Detailed proof of the Jacobian theorem: existence of sufficiently many special sequences.

Note first that if $R$ itself is a field then $S = L$ and $S' = S$, so that $J_{S/R} = J_{S'/R}$ and there is nothing to prove. If $R$ is finite then $R$ must be a field, since $R$ is a domain, and therefore we may assume without loss of generality that $R$ is infinite in the remainder of the proof.

We shall need prime avoidance in the following form (cf. [I. Kaplansky, Commutative Rings, Revised Edition, Univ. of Chicag Press, Chicago, 1974], Theorem 124, p. 90.):

Lemma (prime avoidance for cosets). Let $R$ be any commutative ring, $x \in R$, $I \subseteq R$ an ideal and $P_1, \ldots, P_k$ prime ideals of $R$. Suppose that the coset $x + I$ is contained in $\bigcup_{i=1}^k P_i$. Then there exists $j$ such that $Rx + I \subseteq P_j$.

Proof. If $k = 1$ the result is clear. Choose $k \geq 2$ minimum giving a counterexample. Then no two $P_i$ are comparable, and $x + I$ is not contained in the union of any $k - 1$ of the $P_i$. Now $x = x + 0 \in x + I$, and so $x$ is in at least one of the $P_j$; say $x \in P_k$. If $I \subseteq P_k$, then $Rx + I \subseteq P_k$ and we are done. If not, choose $i_0 \in I - P_k$. We can also choose $i \in I$ such that $x + i \notin \bigcup_{j=1}^{k-1} P_i$. Choose $u_j \in P_j - P_k$ for $j < k$, and let $u$ be the product of the $u_j$. Then $u_0 \in I - P_k$, but is in $P_j$ for $j < k$. It follows that $x + (i + u_0) \in x + I$, but is not in any $P_j$, $1 \leq j \leq k$, a contradiction. □

The following somewhat technical “general position” lemma is needed to prove that Jacobian determinants arising from special sequences generate the Jacobian ideal.

Lemma (general position for generators). Let $R \subseteq T$ be a commutative rings such that $R$ is an infinite integral domain and let $P_1, \ldots, P_r$ be mutually incomparable prime ideals of $T$ contracting to $(0)$ in $R$. Let $N \geq n \geq 1$ be integers and let $M = (g_1, \ldots, g_N)$ be a $1 \times N$ matrix over $T$ with entries in $I = \bigcap_{j=1}^r P_j$. Let $\kappa_j$ denote the field $T_{P_j}/P_j T_{P_j}$ for $1 \leq j \leq r$ and let $V_j$ denote the $\kappa_j$-vector space $P_j T_{P_j}/P_j^2 T_{P_j}$. Suppose that for all $j$, $1 \leq j \leq r$, the $\kappa_j$-span of the images of the $g_i$ under the obvious map $I \subseteq P_j \rightarrow P_j T_{P_j} \rightarrow V_j$ has $\kappa_j$-vector space dimension at least $n$.

Then one may perform elementary column operations on the matrix $M$ over $T$ so as to produce a matrix with the property that, for all $j$, $1 \leq j \leq r$, the images of any $n$ of its distinct entries are $\kappa_j$-linearly independent elements of $V_j$.

Of course, the entries of the new matrix generate the ideal $(g_1, \ldots, g_N)T$.

Proof. First note that the infinite domain $R$ is contained in each of the $\kappa_j$. 

We proceed by induction on the number of primes. If there are no primes there is nothing to prove. Now suppose that \( 1 \leq h \leq r \) and that column operations have already been performed so that any \( n \) entries have \( \kappa_j \)-independent images in \( V_j \) if \( j < h \). (If \( h = 1 \) we may use \( M \) as is, since no condition is imposed.) We need to show that we can perform elementary column operations so that the condition also holds for \( j = h \). Some \( n \) of the entries have \( \kappa_h \)-independent images in \( V_h \); by renumbering we may assume that these are \( g_1, \ldots, g_n \). We now show that by induction on \( a, n + 1 \leq a \leq N \) that we may perform elementary column operations on the matrix so that

(1) The images of the entries of the matrix in each \( V_j \) for \( j < h \) do not change and

(2) Any \( n \) of the images of \( g_1, \ldots, g_a \) in \( V_h \) are independent.

Choose \( t \in T \) so that it is in the primes \( P_j \) for \( j < h \) but not in \( P_h \). Thus, \( t \) has nonzero image \( \tau \) in \( \kappa_h \). Let \( v_j \) denote the image of \( g_j \) in \( V_h \). We may assume that the images of any \( n \) of the elements \( g_1, \ldots, g_{a-1} \) are independent in \( V_h \). Thus, it will suffice to show that there exist \( r_1, \ldots, r_n \in R \) such that the image of \( g_n + tr_1 g_1 + \cdots + tr_n g_n \) is independent of any \( n - 1 \) of the vectors \( v_1, \ldots, v_{n-1} \) in \( V_h \), i.e., such that \( v_n + \tau r_1 v_1 + \cdots + \tau r_n v_n \) is independent of any \( n - 1 \) of the vectors \( v_1, \ldots, v_{n-1} \). (Note that condition (1) is satisfied automatically because the image of \( t \) is 0 in each \( \kappa_j \) for \( j < h \).)

