Examples of integral closure of ideals. Note that whenever \( r \in R \) and \( I \subseteq R \) is an ideal such that \( r^n = i_n \in I^n \), we have that \( r \in \mathfrak{I} \). The point is that \( r \) is a root of \( z^n - i_n = 0 \), and this polynomial is monic with the required form.

In particular, if \( x, y \) are any elements of \( R \), then \( xy \in (x^2, y^2) \), since \((xy)^2 = (x^2)(y^2) \in I^2 \). This holds even when \( x \) and \( y \) are indeterminates.

More generally, if \( x_1, \ldots, x_n \in R \) are any elements and \( I = (x_1^n, \ldots, x_k^n)R \), then every monomial \( r = x_1^{i_1} \cdots x_k^{i_k} \) of degree \( n \) (here the \( i_j \) are nonnegative integers whose sum is \( n \)) is in \( \mathfrak{I} \), since
\[
 r^n = (x_1^n)^{i_1} \cdots (x_k^n)^{i_k} \in I^n,
\]
since every \( x_j^n \in I \) and \( \sum_{j=1}^k i_j = n \).

Now let \( K \) be any field of characteristic \( \neq 3 \), and let \( X, Y, Z \) be indeterminates over \( K \). Let
\[
 R = K[X, Y, Z]/(X^3 + Y^3 + Z^3) = K[x, y, z],
\]
which is a normal domain with an isolated singularity. Here, we are using lower case letters to denote the images of corresponding upper case letters after taking a quotient: we shall frequently do this without explanatory comment. Let \( I = (x, y)R \). Then \( z^3 \in I^3 \), and so \( z \in \mathfrak{I} \). This shows that an ideal generated by a system of parameters in a local ring need not be integrally closed, even if the elements are part of a minimal set of generators of the maximal ideal. It also follows that \( z^2 \in \mathfrak{I}^2 \), where \( I \) is a two generator ideal, while \( z^2 \notin I \). Thus, the Briançon-Skoda theorem, as we stated it for regular rings, is not true for \( R \). (There is a version of the theorem that is true: it asserts that for an \( n \)-generator ideal \( I \), \( \mathfrak{I}^n \subseteq I^* \), where \( I^* \) is the tight closure of \( I \). But we are not assuming familiarity with tight closure here.)

We next want to give a proof that, even when a normal domain \( R \) is not Noetherian, it is an intersection of valuation domains. We first show:

**Lemma.** Let \( L \) be a field, \( R \subseteq L \) a domain, and \( I \subset R \) a proper ideal of \( R \). Let \( x \in L - \{0\} \). Then either \( IR[x] \) is a proper ideal of \( R[x] \) or \( IR[1/x] \) is a proper ideal of \( R[1/x] \).

**Proof.** We may replace \( R \) by its localization at a maximal ideal containing \( I \), which only makes the problem harder. Assume that neither is a proper ideal. Since \( 1 \in IR[x] \) we obtain an equation
\[
(\#) \quad 1 = i_0 + i_1 x + \cdots + i_n x^n,
\]
where all of the \( i_h \in I \). Similarly, we obtain an equation
\[
(\#\#) \quad 1 = j_0 + j_1(1/x) + \cdots j_m(1/x^n),
\]
where all of the \( j_h \in I \).
where all of the $j_h \in I$. We may assume that $n$ and $m$ have been chosen as small as possible. By reversing the roles of $x$ and $1/x$, if necessary, we may assume that $n \geq m$. Then
\[
1 - j_0 = j_1(1/x) + \cdots + j_m(1/x)^m.
\]
Multiplying by the inverse of $1 - j_0$, we have that
\[
1 = j'_1(1/x) + \cdots + j'_m(1/x)^m,
\]
where the $j'_h \in I$. Multiplying through by $x^m$ yields that
\[
x^m = j'_1x^{m-1} + \cdots + j'_m \in I +Ix + \cdots +Ix^{m-1}.
\]
It follows by induction on $k$ that for all $k \geq 0$,
\[
x^k \in I +Ix + \cdots +Ix^{m-1}.
\]
For the inductive step, once we have that
\[
x^{k-1} \in I +Ix + \cdots +Ix^{m-1},
\]
we can multiply by $x$ to get that
\[
x^k \in I +Ix +Ix^2 + \cdotsIx^m,
\]
and we can use the fact that
\[
x^m \in I +Ix + \cdots +Ix^{m-1}
\]
to eliminate the rightmost term on the right. But then we can get rid of the $x^m, \ldots, x^n$ terms in the displayed equation (#), and we have that
\[
1 \in I +Ix + \cdots +Ix^{m-1},
\]
contradicting the minimality of our choice of $n$. □

