Examples. Let \( R = K[[x, y]]/(x^2, xy) \). This ring has a unique minimal prime, \( xR \), and \( m = (x, y)R \) is embedded. The image \( \bar{x} \) of \( x \) in the ring generates a submodule isomorphic to \( R/m \), which has lower dimension. Then \( e(R) = e(R/xR) = e(K[[y]]) = 1 \).

Likewise, if \( R = K[[x, y, z]]/(x, y) \cap (z) \), then \( R \) has two minimal primes, \( (x, y)R \) and \( zR \). Thus, \( \dim(R) = \dim(R/zR) = \dim(K[[x, y]]) \), while the module \( zR \cong R/(x, y) \cong K[[z]] \) is one-dimensional. Thus, \( e(R) = e(R/zR) = e(K[[x, y]]) = 1 \).

These examples illustrate that a local ring of multiplicity 1 need not be regular. In the first example, \( R_{\text{red}} \) is a domain. In the second, \( R \) is reduced, but not equidimensional.

Finally, consider \( R = K[[u, v, x, y, z]]/((u, v) \cap (x, y) \cap z) \). This ring is reduced but not equidimensional. It has dimension 4 (when we kill \( zR \) we get \( K[[u, v, x, y]] \), but has two minimal primes with quotients of dimension 3. Consider the ring obtained when we localize at \( P = (u, v, x, y) \). The localization \( S = K[[u, v, x, y, z]]/(u, v, x, y)T \) is regular of dimension 4, and \( u, v, x, y \) is a regular system of parameters. Thus, \( R_P = S/(u, v) \cap (x, y) \) has two minimal primes with quotients that are regular of dimension 2. It follows that \( e(R) = 1 \) while \( e(R_P) = 2 \). The problem here is that we “localized away” the relevant minimal prime of \( R \) that governed its multiplicity.

Discussion: localization. One expects that under mild conditions, \( e(R_P) \leq e(R) \). But we only expect this for primes \( P \) such that \( \dim(R/P) + \dim(R_P) = \dim(R) \). (We always have \( \dim(R/P) + \dim(R_P) \leq \dim(R) \). The condition of equality means that \( P \) is part of a chain of primes of maximum length, \( \dim(R) \), in \( R \).) It is conjectured that in all local rings, whenever \( \dim(R_P) + \dim(R_P) = \dim(R) \), one has that \( e(R_P) \leq e(R) \).

In studying this problem, one is naturally led to Lech’s Conjecture. The result on localization is true if \( R \) is excellent (and under various weaker hypotheses), but, so far as I know, remains open in the general case. It would follow, however, from a proof of Lech’s Conjecture, which permits a reduction to the case where the ring is complete.

First note:

Lemma. Let \( P \) be a prime ideal of a local ring \( R \). Then:

(a) For every minimal prime \( Q \) of \( P\hat{R} \), height \( (Q) = \text{height (P)} \).

(b) If \( \dim(R/P) + \dim(R_P) = \dim(R) \), then there exists a minimal prime \( Q \) of \( PR \) such that \( \dim(\hat{R}/Q) + \dim(\hat{R}_Q) = \dim(R) \).

(c) If \( \hat{R}/P \) is reduced, then with \( Q \) as in part (b) we have that \( e(R_P) = e(\hat{R}_Q) \).

Proof. (a) \( R_P \to \hat{R}_Q \) is faithfully flat, so that \( \dim(\hat{R}_Q) \geq \dim(R_P) \). The minimality of \( Q \) implies that \( PR_P \) expands to a \( Q\hat{R}_Q \)-primary ideal in \( \hat{R}_Q \), so that a system of parameters for \( R_P \) will be a system of parameters for \( \hat{R}_Q \) as well.
For (b), note that the completion of $R/P$, which is $\hat{R}/P\hat{R}$, has the same dimension as $R/P$, and so has a minimal prime, say $Q/P\hat{R}$, where $Q$ is prime in $\hat{R}$, such that $\dim(\hat{R}/Q) = \dim(\hat{R}/P\hat{R}) = \dim(R/P)$. By part (a), $\dim(\hat{R}_Q) = \dim(R_P)$ as well.

To prove (c), observe that if $\hat{R}/P$ is reduced, then so is $\hat{R}_Q/P\hat{R}_Q$, which means that $PR_P$ expands to the maximal ideal in $\hat{R}_Q$. The equality of multiplicities then follows from the Proposition on p. 6 of the Lecture Notes of October 20. □

Our next objective, which will take a while, is to prove the following:

**Theorem (localization theorem for multiplicities).** If $P$ is a prime ideal of a complete local ring $R$ such that $\dim (R/P) + \dim (R_P) = \dim (R)$, then $e(R_P) \leq e(R)$.

Assuming this for the moment, we have several corollaries.

