We next observe:

**Proposition.** Let \( R \) be a Cohen-Macaulay local ring of characteristic \( p \) and suppose that \( R \) is \( F \)-injective.

(a) For every prime ideal \( P \) of \( R \), \( R_P \) is Cohen-Macaulay and \( F \)-injective.

(b) If \( A \to R \) is a flat local homomorphism, then \( A \) is Cohen-Macaulay and \( F \)-injective.

**Proof.** For part (a), suppose that \( P \) has height \( k \). We can choose \( x_1, \ldots, x_k \in P \) that are part of a system of parameters for \( R \). Their images will be a system of parameters for \( R_P \). Now suppose that \( u \in R \) is such that \( u/1 \in (x_1^p, \ldots, x_k^p)R_P \) (we may assume that \( u \in R \), since every element of \( R_P \) is a unit times an element of \( R \)). Then we can choose \( w \in R - P \) such that \( wu^p \in (x_1^p, \ldots, x_k^p)R \), and it follows that \( (wu)^p \in (x_1^p, \ldots, x_k^p)R \) as well. But then \( wu \in (x_1, \ldots, x_k)R \), and so \( u \in (x_1, \ldots, x_k)R_P \). This proves part (a).

For part (b), note that we immediately know that \( A \) is Cohen-Macaulay. Let \( x_1, \ldots, x_k \) be a system of parameters for \( A \), and suppose that \( u \in A \) is such that \( u^p \in (x_1^p, \ldots, x_k^p)A \). Then the images of \( x_1, \ldots, x_k \) form part of a system of parameters for \( R \), and \( u^p \in (x_1^p, \ldots, x_k^p)R \) implies that \( u \in (x_1, \ldots, x_k)R \cap A = (x_1, \ldots, x_k)A \), as required, since \( R \) is faithfully flat over \( A \). \( \square \)

We can now prove:

**Theorem.** Let \( R \) be a Noetherian ring of positive prime characteristic \( p \), and suppose either that \( (R, m, K) \) is local or that \( R \) is finitely generated \( \mathbb{N} \)-graded over \( R_0 = K \), a field, and that \( m \) is the homogeneous maximal ideal. Let \( I \subseteq m \) be an ideal. Let \( M \) be the maximal ideal of \( \text{gr}_I R \) that is the kernel of the composite surjection \( \text{gr}_I R \to R/I \to R/m \), and suppose that \( (\text{gr}_I R)_M \) is Cohen-Macaulay \( F \)-injective. Then \( R \) is Cohen-Macaulay \( F \)-injective.

**Proof.** The argument is quite similar to the one given for the Cohen-Macaulay and Gorenstein properties in the Theorem on page 3 of the Lecture Notes for September 28. One forms the second Rees ring \( S = R[It, v] \), which maps onto \( S/vS \cong \text{gr}_I R \), and localizes at the contraction of \( M \), which we call \( Q \). Then \( S_Q/(v) \cong (\text{gr}_I R)_M \) is \( F \)-injective Cohen-Macaulay, and so \( S_Q \) is as well. Let \( P \subseteq Q \) be the prime described in the proof of the Theorem on page 3 of the Lecture Notes of September 28. By part (a) of the Proposition above, \( S_P \cong R(t) \) is Cohen-Macaulay \( F \)-injective, since it is a localization of \( S_Q \). Hence \( R \) is Cohen-Macaulay \( F \)-injective, by part (b) of the Proposition above. \( \square \)

**Corollary.** Let \( R \) be a Hodge algebra over a field \( K \) of characteristic \( p > 0 \), and suppose that the corresponding discrete Hodge algebra is Cohen-Macaulay and reduced: the condition that the discrete Hodge algebra be reduced holds whenever \( R \) is an ASL. Then \( R \) is Cohen-Macaulay and \( F \)-injective.