For each set \( D \) of \( n - 1 \) vectors in \( v_1, \ldots, v_{a-1} \), there is a nonzero polynomial \( f_D \) in \( n \) variables over \( \kappa_h \), and whose nonvanishing at the point \((r_1, \ldots, r_n)\) guarantees the independence of \( v_a + \tau r_1 v_1 + \cdots + \tau r_n v_n \) from the vectors in \( D \). To see this, choose a \( \kappa_h \)-basis for the space spanned by all the \( v_j \) and write the vectors in \( D \) and \( v_a + \tau X_1 v_1 + \cdots + \tau X_n v_n \) in terms of this basis. Form a matrix \( C \) from the coefficients. We can choose values of the \( X_i \) in \( R \) that achieve the required independence, and this means that some \( n \times n \) minor of \( C \) does not vanish identically. (If \( v_a \) is independent of the vectors in \( D \) take all the \( X_i \) to be zero. Otherwise, \( v_a \) is in the \( \kappa_h \)-span of \( D \), while at least one of the \( n \) independent vectors \( v_1, \ldots, v_n \) is not, say \( v_\nu \), and we can take all the \( X_i \) except \( X_\nu \) to be 0 and \( X_\nu = 1 \).) This minor gives the polynomial \( f_D \in \kappa_h[X_1, \ldots, X_n] \).

Choose a field extension \( \mathcal{F} \) of \( K = \frac{\mathcal{R}}{(r)} \) that contains isomorphic copies of all of the \( \kappa_j \). The product \( f \) of the \( f_D \) in \( \mathcal{F}[x_1, \ldots, x_n] \) as \( D \) varies through the \( n - 1 \) element subsets of \( v_1, \ldots, v_{a-1} \) is then a nonzero polynomial in \( \mathcal{F}[X_1, \ldots, X_n] \), and so cannot vanish identically on the infinite domain \( R \). Choose \( r_1, \ldots, r_n \in R \) so that \( f(r_1, \ldots, r_n) \neq 0 \). Then every \( f_D(r_1, \ldots, r_n) \neq 0 \). \( \square 

**Lemma.** Let \( g_1, \ldots, g_n \) be elements of a Noetherian ring \( T \) and let \( J \) be an ideal of \( T \) of depth at least \( n \) such that \( (g_1, \ldots, g_n)T + J \) is a proper ideal of \( T \). If \( g_1, \ldots, g_i \) is a regular sequence in \( T \) (i.e., we may be assuming nothing about \( g_1, \ldots, g_n \)) then there are elements \( j_{i+1}, \ldots, j_n \in J \) such that

\[
g_1, \ldots, g_i, g_{i+1} + j_{i+1}, \ldots, g_n + j_n
\]

is a regular sequence in \( T \).

In particular, there are elements \( j_1, \ldots, j_n \in J \) such that \( g_1 + j_1, \ldots, g_n + j_n \) is a regular sequence.
Proof. The last sentence is the case $i = 0$. We proceed by induction on $n - i$. If $i = n$ there is nothing to prove. We may pass to $T/(g_1, \ldots, g_i)T$, and so reduce to the case where $i = 0$. The image $J$ is the same as the image of $J' = J + (g_1, \ldots, g_i)$ modulo $(g_1, \ldots, g_i)$. $J'$ is a proper ideal of depth at least $n$, and so killing a regular sequence of length $i$ in $J'$ produces an ideal of depth at least $n - i > 0$. Thus, we may assume that $i = 0$.

It then suffices to choose $j = j_1$ such that $g_1 + j$ is not a zerodivisor, for we may apply the induction hypothesis to construct the rest of the sequence. But if this were not possible we would have that $g_1 + j$ is contained in the union of the associated primes of $(0)$ in $T$, and this implies that $J$ is contained in an associated prime of $(0)$ in $T$ by the Lemma on prime avoidance for cosets proved at the beginning of this Lecture. This is a contradiction, since the depth of $J$ is positive. \qed

**Theorem (existence of sufficiently many special sequences).** Let $R$ be an infinite Cohen-Macaulay Noetherian domain and let $S$ be a torsion-free generically étale $R$-algebra essentially of finite type over $R$. Let $T$ be a localization of a polynomial ring in $n$ variables over $R$ that maps onto $S$, and let $I$ be the kernel. Let $P_1, \ldots, P_r$ be the minimal primes of $I$ in $T$. Then the Jacobian ideal $J_{S/R}$ is generated by the images of elements $\det (\partial g_j/\partial x_i)$ such that $g_1, \ldots, g_n$ is a special sequence of elements of $I$, i.e., a regular sequence in $I$ such that for every $j$, $1 \leq j \leq r$, $P_jT_{P_j} = (g_1, \ldots, g_n)T_{P_j}$.

Proof. First choose generators $g_1, \ldots, g_N$ for $I$. Think of these generators as forming the entries of a $1 \times N$ matrix as in the Lemma on general position for generators. Each $T_{P_j}$ is regular local of dimension $n$, so that each $P_jT_{P_j}/P_j^2T_{P_j}$ has dimension $n$. It follows from the Lemma cited that we may assume without loss of generality that every $n$ element subset of the generators $g_1, \ldots, g_N$ generates every $P_jT_{P_j}$. We know that the size $n$ minors of the $n \times N$ matrix $(\partial g_j/\partial x_i)$ generate $J_{S/R}$. Fix one of these minors: by renumbering, we may assume that it corresponds to the first $n$ columns. It will suffice to show that the image of this minor in $S$ is the same as the image of a minor coming from a special sequence. We may apply the preceding Lemma to choose elements $h_1, \ldots, h_N$ in $J = I^2$ such that $g_1 + h_1, \ldots, g_n + h_n$ is a regular sequence. This sequence is special: since $J = I^2 \subseteq P_j^2$ for all $j$, the elements generate each $P_jT_{P_j}$, and it was chosen to be a regular sequence. Finally, by the Remark near the top of p. 2 of the Lecture Notes of September 27, the image of the Jacobian determinant of $g_1 + h_1, \ldots, g_n + h_n$ in $S$ is the same as the image of the Jacobian determinant of $g_1, \ldots, g_n$, and the result follows. \qed