Let $R$ be any ring and $I \subseteq R$ any ideal. By the extended Rees ring or second Rees ring of $I$ over $R$ we mean the ring $R[It, 1/t] \subseteq R[t]$. In this context we shall standardly write $v$ for $1/t$. Note that if $I$ is proper, $v$ is not a unit of $R[It, v]$. This ring is $\mathbb{Z}$-graded. Written out as a sum of graded pieces

$$R[It, v] = \cdots + Rv^k + \cdots + Rv^2 + Rv + R + It + I^2t^2 + \cdots + I^n t^n + \cdots.$$  

The element $v$ generates a homogeneous principal ideal, and

$$vR[It, v] = \cdots + Rv^k + \cdots + Rv^2 + Rv + I + I^2t + I^3t^2 + \cdots + I^n t^{n-1} + \cdots.$$  

From this it follows easily that $R[It, v]/(v) \cong \text{gr}_I R$. There is a composite surjection

$$R[It, v] \twoheadrightarrow \text{gr}_I R \twoheadrightarrow R/I.$$  

When $I$ is the unit ideal of $R$ we have that $R[It, v] = R[t, t^{-1}]$.

When $(R, m, K)$ is local and $I$ is proper we further have a composite surjection

$$R[It, v] \twoheadrightarrow R/I \twoheadrightarrow R/m = K,$$

and the kernel is a maximal ideal $\mathcal{M}$ of $R[It, v]$. Explicitly,

$$\mathcal{M} = \cdots + Rv^k + \cdots + Rv^2 + Rv + m + It + I^2t^2 + \cdots + I^n t^n + \cdots.$$  

**Theorem.** Let $(R, m, K)$ be local, let $I \subseteq R$ be proper, and let $R[It, v]$ and $\mathcal{M}$ be as in the paragraphs just above,

(a) The Krull dimension of $R[It, 1/t]$ is $\dim (R) + 1$, and this is the height of $\mathcal{M}$.

(b) $\dim (\text{gr}_I(R)) = \dim (R)$.

**Proof.** Let

$$\mathcal{P} = \cdots + mv^k + \cdots + mv^2 + mv + m + It + I^2t^2 + \cdots + I^n t^n + \cdots,$$

which is the contraction of $mR[t, 1/t]$ to $R[It, v]$. Then $\mathcal{P} \subseteq \mathcal{M}$ and $R[It, v]/\mathcal{P} \cong K[v]$, a polynomial ring in one variable over a field. The height of $\mathcal{P}$ is the same as the height of $m$: when we localize at $\mathcal{P}$ in $R[It, v]$, $v$ becomes invertible, so that $t = 1/v$ becomes an element of the localized ring. But $R[It, v][t] = R[t, v]$, and the expansion of $\mathcal{P}$ is $mR[t, 1/t]$. The localization at the expansion is just $R(t)$ (note that when we localize $R[t]$ at $mR[t]$, $v$ becomes an element of the ring), which we already know has the same dimension as $R$. Thus, height $\mathcal{P} = \dim (R)$. Since $\mathcal{M} = \mathcal{P} + vR[It, v]$ is strictly larger than $\mathcal{P}$, we have

$$\dim (R[It, v]/\mathcal{P}) = \dim (R).$$
that height $\mathcal{M} \geq \dim (R) + 1$. To complete the proof of (a), it will suffice to show that $\dim (R[v]) \leq \dim (R) + 1$, for then height $\mathcal{M} \leq \dim (R) + 1$ as well.