**Proof.** When it is reduced, the corresponding discrete Hodge algebra is a face ring, and we have seen that face rings over a field are \( F \)-split and therefore \( F \)-injective in characteristic
Let \( X = (x_{ij}) \) be an \( r \times s \) matrix of indeterminates, where \( 1 \leq r \leq s \), over a base ring \( K \), and let \( K[X/r] \) be the subring of the polynomial ring \( K[X] \) in the indeterminates generated by the \( r \times r \) minors of \( X \). As mentioned earlier, this is the homogeneous coordinate ring of the Grassmann variety of \( r \)-dimensional subspaces of \( K^s \). We want to prove that this ring is an ASL on the poset \( H \) of minors. We shall write \( X[a_1, \ldots, a_r] \) for the determinant of the matrix formed from the columns of \( X \) indexed by the integers \( a_1, \ldots, a_r \), which are required to be integers satisfying \( 1 \leq a_j \leq s \). In the standard description of a minor we shall assume that \( a_1 < a_2 < \cdots < a_r \). However, the symbol \( X[a_1, \ldots, a_r] \) has meaning in any case: if \( a_j = a_k \) for \( j \neq k \), then \( X[a_1, \ldots, a_r] = 0 \), and if \( \pi \) is a permutation of the integers \( 1, \ldots, r \), then \( X[a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(r)}] = \text{sgn} (\pi) X[a_1, \ldots, a_r] \), where \( \text{sgn} (\pi) \in \{\pm 1\} \) is the sign of the permutation \( \pi \). Recall that \( H \) is partially ordered so that when \( a_1 < a_2 < \cdots < a_r \) and \( b_1 < b_2 < \cdots < b_r \), \( X[a_1, \ldots, a_r] \leq X[b_1, \ldots, b_r] \) means that \( a_j \leq b_j \) for \( 1 \leq j \leq r \). The standard monomials are those such that the set of minors occurring is linearly ordered.

We first want to show that the standard monomials are linearly independent over \( K \). In order to prove this, we introduce several matrices \( Y_h \), one for each element \( h \in H \), the poset of minors. Specifically, let \( Y = (y_{ij}) \) be a matrix of indeterminates, and suppose that we are given \( h \in H \), say \( h = X[a_1, \ldots, a_r] \) where \( a_1 < a_2 < \cdots < a_r \). We define \( Y_h \) to be the matrix obtained from \( Y \) by replacing the \( a_i - 1 \) leftmost variables \( y_{i1}, \ldots, y_{ia_i-1} \) of the \( i \)th row by 0, while leaving all other entries of the \( i \)th row unchanged. Then there is a \( K \)-algebra homomorphism \( K[X] \to K[Y_h] \) that maps each entry of \( X \) to the corresponding entry of \( Y_h \): \( x_{ij} \mapsto 0 \) if \( j < a_i \), and \( x_{ij} \mapsto y_{ij} \) if \( j \geq a_i \). This map restricts to a map \( \theta_h : K[X/r] \to K[Y_h/r] \). Also note that if \( h \leq h' \), where \( h' = X[b_1, \ldots, b_r] \) with \( b_1 < \cdots < b_r \), then there is a \( K \)-algebra map \( K[Y_h] \to K[Y_{h'}] \) that sends \( y_{ij} \mapsto 0 \) if \( a_i \leq j < b_i \) and \( y_{ij} \mapsto y_{ij} \) if \( j \geq b_i \). Again, this induces a \( K \)-algebra homomorphism \( \lambda_{h,h'} : K[Y_h] \to K[Y_{h'}] \) when \( h \leq h' \), and it is clear that \( \lambda_{h,h'} \circ \theta_h = \theta_{h'} \).

