Math 711: Lecture of October 19, 2005

To prove that the Plücker relations hold, we think of the columns indexed by $i_1, \ldots, i_a$ and $k_1, \ldots, k_b$ as fixed. The left hand side may then be thought of as a function of m vector variables in K^r, namely the columns indexed by $j_1, \ldots, j_m$. We first show that if we let π range over all permutations of $1, \ldots, m$ (writing π_c for $\pi(c)$), we have that

$$(\#) \qquad \sum_{\pi \in S_m} \mathrm{sgn}\,(\pi) X[i_1, \ldots, i_a, j_{\pi_1}, \ldots, j_{\pi_t}] X[j_{\pi_{t+1}}, \ldots, j_{\pi_s}, k_1, \ldots, k_b] = 0.$$

To prove this we observe that the left hand side is multilinear in the m variable columns indexed by $j_1, \ldots, j_m$. It is also alternating: if two columns are equal, any summand in which they both occur in the first factor or in the second factor is 0, while for each term where one occurs in the first factor and one in the second factor, there is a corresponding term, which results from switching the relevant indices, which has the opposite sign. Therefore, the left hand side represents a map from $\bigwedge^m(K^r)$ to K, and is therefore 0, because $m > r$. We shall complete the proof that the Plücker relation holds by showing that the left hand side of (#) is $t!u!$ times the left hand side of the Plücker relation. The point is that for each t element subset T consisting of $j_{\nu_1} < \cdots < j_{\nu_t}$ in $\{1, \ldots, m\}$, there is a unique summand in the Plücker relation in which that subset occurs, but there are $t!u!$ permutations π such that the subscripts with those columns occur on the left. To bring the term involving π to the term involving ν, we need to apply the permutation $\nu \circ \pi^{-1}$. Note that π, ν and, hence, $\nu \circ \pi^{-1}$ all permute T and permute its complement T' in $\{1, \ldots, m\}$. The sign of $\nu \circ \pi^{-1}$ is the product of the signs of the separate permutations it induces on T and T'. Thus, all $t!u!$ terms can be written as $\mathrm{sgn}\,(\pi)\mathrm{sgn}\,(\nu \circ \pi^{-1}) = \mathrm{sgn}\,(\nu)$ times the product of the two minors in the term corresponding to ν, and so all of these terms are the same as the term in the Plücker relation. Thus, when we multiply the left hand side of the Plücker relation by $t!u!$, we get 0. Since we are working in equal characteristic 0, we have proved what we need. $\square$

We shall next prove that the Plücker relations can be used to write any product hh' of incomparable minors as a linear combination of standard quadratic monomials in which, for every term, the smaller of the two minors occurring is $< h$. This will suffice, because the same reasoning applied to $h'h$ then shows that it can be written in the same way with the smaller minor in each term $\le h'$. By the linear independence of standard monomials, these two reperesentations are actually the same. Therefore, the smaller minor in each term is less than both h and h'. (The argument actually shows as well that the larger minor in each terms is greater than both h and h'.)

Before giving the argument we note the following fact: if $i_1 < \cdots < i_r$, and one or more of these integers is replaced by a smaller integer (distinct from the remaining i_t), the new minor (with its columns arranged with indices in increasing order), is strictly less than $X[i_1, \ldots, i_r]$. It suffices to see this for just one replacement. Suppose that i_n is replaced by i, where $i_c < i < i_{c+1} \le i_n$. The i_j do not change for $j \le c$ or $j > n$, but i_{c+1} is replaced by i, decreasing, and i_j is replaced by i_{j-1} for $c+1 < j \leq n$. Entirely similarly, if one or more of the i_j is replaced by a larger integer (distinct from the remaining i_t), the new minor (with its columns arranged with indices in increasing order) is larger than h.

We are now ready to carry through the proof that Plücker relations suffice for straightening. Suppose that $h = X[i_1, \ldots, i_r]$ and $h' = [k_1, \ldots, k_r]$ give a counter-example, where $i_1 < \cdots < i_r$, $k_1 < \cdots < k_r$ and for some $a < r$ we have that $i_n \leq b_n$ for $1 \leq n \leq a$ while $i_{a+1} > k_{a+1}$ (a may be 0). Choose the counter-example with a as large as possible. Take $m = r+1$, and apply the Plücker relation for

$$X[i_1, \ldots, i_a, i_{a+1}, \ldots, i_r]X[k_1, \ldots, k_{a+1}, k_{a+2}, \ldots, k_r],$$

where the sequence $k_{a+2}, \ldots, k_r$ is empty if $a = r-1$. Here, the elements $j_1, \ldots, j_m$ are $i_{a+1}, \ldots, i_r, k_1, \ldots, k_{a+1}$. The terms in the Plücker relation other than the one displayed above all have some of the elements $i_{a+1}, \ldots, i_r$ replaced by a corresponding number of elements from $k_1, \ldots, k_{a+1}$, all of which are $\leq i_{a+1}$, and all of which are therefore smaller than the elements they are replacing. It follows that the smaller minor in each term is $< h$. Similarly, the larger minor is $> h'$. Suppose that a typical summand is, up to sign, $X[f_1, \ldots, f_r]X[g_1, \ldots, g_r]$.