**Corollary.** If $P$ is a prime of a local ring $R$ such that $\dim (R/P) + \dim (R_P) = \dim (R)$ and the completion of $R/P$ is reduced,\(^1\) then $e(R_P) \leq e(R)$.

**Proof.** Choose a minimal prime $Q$ of $P\hat{R}$ such that $\dim(\hat{R}/Q) + \dim(\hat{R}_Q) = \dim(\hat{R})$, as in part (b) of the Lemma. Then by part (c),

$$e(R_P) = e(\hat{R}_Q) \leq e(\hat{R}) = e(R).$$

**Corollary.** If Lech’s conjecture holds, then for every prime $P$ of a local ring $R$ such that $\dim (R/P) + \dim (R_P) = \dim (R)$, $e(R_P) \leq e(R)$.

**Proof.** Choose $Q$ as in part (b) of the Lemma. Then $R_P \to \hat{R}_Q$ is flat local, and so by Lech’s conjecture

$$e(R_P) \leq e(\hat{R}_Q) \leq e(\hat{R}) = e(R). \quad \square$$

We also get corresponding results for modules.

**Corollary.** If $R$ is a local ring, $M$ a finitely generated $R$-module, and $P$ is a prime of the support of $M$ such that $\dim (R/P) + \dim (M_P) = \dim (M)$, then:

(a) If the completion of $R/P$ is reduced, then $e(M_P) \leq e(M)$.

(b) If Lech's conjecture holds, then $e(M_P) \leq e(M)$.

**Proof.** Note that we can replace $R$ by $R/\text{Ann}_R M$, so that we may assume that $M$ is faithful and $\dim (R) = \dim (M) = d$, say. Note that $M$ is faithful if and only if for some (equivalently, every) finite set of generators $u_1, \ldots, u_h$ for $M$, the map $R \to M^{\oplus h}$ such

\(^1\)This is always true if $R$ is excellent: the completion of an excellent reduced local ring is reduced.
that \( r \mapsto (ru_1, \ldots, ru_h) \) is injective. This condition is obviously preserved by localization. Now,

\[
\text{(*) } e(M) = \sum_{1 \leq i \leq h, \dim(R/P_i) = d} \ell_{R_{P_i}}(M_{P_i})e(R/P_i).
\]

Note that once we have that \( M \) is faithful, \( \dim(R/P) + \dim(M_P) = \dim(M) \) is equivalent to \( \dim(R/P) + \dim(R_P) = \dim(R) \), since \( M_P \) is faithful over \( R_P \). The minimal primes of \( M \) and \( R \) are the same, and so are the minimal primes of \( M_P \) and \( R_P \): the latter correspond to the minimal primes of \( R \) that are contained in \( P \). There is a formula like (*) for \( e(M_P) \), where the summation is extended over minimal primes \( p \) of the support of \( M_P \), i.e., of \( R_P \), such that \( \dim(R/P) / p = \dim(M_P) \), which is \( \dim(R_P) \). Let \( p \) be such a minimal prime. Then there is a chain of primes from \( p \) to \( P \) of length \( \text{height}(P) \), and this can be concatenated with a chain of primes of length \( \dim(R/P) \) from \( P \) to \( m \), producing a chain of length \( \dim(R) \). It follows that \( \dim(R/P) \geq d \), and the other inequality is obvious. Therefore, \( p \) is one of the \( P_i \). Moreover, in \( R/P_i \), we still have

\[
\dim \left( \frac{(R/P_i)}{(P/P_i)} \right) + \text{height} \left( \frac{(R/P_i)}{P/P_i} \right) = \dim(R/P_i) = d.
\]

Thus, the terms in the formula corresponding to (*) for \( M_P \) correspond to a subset of the the terms occurring in (*), but have the form

\[
\ell_{R_{P_i}}(M_{P_i})e(R/P_i)R_{P_i}.
\]

Note that each \( P_i \) occurring is contained in \( P \), and localizing first at \( P \) and then at \( P_i R_P \) produces the same result as localizing at \( P_i \). Using either (a) or (b), whichever holds, we have that every \( e((R/P_i)P) \leq e(R/P_i) \). □

We next want to understand multiplicities in the hypersurface case.

**Theorem.** Let \((R, m, K)\) be a regular local ring of dimension \( d \) and let \( f \in m \). Let \( S = R/fR \). The \( e(S) \) is the \( m \)-adic order of \( f \), i.e., the unique integer \( k \) such that \( f \in m^k - m^{k+1} \).