We first reduce to the case where $R$ is a domain. To do so, we want to understand the minimal primes of $S = R[t, v]$. If $q$ is any prime of $S$, it lies over some prime of $R$, and this prime contains a minimal prime $p$ of $R$. We shall show that there is a unique minimal prime $\mathfrak{p}$ of $S$ containing $p$, and it will follow that every minimal prime has the form $\mathfrak{p}$. To see this, note that $q$ cannot contain $v$, for $v$ is not a zerodivisor in $S$. Hence, $q$ corresponds via expansion to a minimal prime of $S_v$ containing $p$. But $S_v \cong R[t, v]$, and $pR[t, 1/t]$ is already a minimal prime of $R[t, 1/t]$. It follows that $q = pR[t, 1/t] \cap S$, and this is the minimal prime $\mathfrak{p}$. Note that $R[t, 1/t]/\mathfrak{p}$ embeds in $(R/p)[t, 1/t]$, and that the image is the extended Rees ring of $I(R/p)$. Therefore, it suffices to show that the dimension of each of these Rees rings over a domain $D$ obtained by killing a minimal prime of $R$ has dimension at most $\dim (D) + 1 \leq \dim (R) + 1$, and we may therefore assume without loss of generality that $R$ is a local domain.

But $S$ is then a domain finitely generated over $R$. If the fraction field of $R$ is $\mathcal{F}$, then the fraction field of $S$ is $\mathcal{F}(t)$. If $Q$ is any prime ideal of $S$, $Q$ lies over, say, $P$ in $R$, and the residue class fields of $R_P$ and $S_Q$ are $\kappa_P$ and $\kappa_Q$ respectively, then the dimension formula yields

$$\text{height } Q \leq \text{height } P + \text{tr. deg.}(\mathcal{F}(t)/\mathcal{F}) - \text{tr. deg.}(\kappa_Q/\kappa_P) \leq \text{height } P + 1 \leq \dim (R) + 1,$$

as required.

For part (b), note that the height of $\mathcal{M}$, which is $\dim (R) + 1$ drops exactly 1 when we kill the nonzerodivisor $v$. This shows that $\dim (\text{gr}_I(R)) \geq \dim (R)$. But killing a nonzerodivisor in a Noetherian domain of finite Krull dimension drops the dimension by at least one, so that $\dim (\text{gr}_I(R)) \leq \dim (S) - 1 = \dim (R)$. □

**Corollary.** Let $x_1, \ldots, x_n$ be a system of parameters in a local ring $(R, m, K)$. Let $F$ be a homogenous polynomial of degree $d$ in $R[X_1, \ldots, X_n]$ such that $F(x_1, \ldots, x_n) = 0$. That is, $F$ gives a relation over $R$ on the monomials of degree $d$ in $x_1, \ldots, x_n$. Then all coefficients of $F$ are in $m$.

*Proof.* Consider the associated graded ring $\text{gr}_I(R)$, where $I = (x_1, \ldots, x_n)R$. This ring is generated by the images $\overline{x}_1, \ldots, \overline{x}_n$ of $x_1, \ldots, x_n$ in $I/I^2 = [\text{gr}_I(R)]_1$. Let $A = R/I$, an Artin local ring. By the preceding Theorem, $\dim (\text{gr}_I(R)) = n$. But $\text{gr}_I(R) = A[\overline{x}_1, \ldots, \overline{x}_n]$. Killing the maximal ideal $m/I$ of $A$ does not affect the dimension of this ring. It follows that quotient has dimension $n$, so that $K[\overline{x}_1, \ldots, \overline{x}_n]$ is a polynomial ring in $\overline{x}_1, \ldots, \overline{x}_n$. If $F(x_1, \ldots, x_n) = 0$ and has a coefficient outside $m$, we find the $\overline{F}(\overline{x}_1, \ldots, \overline{x}_n) = 0$ in $K[\overline{x}_1, \ldots, \overline{x}_n]$, where $\overline{F}$ is the image of $F$ mod $m$ and so is a nonzero polynomial in the $K[\overline{x}_1, \ldots, \overline{x}_n]$. This forces the dimension of $K \otimes_R \text{gr}_I(R)$ to be smaller than $n$, a contradiction. □

