Remark. The quadratic transform of \((R, m, K)\) along \(V\) is independent of the choice of regular system of parameters. If \(x \in m\) has minimum order in \(V\), the quadratic transform is the localization of \(R[m/x]\) at the contraction \(P\) of the maximal ideal of \(V\). If \(y\) also has minimum order, then \(y/x \notin P\), and so \(x/y \in R[m/x]_p\). Since \((u/x)(x/y) = u/y\) for all \(u \in m\), it follows that

\[
R[m/y] \subseteq R[m/x]_p.
\]

If \(Q\) is the contraction of the maximal ideal of \(V\) to \(R[m/y]\), we have that elements of \(R[m/y] - Q\) are not in \(PR[m/x]_p\), and therefore have inverses in \(R[m/x]_p\). Thus,

\[
R[m/y]Q \subseteq R[m/x]_p.
\]

The other inclusion follows exactly similarly. □

Remark. The first quadratic transform \((T_1, m_1, K_1)\) of the regular local ring \((R, m, K)\) has dimension at most \(\dim(R)\), and \(\dim(T) = \dim(R) - \text{tr.deg.}(K_1/K)\). In fact, since \(R\) is regular, it is universally catenary and the dimension formula holds. Since the two fraction fields are equal, the transcendence degree of the the extension of fraction fields is 0. □

Remark. A local inclusion of valuation domains with the same fraction field must be the identity map. For say that \((W, m_W) \subseteq (V, m_V)\) with \(m_W \subseteq m_V\). If \(f \in V - W\), then \(1/f \in W\). Since \(f \notin W\), \(1/f \in m_W\). But then \(1/f \in m_V\), which contradicts \(f \in V\). □

Lemma. Let \(x_1, \ldots , x_d\) be a regular sequence in the ring \(R\) with \(d \geq 2\).

(a) Consider \(T = R[x_2/x_1] \subseteq R_{x_1}\). Then \(T \cong R[X]/(x_1X - x_2)\). Moreover, \(x_1, x_3, \ldots , x_d\) is a regular sequence in \(R[x_2/x_1]\).

(b) Consider \(T = R[x_2/x_1, \ldots , x_d/x_1] \subseteq R_{x_1}\). Then

\[
T \cong R[X_2, \ldots , X_d]/(x_1X_1 - x_i: 2 \leq i \leq d).
\]

Moreover, \(J_{T/R} = x_1^{d-1}T\).

Proof. (a) Since \(x_1, x_2\) is a regular sequence in \(R\), \(x_1X - x_2\) is a regular sequence in \(R[X]\): killing \(x_1\) produces \((R/x_1R)[X]\), and the image of the second element is \(-x_2\).

We claim that \(x_1\) is not a zerodivisor modulo \((x_1X - x_2)\), for if \(x_1f = (x_1X - x_2)g\) in \(R[X]\), then \(g = hx_1\) by the paragraph above. Since \(x_1\) is not a zerodivisor in \(R[X]\), we find that \(f = (x_1X - x_2)h\).

This means that \(R[X]/(x_1X - x_2)\) injects into its localization at \(x_1\), which we may view as \(R_{x_1}/(x_1X - x_2)\). Since \(x_1\) is a unit, we may take \(X - x_2/x_1\) as a generator of the ideal
in the denominator, and so the quotient is simply $R_{v_1}$. The image of $R[X]/(x_1X - x_2)$ is then $R[x_2/x_1] \subseteq R_{v_1}$.

The last statement in part (a) follows because if we kill $x_1$ in $R[X]/(x_1X - x_2)$, we simply get $(R/(x_1, x_2))[X]$. The images of $x_3, \ldots, x_d$ form a regular sequence in this polynomial ring, because that was true in $R/(x_1, x_2)$.

Part (b) now follows by induction on $d$: as we successively adjoin $x_2/x_1, x_3/x_1$ and so forth, the hypothesis we need for the next fraction continues to hold. The final statement is then immediate, because the Jacobian matrix calculated from the given presentation is $x_1$ times the size $d - 1$ identity matrix. □

We also note:

**Proposition.** Let $(R, m, K) \subseteq V$ be an inclusion of a regular local ring in a DVR, let $v \in m$ be such that $R/vR$ is regular, and let $x \in m$ have minimal order in $V$. Let $T$ be the first quadratic transform of $R$ along $V$. Then either $v_1 = v/x$ is a unit of $T$, or else $T/v_1T$ is regular. If the first quadratic transform is a DVR, it is always the case that $v_1$ is a unit.