Let \( \mathcal{M}_h \) denote the set of standard monomials that are \( \geq h \). We shall prove that for all \( h \in H \), the elements \( \{\theta_h(\mu) : \mu \in \mathcal{M}_h\} \) is a \( K \)-linearly independent set indexed by \( \mathcal{M}_h \). If we take \( h_0 = X[1, \ldots, r] \), the minimum element of \( H \), we find that the images of the standard monomials under \( \theta_{h_0} \) are linearly independent over \( K \), and it follows that the standard monomials themselves are linearly independent over \( K \).

We first note that \( \theta_h \) has the following critical property:

\[ (** \, \theta_h \text{ kills every minor } h' = X[b_1, \ldots, b_r] \text{ with } b_1 < \cdots < b_r \text{ such that } h' \text{ is not } \geq h. \]

The reason is that for some \( i \), we have that \( b_i < a_i \). This implies that the \( i \)th row of the matrix consisting of the columns of \( Y_h \) indexed by \( b_1, \ldots, b_i \) is 0, and so this matrix, which has \( i \) columns, has rank \( \leq i - 1 \). But then the \( r - i \) additional columns indexed \( b_{i+1}, \ldots, b_r \) can increase the rank at most to \( i - 1 + (r - i) = r - 1 \), and so \( Y_h[b_1, \ldots, b_r] = 0 \).

To prove the result, we use a sort of reverse induction on \( h \). Choose \( h \) maximal in \( H \) for which the result is false, and suppose there is nonzero \( K \)-relation on the images of certain standard monomials \( \mu_1, \ldots, \mu_n \): we may take these of smallest possible degree, and we may assume that every \( \mu_j \) occurs with nonzero coefficient.
We consider two cases. The first case is that each of the $\mu_j$ has $h$ as a factor and can be written $h\nu_j$. Note that $\theta_h(h) = y_{a_1} \cdots y_{a_r}$ is not a zerodivisor in $K[Y_h]$, nor in $K[Y_h/r]$. It follows that we get a $K$-relation on the elements $\theta_h(\nu_j)$, and the degrees have decreased.

Therefore we may assume that there is at least one element $\mu'$ that is not divisible by $h$: call its smallest factor $h'$. We now apply $\lambda_{h,h'}$ to this relation. This has the same effect as applying $\theta_{h'}$ to the original relation. This does not kill the term in the linear combination that is the image of a multiple of $\mu'$ with nonzero coefficient from $K$, but it does kill all terms that involve an element $h'' \in H$ that is not $\geq h'$ by property (**) proved above. This gives a nonzero relation on elements that are in the image of $\mathcal{M}_{h'}$ under $\theta_{h'}$, a contradiction. □

Our next objective is to describe the Plücker relations on the minors of a matrix. We assume that we are given nonnegative integers $a, t, u, b$ such that $a + t = r$, $u + b = r$, $t, u > 0$, and $t + u = m > r$. We also assume given indices $i_1, \ldots, i_a, j_1, \ldots, j_m, k_1, \ldots, k_b$. Let $\mathcal{N}$ denote the set of permutations $\nu$ of $1, \ldots, m$ such that, writing $\nu_c$ for $\nu(c)$, we have $j_{\nu_1} < \cdots < j_{\nu_t}$ and $j_{\nu_t+1} < \cdots < j_{\nu_m}$. Then

$$\sum_{\nu \in \mathcal{N}} \text{sgn}(\nu)X[i_1, \ldots, i_a, j_{\nu_1}, \ldots, j_{\nu_t}]X[j_{\nu_t+1}, \ldots, j_{\nu_m}, k_1, \ldots, k_b] = 0.$$ 

This is a typical Plücker relation. We shall prove the validity of these determinantal identities, and then show that they suffice to give straightening relations for $K[X/r]$. Note that in order to prove these relations, it suffices to do the case where the entries of the matrix $X$ are indeterminates over $\mathbb{Z}$, and then we may pass to the field of fractions $\mathbb{Q}(X)$ of $\mathbb{Z}[X]$. Therefore, it suffices to prove that these identities when the matrix has entries in a field $L$ of characteristic 0.