We next want to prove two consequences of the Briançon-Skoda Theorem that were stated without proof in as Corollaries at the bottom of p. 1 and the top of p. 2 of the Lecture Notes of September 6. The next result generalizes the first Corollary.
Theorem (corollary of the Briançon-Skoda Theorem). Let \( R \) be a regular Noetherian ring of Krull dimension \( n \) and let \( f_1, \ldots, f_{n+1} \) be elements of \( R \). Then
\[
f_1^n \cdots f_{n+1}^n \in (f_1^{n+1}, \ldots, f_{n+1}^{n+1})R.
\]

Proof. Call the product on the left \( g \) and the ideal on the right \( I \). If \( g \not\in I \), then \((I + Rg)/I\) is not zero, and we can localize at a prime in its support. Therefore, we may without loss of generality that assume that \((R, m, \mathbb{K})\) is a regular local ring of dimension at most \( n \). Second, if \( g \notin I \) this remains true when we replace \( R \) by \( R(t) \), since \( R(t) \) is faithfully flat over \( R \). We also have that \( R \) and \( R(t) \) have the same dimension. Thus, we may assume that \( R \) has an infinite residue class field. Let \( h = f_1 \cdots f_n \), so that \( g = h^n \). Since \( h^{n+1} \in I^{n+1} \), \( h \in I \). Since \( \text{ann}(I) \leq \dim(R) \leq n \) and the residue class field is infinite, \( I \) is integral over an ideal \( I_0 \) with at most \( n \) generators. Then \( h \in I_0 \), and it follows from the Briançon-Skoda theorem that \( h^n \in I_0 \subseteq I \), as required. \( \square \)

We next observe:

Theorem. Let \( R \) denote \( \mathbb{C}\{z_1, \ldots, z_n\} \) or \( \mathbb{C}\llbracket z_1, \ldots, z_n \rrbracket \), the convergent or formal powers series ring in \( n \) variables. Let \( f \) be in the maximal ideal of \( R \), and let \( I \) be the ideal generated by the partial derivatives \( \partial f/\partial z_i \) of \( f \). Then \( f \) is integrally dependent on \( I \).

Proof. We assume the result from the first Problem Set, Problem #6, that the integral closure of \( I \) is an intersection of integrally closed \( m \)-primary ideals (but we do not need this result for the case where the \( \partial f/\partial z_i \) generate an \( m \)-primary ideal). Choose an integrally closed \( m \)-primary ideal \( \mathfrak{A} \supseteq I \) with \( f \not\in \mathfrak{A} \). Then we can map \( R \) to a discrete valuation ring \( V \) in such a way that the image \( f \) is not in \( \mathfrak{A}V \) (and, hence, not in \( IV \)), and it follows that \( m \) maps into the maximal ideal of \( V \). Note that \( V \) cannot be just a field here, for then \( f \) maps to 0. Replace \( V \) by its completion: we may assume that \( V \) is complete. Since we are in the equal characteristic 0 case, the image of \( \mathbb{C} \) in \( V \) can be extended to a coefficient field. Thus, we may assume that \( V = L[[x]] \), where \( \mathbb{C} \subseteq L \) and \( m \) maps into \( (x) \).

Let \( h : R \to L[[x]] \) be the map, and \( h(z_i) = g_i(x) \), \( 1 \leq i \leq n \). Then \( f \) maps to \( f(g_1(x), \ldots, g_n(x)) \). The key point is that the chain rule holds here, by a formal calculation. Thus,
\[
\frac{d}{dx}(h(f)) = \sum_{i=1}^n h(\partial f/\partial z_i) \frac{dg_i(x)}{dx}.
\]
It follows that the derivative of \( h(f) \) is in \( IV \). But over a field of characteristic 0, the derivative of a nonzero non-unit \( v \) has order exactly one less than that of \( v \). Hence, \( f \in IV \) as well. \( \square \)

Theorem (corollary of the Briançon-Skoda theorem). With hypotheses as in the preceding Theorem, \( f^n \) is in the ideal generated by its partial derivatives.

Proof. This is immediate from the preceding Theorem and the Briançon-Skoda Theorem. \( \square \)