We prove the Corollary stated at the end of the Lecture of September 27.

Proof. If the ideals are different, we may localize at a prime of $R_2$ in the support of the module which is their sum modulo their intersection. Thus, we may assume without loss of generality that $R_2$ is local. We may replace $R_1$ by its localization at the contraction of the maximal ideal of $R_2$, and $R_0$ by its localization at the contraction of the maximal ideal of $R_2$ (or of $R_1$: the contractions are the same). Thus, we may assume that we are in the case where $R_0 \hookrightarrow R_1 \hookrightarrow R_2$ are local and the homomorphisms are local. By the Lemma near the bottom of the page 4 of the Lecture Notes of September 27, we have that

$$R_1 \cong R_0[X_1, \ldots, X_m]_P/(F_1, \ldots, F_m)$$

for some $m$ and prime $P$ of $R_0[X_1, \ldots, X_m]$. Then $J_{R_1/R_0}$ is generated by the image of $\frac{\partial(F_1, \ldots, F_m)}{\partial(X_1, \ldots, X_m)}$. Likewise,

$$R_2 \cong (R_1[Y_1, \ldots, Y_s])_Q/(G_1, \ldots, G_s).$$

Again, $J_{R_2/R_1}$ is generated by the image of $\frac{\partial(G_1, \ldots, G_s)}{\partial(Y_1, \ldots, Y_s)}$. It follows that we can write $R_2$ as a localization of

$$R_0[X_1, \ldots, X_m, Y_1, \ldots, Y_s]/(F_1, \ldots, F_m, G_1, \ldots, G_s),$$

where the $F_j$ do not involve the $Y_i$. This means that the Jacobian matrix has the block form

$$\begin{pmatrix} M_F & N \\ 0 & M_G \end{pmatrix}$$

where $M_F$ is $(\partial F_j/\partial X_i)$ and $M_G$ is $(\partial G_j/\partial Y_i)$. We have that $J_{R_2/R_1}$ is generated by the image of the determinant of this matrix. No matter what $N$ is, this determinant is

$$\det(M_F) \det(M_G) = \frac{\partial(F_1, \ldots, F_m)}{\partial(X_1, \ldots, X_m)} \frac{\partial(G_1, \ldots, G_s)}{\partial(Y_1, \ldots, Y_s)}$$

as required. □

There are now three results whose proof are hanging: one is the proof that $S'$ is module-finite over $S$, the second is the proof of the Key Lemma, which is stated on p. 3 of the Lecture Notes of September 27, and the third is the proof of the Jacobian Theorem itself. We begin with the proof of the Key Lemma. This will involve studying quadratic
transforms of a regular local ring along a valuation. We first indicate our approach to the proof of the Key Lemma.

**Proof of the Key Lemma:** step 1. We are trying to show that \( \mathcal{J}_{S'/R} \subseteq \mathfrak{v}S' \). Assume the contrary. Consider the primary decomposition of \( \mathfrak{v}S' \). Since we are assuming that \( S' \) is module-finite over \( S \) (we still need to prove this), \( S' \) is a normal Noetherian ring, and the associated primes of \( \mathfrak{v}S' \) have height one. We may choose such a prime \( Q \) such that \( \mathcal{J}_{S'/R} \) is not contained in the corresponding primary ideal in the primary decomposition of \( \mathfrak{v}S' \). Since the elements of \( S' - Q \) are not zerodivisors on this primary ideal, we also have that \( \mathcal{J}_{S'/Q} \) is not contained in \( S'Q = V \). Thus, for the purpose of proving the Key Lemma, we may replace \( S' \) by \( V \), which is a discrete valuation ring, and we may replace \( R \) by its localization at the contraction of \( Q \) to \( R \). In the remainder of the argument we may therefore assume\(^1\) that \((R, m, K)\) is regular local.

We now digress to discuss quadratic transforms. We first want to prove:

**Lemma.** Let \((R, m, K)\) be regular local with regular system of parameters \(x_1, \ldots, x_d\). Then \( T = R[x_2/x_1, \ldots, x_d/x_1] = R[m/x_1] \) is regular, and the images of \(x_2, \ldots, x_d\) are algebraically independent over \(K\) in \(T/x_1T \cong K[\overline{x}_2, \ldots, \overline{x}_d]\).

**Proof.** If we localize \( T \) at a prime that does not contain \( x_1 \), the resulting ring is a localization of \( R_{x_1} \), and is therefore regular. For primes that contain \( x_1 \), it suffices to show that the localization is regular after killing \( x_1 \), and this follows from the fact that \( T/x_1T \) is regular even without localizing. It therefore suffices to prove that \( T/x_1T \) is a polynomial ring.

First note that

\[
T = \bigcup_k m^k/x_1^k,
\]

where \( m^k/x_1^k = \{u/x_1^k : u \in m^k\} \), and this is an increasing union since \( m^k/x_1^k = x_1m^k/x_1^{k+1} \). (It is clear that \( m^k/x_1^k \subseteq T \), and the product of the \( j \)th and \( k \)th terms in the union is the \((j + k)\)th term.) Hence, if there is a relation among the \( \overline{x}_j \), \( 2 \leq j \leq n \), we may lift it to obtain a nonzero polynomial \( g \) whose nonzero coefficients are units of \( R \) (we get them by lifting elements of \( K \) to \( R \)) such that \( g(x_2/x_1, \ldots, x_d/x_1) = x_1t \) where \( t \in m^k/x_1^k \). We multiply both sides by \( x_1^N \) where \( N \geq k \) is also larger than the absolute value of any negative exponent on \( x_1 \) occurring on the left. The left hand side becomes a nonzero homogeneous polynomial \( G \) of degree \( N \) in \( x_1, \ldots, x_d \) whose nonzero coefficients are units. The right hand side is in \( x_1^N \cdot x_1^{N-k}m^k \subseteq x_1^{1+N-k}m^k \subseteq m^{N+1} \). This gives a nonzero relation of degree \( N \) on the images of \( x_1, \ldots, x_d \) in \( \text{gr}_m(R) \), a contradiction, since when \( R \) is regular this is a polynomial ring in the images of the \( x_j \). \( \square \)

**Definition.** Let \((R, m, K)\) be a regular local ring of Krull dimension \( d \geq 2 \) and let suppose that \( R \subseteq V \) is a local map to discrete valuation ring \( V \). Let \( \text{ord}_V = \text{ord} \) denote

\(^1\)Note that in the refined version of the Jacobian theorem, it was assumed that \( R_P \) is regular if \( P \) lies under a height one prime of \( S' \), so that we may make this reduction even in that case.
the corresponding valuation. By the immediate or first quadratic transform of $R$ along $V$ we mean the following: let $x_1, \ldots, x_d$ be a regular system of parameters for $R$ numbered so that $\text{ord}(x_1) \leq \text{ord}(x_j)$ for $j \geq 2$, let $T = R[x_2/x_1, \ldots, x_d/x_1]$, and then the first quadratic transform is $T_P$, where $P$ is the contraction to $T$ of the maximal ideal of $V$. Then $T_P$ is again regular, and we have a local map $T_P \subseteq V$. We may therefore iterate to obtain a sequence of quadratic transforms of $R$, called the quadratic sequence of $R$ along $V$. The sequence is finite if it eventually contains a ring of dimension 1. We are aiming to prove that the sequence is finite whenever the transcendence degree of the residue class field of $V$ is of $d - 1$, in which case the ring $V \cap \text{frac}(R)$ occurs as the final term.