(G''),\, M\big) \to \mathrm{Hom}_A\big(F^e(G),\, M\big) \to \mathrm{Hom}_A\big(F^e(G'),\, M\big)$$

at the middle spot. Then there is a positive constant C such that

$$\ell\big(H^e(M)\big) \le Cp^{ke}$$

for all e.

Proof. It will be convenient to think in terms of the longer complex

$$0 \to G' \to G \to G'' \to 0$$

for the purpose of applying the snake lemma. We use induction on the dimension of M. If we have a short exact sequence of modules $0 \to M' \to M \to M'' \to 0$ we get a short exact sequence of complexes

$$0 \to \mathrm{Hom}_A\big(F^e(G_\bullet),\, M\big) \to \mathrm{Hom}_A\big(F^e(G_\bullet),\, M\big) \to \mathrm{Hom}_A\big(F^e(G_\bullet),\, M\big) \to 0,$$

and the snake lemma gives a long exact sequence for homology which includes

$$H^{(e)}(M') \to H^{(e)}(M) \to H^{(e)}(M'').$$

It follows that
$$\ell\big(H^{(e)}(M)\big) \le \ell\big(H^{(e)}(M')\big) + \ell\big(H^{(e)}(M'')\big).$$

By induction on the length s of a filttration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M$, one has that
$$\ell\big(H^{(e)}(M)\big) \le \sum_{i=1}^{s} \ell\big(H^{(e)}(M_i/M_{i-1})\big).$$
Since M has a finite filtration in which the factors are prime cyclic modules of at most the same dimension as M, we may reduce to the case where $M = A/P$ is prime cyclic module. For any A-algebra B, the commutativity of the diagram
$$\begin{array}{ccc} B & \xrightarrow{F^e} & B \\ \uparrow & & \uparrow \\ A & \xrightarrow{F^e} & B \end{array}$$
and the associativity of tensor gives and isomorphism of functors $F^e_B(B \otimes_A \_\,) \cong B \otimes F^e_A(\_\,)$ from A-modules to B-modules. Moreover, taking $B = A/P$, we have that $\mathrm{Hom}_A\big(F^e(G_\bullet),\, B\big)$ may be identified with
$$\mathrm{Hom}_A\big(B \otimes_A F^e_A(G_\bullet),\, B\big) \cong \mathrm{Hom}_B\big(F^e_B(B \otimes_A G_\bullet),\, B\big).$$
Thus, we may replace $G_\bullet$ by $B \otimes_A G_\bullet$ and A by B. This mean that we may assume without loss of generality that $M = A$ is a local domain, say of dimension n. Recall that we may assume from the induction hypothesis that the result holds for modules of dimension smaller than that of A.

If A is a field, the result is clear: the rank of G gives the required bound. If not, we may localize at nonzero element u of the maximal ideal of A. The complex becomes an exact sequence of free modules over A_u, and so we get maps α, β such that $f \circ \alpha + \beta \circ g = u^s \,\mathrm{id}_G = x\,\mathrm{id}_G$ for some nonzero element $x \in A$. Applying the eth iterate F^e of the Frobenius endomorphism, we have the same situation with $y = x^q$ replacing x. This remains the case when we apply $\mathrm{Hom}_A(\_\,,\, N)$ for any A-module N.

This implies that $y = x^q$ kills the homology no matter what module we tensor with. Then the long exact sequence associated with
$$0 \longrightarrow A \xrightarrow{y} A \longrightarrow A/yA \longrightarrow 0$$
gives
$$H^{(e)}(A) \xrightarrow{y} H^{(e)}(A) \longrightarrow H^{(e)}(A/yA),$$
and since multiplication by y is the zero map on $H^{(e)}(A)$, we obtain an embedding of $H^{(e)}(A)$ in $H^{(e)}(A/yA)$. Thus, $\ell\big(H^{(e)}(A)\big) \le \ell\big(H^{(e)}(A/yA)\big)$. Since $y = x^q$ and x is a nonzerodivisor on the domain A, A/yA has a filtration by q copies of A/xA. Hence, $\ell\big(H^{(e)}(A)\big) \le q\ell\big(H^{(e)}(A/xA)\big)$ for all q. By the induction hypothesis, we have $C > 0$ such that $\ell\big(H^{(e)}(A/xA)\big) \le Cp^{e(n-1)}$ for all e. Since $q = p^e$, we then have that $\ell\big(H^{(e)}(A)\big) \le qCq^{e(n-1)} = Cq^{en}$, as required. $\square$

The following result now completes our treatment of the proof of the new intersection theorem.

Lemma. *If $G_\bullet$ is any complex*

$$\cdots \to G_1 \to G_0 \to 0$$

over a local ring A of dimension n and prime characteristic $p > 0$ such that $H_0(G_\bullet)$ is nonzero of finite length, then $\ell\big(H_0\big(F^e(G_\bullet)\big)\big)/p^{ne}$ is positive and bounded away from 0.

Proof. Let $H_0(G_\bullet) = M$. Then $H_0\big(F^e(G_\bullet)\big) = F^e(M)$, by the right exactness of tensor. Since $M \neq 0$, we have a surjection $M \twoheadrightarrow K$, and so we have a surjection $F^e(M) \twoheadrightarrow F^e(K)$, and it suffices to prove that $\ell\big(F^e(K)\big)/p^{en}$ is bounded away from 0. But $F^e(K) = A/m^{[p^e]}$, which maps onto A/m^{p^e}, whose length is given by $H(p^e)$ for large e, where H is the Hilbert polynomial of A. This is a polynomial of degree n, with leading coefficient $\mu/n!$ for some positive integer μ. Thus, we have that $\ell\big(H_0\big(F^e(M)\big)\big)/p^{en}) \geq \ell(A/m^{p^e})/(p^e)^n$, and the limit of the right hand side is $\mu/n!$ as $e \to \infty$. $\square$