**Corollary.** Let $R \subseteq L$, a field, and let $I \subseteq R$ be a proper ideal of $R$. Then there is a valuation domain $V$ with $R \subseteq V \subseteq L$ such that $IV \neq V$.

**Proof.** Consider the set $S$ of all rings $S$ such that $R \subseteq S \subseteq L$ and $IS \neq S$. This set contains $R$, and so is not empty. The union of a chain of rings in $S$ is easily seen to be in $S$. Hence, by Zorn’s lemma, $S$ has a maximal element $V$. We claim that $V$ is a valuation domain with fraction field $L$. For let $x \in L - \{0\}$. By the preceding Lemma, either $IV[x]$ or $IV[1/x]$ is a proper ideal. Thus, either $V[x] \in S$ or $V[1/x] \in S$. By the maximality of $V$, either $x \in V$ or $1/x \in V$. □

We now can prove the result we were aiming for.
**Corollary.** Let $R$ be a normal domain with fraction field $L$. Then $R$ is the intersection of all valuation domains $V$ with $R \subseteq V \subseteq L$.

**Proof.** Let $x \in L - R$. It suffices to find $V$ with $R \subseteq V \subseteq L$ such that $x \notin V$. Let $y = 1/x$. We claim that $y$ is not a unit in $R[y]$, for its inverse is $x$, and if $y$ were a unit we would have

$$x = r_0 + r_1(1/x) + \cdots + r_n(1/x)^n$$

for some positive integer $n$ and $r_j \in R$. Multiplying through by $x^n$ gives an equation of integral dependence for $x$ on $R$, and since $R$ is normal this yields $x \in R$, a contradiction. Since $yR[y]$ is a proper ideal, by the preceding Corollary we can choose a valuation domain $V$ with $R[y] \subseteq V \subseteq K$ such that $yV$ is a proper ideal of $V$. But this implies that $x \notin V$. □

The following important result can be found in most introductory texts on commutative algebra, including [M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, Massachusetts, 1969], which we refer to briefly as Atiyah-Macdonald.

**Theorem.** If $R$ is a normal Noetherian domain, then the integral closure $S$ of $R$ in a finite separable extension $G$ of its fraction field $F$ is module-finite over $R$.

**Proof.** See Proposition 5.19 of Atiyah-Macdonald for a detailed argument. We do mention the basic idea: choose elements $s_1, \ldots, s_d$ of $S$ that are basis for $G$ over $F$, and then the discriminant $D = \det(\text{Trace}_G/F s_is_j)$, which is nonzero because of the separability hypothesis, multiplies $S$ into the Noetherian $R$-module $\sum_{i=1}^d Rs_i$. □

**Theorem (Nagata).** Let $R$ be a complete local domain. Then the integral closure of $R$ in a finite field extension of its fraction field is a finitely generated $R$-module.

**Proof.** Because $R$ is module-finite over a formal power series ring over a field, or, if $R$ does not contain a field, over DVR whose fraction field has characteristic zero, we may replace the original $R$ by a formal power series ring, which is regular and, hence, normal. Unless $R$ has characteristic $p$ the extension is separable and we may apply the Theorem just above.