**Proof.** Extend $v$ to a regular system of parameters $S$ for $R$. If $v$ itself has minimum order in $V$, then $v/x$ is a unit of $T$. If not, then $x$ is a unit of $T$ times some element $x_1 \in S$, and $v_1T = v^*_1T$ if $v^*_1 = v/x_1$. Hence, we may assume without loss of generality that $x = x_1$ and $v = x_2$ in the regular system of parameters $S$. Then $x_1$ and $x_2/x_1$ are in the maximal ideal of $T$, and to show that they form a regular sequence in $T$, it suffices to show that they form a regular sequence in the ring

$$R[x_1, x_2/x_1, \ldots, x_d/x_1].$$

However, mod $(x_1)$, this ring becomes the polynomial ring $K[\overline{x}_2, \ldots, \overline{x}_d]$, and the image of $x_2/x_1$ is $\overline{x}_2$. The quotient of $T$ by this regular sequence is a localization of the polynomial ring $K[\overline{x}_3, \ldots, \overline{x}_d]$, and so is regular. Hence, $x_1, x_2/x_1$ is part of a regular system of parameters for $T$, and so $x_2/x_1 = v/x = v_1$ is a regular parameter. Note that in this case $\text{di}(T) \geq 2$, so that if $T$ is a DVR we must have that $v_1$ is a unit. □

We are now ready to prove the result mentioned at the end of the Lecture Notes for September 29 concerning finiteness of the sequence of quadratic transforms under certain conditions: we follow the treatment in [S. Abhyankar, *Ramification theoretic methods in algebraic geometry*, Annals of Mathematics Studies Number 43, Princeton University Press, Princeton, New Jersey, 1959], Proposition 4.4, p. 77.

**Theorem (finiteness of the quadratic sequence).** Let $(R, m, K) \subseteq (V, n, L')$ be a local inclusion of a regular local ring $R$ of dimension $d$ with fraction field $K$ in a discrete valuation ring. Suppose that $\text{tr. deg.}(L'/K) = d - 1$. Let $T_0 = R$, and let $(T_i, m_i, K_i)$ denote the $i$th quadratic transform of $R$ along $V$, so that for each $i \geq 0$, $T_{i+1}$ is the first quadratic transform of $T_i$ along $V$. Then this sequence is finite, and terminates at
some $T_h$ (which means precisely that $T_h$ has dimension 1). Moreover, for this value of $h$, $T_h = V \cap K$, so that $V \cap K$ is essentially of finite type over $R$, and the transcendence degree of the residue field of $V \cap K$ over $K$ is $d - 1$.

**Proof.** Assume that the sequence is infinite. By the dimension formula, for every $i$,

$$\dim (T_i) = \dim (R) - \text{tr. deg.}(K_i/K),$$

so that

$$\text{tr. deg.}(L'/K_i) = \text{tr. deg.}(L'/K) - \text{tr. deg.}(K_i/K) = d - 1 - \text{tr. deg.}(K_i/K) = (\dim (R) - \text{tr. deg.}(K_i/K)) - 1 = \dim (T_i) - 1.$$ 

Therefore, at every stage, we have that $T_i \subseteq V$ satisfies the same condition that $R \to V$ did. Since the dimension is non-increasing it is eventually stable, and, by replacing $R$ by $T_j$ for $j \gg 0$, we might as well assume that the dimension of $T_i$ is stable throughout. We call the stable value $d$, and we may assume that $d \geq 2$. It follows that every $K_i$ is algebraic over $K$. We have a directed (in fact, non-decreasing) union $\bigcup T_j$ of local rings inside $V$: call the union $(W, N, L)$. Here, $N$ is the union of the $m_i$ and $L$ is the union of the $K_i$, and so is algebraic over $K$.