Since $X[f_1, \ldots, f_r] < h$ and $h' < X[g_1, \ldots, g_r]$, we certainly have $f_i \leq g_i$ for $i \leq a$. But it is also the case that (after arranging the indices for each factor so that they are increasing) one has that $f_{a+1} \leq g_{a+1}$. To see why, note that the elements $f_1, \ldots, f_r$ include at least $a+1$ terms tthat are $\leq \max\{i_a, k_{a+1}\}$ so that $f_{a+1} \leq \max\{i_a, k_{a+1}\}$, while the elements $g_1, \ldots, g_r$ include at least $r-a$ terms that are $\geq \min\{i_{a+1}, k_{a+2}\}$, so that $g_{a+1} \geq \min\{i_{a+1}, k_{a+2}\} > \max\{i_a, k_{a+1}\} \geq f_{a+1}$, as required.

By induction on a we can reach the required form. □

A poset H is called *bounded* if it has a greatest and least element. A poset H is called a *distributive lattice* if it is bounded, if any two elements x, y have a least upper bound $x \vee y$ and and a greatest lower bound $x \wedge y$, and the following axioms are satisfied:

(1) For all $x, y, z \in H$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
(2) For all $x, y, z \in H$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Either axiom implies the other. E.g., assume (1). Then with $w = x \vee y$, we have that $(x \vee y) \wedge (x \vee z) = w \wedge (x \vee y) = (w \wedge x) \vee (w \wedge z) = x \vee (w \wedge z) = x \vee \big((x \vee y) \wedge z\big) = x \vee \big((x \wedge z) \vee (y \wedge z)\big) = \big(x \vee (x \wedge z)\big) \vee (y \wedge z) = x \vee (y \wedge z)$, as required.

We say that y is a *cover* of x in a poset H if $y > x$ and no element is strictly between y and x.

A poset is called *locally upper semimodular* if whenever y and z are covers of x and there exists v such that $x < v$ and $y < v$, then there exists $w \leq v$ such that w is a cover of both x and y.

Note that the poset of $r \times r$ minors of the $r \times s$ matrix H is a distributive lattice. In fact, if $h = X[i_1, \ldots, i_r]$ and $h' = X[j_1, \ldots, j_r]$, where $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_r$, it is not diificult to see that $h \vee h' = X[i_1 \vee j_1, \ldots, i_r \vee j_r]$ and $h \wedge h' = X[i_1 \wedge j_1, \ldots, i_r \wedge j_r]$. The only point of interest is that one needs to see that the sequences produced in this way are still strictly increasing, which is straightforward. Note that H has least element $X[1, \ldots, r]$ and greatest element $X[s-r+1, \ldots, s]$.

We shall prove that a distributive lattice is locally upper semimodular and that an ALS over a field K such that the poset is bounded and locally upper semimodular is Cohen-Macaulay. Along with the comments in the preceding paragraph, this will establish that $K[X/r]$ is Cohen-Macaulay when K is any field and, hence, when K is any Cohen-Macaulay ring. Before doing so, we analyze the dimension of $K[X/r]$.

The *rank* or height of an element of a poset is the supremum of lengths of chains descending from that element. We claim that the height of $X[i_1, \ldots, i_r]$ is

$$\sum_{t=1}^{r} i_t - \sum_{t=1}^{r} r = \sum_{t=1}^{r} i_t - \frac{r(r+1)}{2}.$$

Any maximal chain descending from $X[i_1, \ldots, i_r]$ must end with $X[1, \ldots, r]$ The assetion follows if we can show that given consecutive elements in the chain, the sum of the entries decreases by exactly one from the larger to the smaller. Suppose $X[j_1, \ldots, j_r] > X[k_1, \ldots, k_r]$ are consecutive. Consider the smallest t such that $k_t < j_t$. Then

$$X[j_1, \ldots, j_{t-1}, j_t - 1, j_{t+1}, \ldots, j_r]$$

will be strictly between $X[j_1, \ldots, j_r]$ and $X[k_1, \ldots, k_r]$ unless it is equal to $X[k_1, \ldots, k_r]$ (note that $j_{t-1} = k_{t-1} < k_t \leq j_t - 1$ here).

It follows that the length of a longest chain in H is the height of $X[s-r+1, \ldots, s]$, which is $\sum_{t=1}^{r}(s-r+t-t) = r(s-r)$, and this is the dimension of the order complex.

Hence:

Corollary. *For any ring K, if X is an $r \times s$ matrix of indeterminates over K with $1 \leq r \leq s$, $\dim(K[X/r]) = \dim(K) + r(s-r) + 1$.*