Thus, we may assume that $R$ is a formal power series ring $K[[y_1, \ldots, y_n]]$ over a field $K$ of characteristic $p$. If we prove the result for a larger finite field extension, we are done, because the original integral closure will be an $R$-submodule of a Noetherian $R$-module. This enables us to view the field extension as a purely inseparable extension followed by a separable extension. The separable part may be handled using the Theorem just above. It follows that we may assume that the field extension is contained in the fraction field of $K^{1/q}[x_1, \ldots, x_n]$ with $x_i = y_i^{1/q}$ for all $i$. We may adjoin the $x_i$ to the given field extension, and it suffices to show that the integral closure is module-finite over $K[[x_1, \ldots, x_n]]$, since this ring is module-finite over $K[[y_1, \ldots, y_n]]$. Thus, we have reduced to the case where $R = K[[x_1, \ldots, x_n]]$ and the integral closure $\bar{S}$ will lie inside $K^{1/q}[x_1, \ldots, x_n]$, since this ring is regular and, hence, normal.
Now consider the set $\mathcal{L}$ of leading forms of the elements of $S$, viewed in the ring $K^{1/q}[x_1, \ldots, x_n]$. Let $d$ be the degree of the field extension from the fraction field of $R$ to that of $S$. We claim that any $d + 1$ or more $F_1, \ldots, F_N$ of the leading forms in $\mathcal{L}$ are linearly dependent over (the fraction field of) $R$ for, if not, choose elements $s_j$ of $S$ which have them as leading forms, and note that these will also be linearly independent over $R$, a contradiction (if a non-trivial $R$-linear combination of them were zero, say $\sum_j r_j s_j = 0$, where the $r_j$ are in $R$, and if $F_j$ has degree $d_j$ while the leading form $g_j$ of $r_j$ has degree $d'_j$, then one also gets $\sum_j g_j F'_j = 0$, where the sum is extended over those values of $j$ for which $d_j + d'_j$ is minimum). Choose a maximal set of linearly independent elements $f_j$ of $\mathcal{L}$. Let $K'$ denote the extension of $K$ generated by all of their coefficients. Since there are only finitely many, $T = K'[x_1, \ldots, x_n]$ is module-finite over $R$. But $T$ contains every element $L$ of $\mathcal{L}$, for each element of $\mathcal{L}$ is linearly dependent over $R$ on the $f_j$, and so is in the fraction field of $T$, and has its $q$ th power in $R \subseteq T$. Since $T$ is regular, it is normal, and so must contain $L$.

Thus, the elements of $\mathcal{L}$ span a finitely generated $R$-submodule of $T$, and so we can choose a finite set $L_1, \ldots, L_k \subseteq \mathcal{L}$ that span an $R$-module containing all of $\mathcal{L}$. We can then choose finitely many elements $s_1, \ldots, s_k$ of $S$ whose leading forms are the $L_1, \ldots, L_k$.

Let $S_0$ be the module-finite extension of $R$ generated by the elements $s_1, \ldots, s_k$. We complete the proof by showing that $S_0 = S$. We first note that for every element $L$ of $\mathcal{L}$, $S_0$ contains an element $s$ whose leading form is $L$. To see this, observe that if we write $L$ as an $R$-linear combination $\sum_j r_j L_j$, the same formula holds when every $r_j$ is replaced by its homogeneous component of degree $\deg L - \deg L_j$. Thus, the $r_j$ may be assumed to be homogeneous of the specified degrees. But then $\sum_j r_j s_j$ has $L$ as its leading form.

Let $s \in S$ be given. Recursively choose $u_0, u_1, \ldots, u_n, \ldots \in S_0$ such that $u_0$ has the same leading form as $s$ and, for all $n, u_{n+1}$ has the same leading form a $s - (u_0 + \cdots + u_n)$. For all $n \geq 0$, let $v_n = u_0 + \cdots + u_n$. Then $\{v_n\}_n$ is a Cauchy sequence in $S_0$ that converges to $s$ in the topology given by the powers $m_T^n$ of the maximal ideal of $T = K'[x_1, \ldots, x_n]$. Since $S_0$ is module-finite over $K[[x_1, \ldots, x_n]]$, $S_0$ is complete. By Chevalley’s lemma, which is discussed below, when we intersect the $m_T^n$ with $S_0$ we obtain a sequence of ideals cofinal with the powers of the maximal ideal of $S_0$. Thus, the sequence, which converges to $s$, is Cauchy with respect to the powers of the maximal ideal of $S_0$. Since, as observed above, $S_0$ is complete, we have that $s \in S_0$, as required. □