We claim that $W$ must be a valuation domain of $K$. If not, choose $x \in K$ such that neither $x$ nor $1/x$ is in $W$, i.e., neither is in any $T_i$. Write $x = y_0/z_0$ where $y_0$, $z_0 \in T_0 = R$. These are both in the maximal ideal of $T_0$ (if $z_0$ were a unit, we would have $x \in T_0$, while if $y_0$ were a unit, we would have $1/x \in T_0$). If $u_0$ is a minimal generator of $m_0$ of minimum order in $V$, then $x = y_1/z_1$ where $y_1 = y_0/u_0$ and $z_1 = z_0/u_0$ are in $T_1$. We once again see that $y_1$ and $z_1$ must both belong to $m_1$. These have positive order in $V$, but $\text{ord}_V(y_0) > \text{ord}_V(y_1)$ and $\text{ord}_V(z_0) > \text{ord}_V(z_1)$. We recursively construct $y_i$ and $z_i$ in $m_i$ such that $x = y_i/z_i$, and $\text{ord}_V(y_0) > \cdots > \text{ord}_V(y_i)$ while $\text{ord}_V(z_0) > \cdots > \text{ord}_V(z_i)$. At the recursive step let $u_i$ be a minimal generator of $m_i$ such that $\text{ord}_V(u_i)$ is minimum. Then $x = y_{i+1}/z_{i+1}$ where $y_{i+1} = y_i/u_i \in T_{i+1}$ and $z_{i+1} = z_i/u_i \in T_{i+1}$. As before, the fact that $x \notin T_{i+1}$ and $1/x \notin T_{i+1}$ yields that $y_{i+1}$ and $z_{i+1}$ are both in $m_{i+1}$, as required, and we also have that $\text{ord}_V (y_i) > \text{ord}_V (y_{i+1})$ and $\text{ord}_V (z_i) > \text{ord}_V (z_{i+1})$. This yields that $\{\text{ord}_V (y_i)\}_i$ is a strictly decreasing sequence of nonnegative integers, a contradiction. It follows that $W$ is a valuation domain.

Since $W \subseteq V \cap K$ is a local inclusion of valuation domains with the same fraction field, we can conclude from the third Remark on the first page of the Notes for this Lecture that $W = V \cap K$. Therefore, $V \cap K$ is essentially of finite type over $R = T_0$, and we have from the dimension formula that $\text{tr. deg.}(L/K) = \dim (R) - \dim (W) = d - 1 \geq 1$, contradicting our earlier conclusion that $L$ is algebraic over $K$. This contradiction establishes the result. □

**Proof of the Key Lemma, second step.** We recall the situation: $R$ is a regular domain with fraction field $K$, $v$ is an element such that $R/vR$ is regular, $S$ is a reduced torsion-free algebra essentially of finite type over $R$ that is generically étale, $y \in R - vR$ is such
that \( y/v \in S \), and we want to prove that \( \mathcal{J}_{S'/R} \subseteq vS' \). In the first step, we replaced \( S' \) by its localization \((V, n, L')\) at a minimal prime of \( vS' \) and \( R \) by its localization at the contraction of that minimal prime. Thus, we have that \((R, m, K) \subseteq V\) is local, where \( \dim (R) = d \). Since \( V \) is essentially of finite type over \( R \), the dimension formula yields that the tr. deg. \((L'/K) = d - 1\). It will suffice to show that \( \mathcal{J}_{V/R} \subseteq vV \). Note that if \( d = 1 \), then adjoining \( y/v \) makes \( v \) invertible in \( V \), and there is nothing to prove. Thus, we may assume that \( d \geq 2 \). Consider the sequence of quadratic transforms

\[
R = (T_0, m_0, K_0) \subseteq \cdots \subseteq (T_i, m_i, K_i) \subseteq \cdots \subseteq (T_h, m_h, K_h) \subseteq \cdots
\]

By the Theorem on the finiteness of the quadratic sequence, we have that for some \( h \), \( T_h = V \cap K = W \), which is therefore essentially of finite type over \( R \). The condition \( y/v \in S - R \) shows that \( h \geq 1 \). By the multiplicative property of Jacobian ideals stated in the Corollary at the end of the Lecture of September 27, we have that \( \mathcal{J}_{V/R} = \mathcal{J}_{V/W} \mathcal{J}_{W/R} \). It therefore suffices to prove that \( \mathcal{J}_{W/R} \subseteq vW \), and so we may henceforth assume that \( V = W \) has fraction field \( K \) and is obtained from \( R \) by a finite sequence of quadratic transforms along \( V \).