**Proof.** We use induction on \( \dim(R) \). If \( \dim(R) = 1 \) the result is obvious. Suppose \( \dim(R) > 1 \). We replace \( R \) by \( R(t) \) if necessary so that we may assume the residue class field is infinite. Choose a regular system of parameters \( x_1, \ldots, x_d \) for \( R \). By replacing these by linearly independent linear combinatons we may assume that \( x_1 \) is such that

1. \( x_1 \) does not divide \( f \), so that the image of \( x_1 \) is not a zerodivisor in \( S \).
2. The image of \( x_1 \) in \( m/m^2 \) does not divide the leading form of \( f \) in \( \text{gr}_m(R) \).
3. The image of \( x_1 \) in \( S \) is part of a minimal set of generators for a minimal reduction of \( m/fR \), the maximal ideal of \( S \).

Let \( \pi \) be the image of \( x_1 \) in \( S \). Then \( e(S) = e(S/\pi S) \), and this is the quotient of the regular ring \( R/x_1 R \) by the image of \( f \). Moreover, the \((m/x_1 R)\)-adic order of the image of
Theorem (symbolic powers in regular rings). Let $R$ be a regular ring. Then $R^{(n)} \subseteq Q^{(n)}$ for every positive integer $n$.

We next want to reduce the problem of proving the localization result for complete local domains to proving the following statement:

Theorem (symbolic powers in regular rings). Let $P \subseteq Q$ be prime ideals of a regular ring $R$. Then $P^{(n)} \subseteq Q^{(n)}$ for every positive integer $n$.

We postpone the proof for the moment. Note, however, that one can reduce at once to the local case, where $Q$ is the maximal ideal, by working with $(R_Q, Q_R)$ instead of $R$.

Discussion: the symbolic power theorem for regular rings implies that multiplicities do not increase under localization. Let $R$ be a complete local, and let $P$ be a prime ideal of $R$ such that $\dim (R/P) + \text{height } (R_P) = \dim (R)$. We want to show that $e(R_P) \leq e(R)$. Exactly as in the discussion of the module case in the proof of the Corollary, one can reduce to the case where $R$ is a domain. As usual, one may assume without loss of generality that the residue field is sufficiently large for $R$ to have a system of parameters $x_1, \ldots, x_d$ that generates a minimal reduction of $m$. Then in the equicharacteristic case (respectively, the mixed characteristic case), we can map $K[[X_1, \ldots, X_d]] \to R$ (respectively, $V[[X_1, \ldots, X_d]] \to R$), where $K \subseteq R$ (respectively $V \subseteq R$) is a coefficient field (respectively, a complete DVR that is a coefficient ring) and so that $X_i \mapsto x_i$, $1 \leq i \leq d$. In both cases, $R$ is module-finite over the image $A$: in the equicharacteristic case, $A = K[[x_1, \ldots, x_d]]$ is regular, while in mixed characteristic the kernel of $V[[X_1, \ldots, X_d]] \to R$ must be a height one prime, and therefore principle, so that $A \cong V[[x_1, \ldots, x_d]]/(f)$. Since the maximal ideal of $R$ is integral over $(x_1, \ldots, x_d)R$ and $R$ is module-finite over $A$, the maximal ideal of $A$ is also integral over $(x_1, \ldots, x_d)A$. Let $\rho$ denote the torsion-free rank of $R$ as an $A$-module, which is the same as the degree of the extension of fraction fields. Suppose that $P$ is a prime of $R$ and let $p$ be its contraction to $A$. Let $I$ be the ideal $(x_1, \ldots, x_d)A$. Then $e(R) = e_{1/R}(R)$, which is the same as $e_{I}(R)$ with $R$ thought of as an $A$-module. This is $\rho e(I)(A) = \rho e(A)$. The result on symbolic powers gives the result on localization of multiplicities for $A = T/(f)$, when $T$ is regular: one multiplicity is the order of $f$ in $T$ with respect to the maximal ideal, while the other is the order of $f$ in a localization of $T$. (In the equicharacteristic case, both $A$ and its localization are regular, and both multiplicities are 1.) Thus, $\rho e(A_p) \leq \rho e(A) = e(R)$. But we shall see in the sequel that $e(R_P) \leq e_p(R_p)$, with $R_p$ is viewed as an $A_p$ module. Since $R$ is module-finite over $A$, $R_p$ is module-finite over $A_p$, and $R_p/p^nR_p$ is an Artin ring, and is a product of local rings one of which is $R_p/(p^nR_p)$. Then

$$\ell_{A_p}(R_p/p^nR_p) \geq \ell_{A_p}(R_P/p^nR_P) \geq \ell_{A_p}(R_P/P^nR_P) \geq \ell_{R_P}(R_P/P^nR_P)$$

for all $n$, so that the multiplicity of $R_p$ as an $A_p$-module is greater than or equal to $e(R_P)$. But then

$$e(R_P) \leq e_p(R_p) = \rho e(A_p) \leq \rho e(A) = e(R),$$

as required. □

Thus, all that remains is to prove the theorem on symbolic powers in regular rings.