Math 711: Lecture of September 14, 2007

The following result is very useful in thinking about tight closure.

Proposition. *Let R be a Noetherian ring of prime characteristic $p > 0$, let $N \subseteq M$ be R-modules, and let $u \in M$. Then $u \in N^*_M$ if and only if the image $\overline{u}$ of u in the quotient M/N is in $0^*_{M/N}$.*

Hence, if we map a free module G onto M, say $h : G \twoheadrightarrow M$, let $H = h^{-1}(N) \subseteq G$, and let $v \in G$ be such that $h(v) = u$, then $u \in N^*_M$ if and only if $v \in H^*_G$.

Proof. For the first part, let $c \in R^0$. Note that, by the right exactness of tensor products, $\mathcal{F}^e(M/N) \cong \mathcal{F}^e(M)/N^{[q]}$. Consequently, $cu^q \in N^{[q]}$ for all $q \geq q_0$ if and only if $c\overline{u}^q = 0$ in $\mathcal{F}^e(M/N)$ for $q \geq q_0$.

For the second part, simply note that the image of v in $G/H \cong M/N$ corresponds to $\overline{u}$ in M/N. $\square$

It follows many questions about tight closure can be formulated in terms of the behavior of tight closures of submodules of free modules. Of course, when M is finitely generated, the free module G can be taken to be finitely generated with the same number of generators.

Given a free module G of rank n, we can choose an ordered free basis for G. This is equivalent to choosng an isomorphism $G \cong R^n = R \oplus \cdots \oplus R$. In the case of R^n, one may understand the action of Frobenius in a very down-to-earth way. We may identify $\mathcal{F}^e(R^n) \cong R^n$, since we have this identification when $n = 1$. Keep in mind, however, that the identification of $\mathcal{F}^e(G)$ with G depends on the choice of an ordered free basis for G. If $u = r_1 \oplus \cdots \oplus r_n \in R^n$, then $u^q = r_1^q \oplus \cdots \oplus r_n^q$. With $H \in R^n$, $H^{[q]}$ is the R-span of the elements u^q for $u \in H$ (or for u running through generators of H). Very similar remarks apply to the case of an infinitely generated free module G with a specified basis b_λ. The elements b_λ^q give a free basis for $\mathcal{F}^e(G)$, and if $u = r_1 b_{\lambda_1} + \cdots + r_s b_{\lambda_s}$, then $u^q = r_1^q b_{\lambda_1}^q + \cdots + r_s^q b_{\lambda_s}^q$ gives the representation of u^q as a linear combination of elements of the free basis $\{b_\lambda^q\}_\lambda$.

We could have defined tight closure for submodules of free modules using this very concrete description of u^q and $H^{[q]}$. The similarity to the case of ideals in the ring is visibly very great. But we are then saddled with the problem of proving that the notion is independent of the choice of free basis. Moreover, if we take this approach, we need to define N^*_M by mapping a free module G onto M and replacing N by its inverse image in G. We then have the problem of proving that the notion we get is independent of the choices we make.

Our next objective is to prove that in a regular ring, every ideal is tightly closed. This depends on knowing that $F : R \to R$ is flat for regular rings of prime characteristic $p > 0$.

Eventually we sketch below a proof of the flatness of F that depends on the structure theory for complete local rings of prime characteristic $p > 0$. Later, we shall give a different proof, based on the following result, which is valid without restriction on the characteristic:

Theorem. *Let $(R,\, m,\, K)$ be a regular local ring and M and R-module. Then M is a big Cohen-Macaulay module for R if and only if M is faithfully flat over R.*

We postpone the proof of this result for a while: it makes considerable use of the properties of the functor Tor. However, we do want to make several comments.

First note that it immediately implies that when R is regular, $F : R \to R$ is flat. In general, $R \to S$ is flat if and only if for every prime ideal Q of S with contraction P to R, the map $R_P \to S_Q$ is flat.

To see this, note that for R_P-modules M, the natural map $S_Q \otimes_R M \to S_Q \otimes_{R_P} M$ is an isomorphism, because $M \to R_P \otimes_{R_P} M$ is an isomorphism, and we have

$$S_Q \otimes_{R_P} M \cong S_Q \otimes_{R_P} (R_P \otimes_R M) \cong S_Q \otimes_R M.$$

The latter is also $S_Q \otimes_S (S \otimes_R M)$. If S is flat over R, since S_P is flat over S we have that S_P is flat over R. On the other hand, if $N \hookrightarrow M$ is an injection of R-modules and $S \otimes_R N \to S \otimes_R M$ is not injective, we can localize at a prime Q of S in the support of the kernel. This yields a map $S_Q \otimes_R N \to S_Q \otimes_R M$ that is not injective. But if Q contracts to P, we do have that $N_P \to M_P$ is injective. This shows that S_Q is not flat over R_P. $\square$

Note that when $S = R$ and the map is F, the contraction of $P \in \mathrm{Spec}\,(R)$ is P. Thus, it suffices to show that F is flat on R_P for all primes P. This is now obvious given the Theorem above: any regular sequence (equivalently, system of parameters) in R, say $x_1,\, \dots\,, x_n$, maps to $x_1^p,\, \dots\,,\, x_n^p$ in R, which is again a regular sequence. Hence, R is a big Cohen-Macaulay algebra for R under the map $F : R \to R$, and this proves that R is faithfully flat over R.

We have the following additional comments on the Theorem. Suppose that M is a module over a local ring $(R,\, m,\, K)$ and suppose that we know that $x_1,\, \dots\,, x_n$ is a system of parameters that is a regular sequence on M. Let $\mathfrak{A} = (x_1,\, \dots\,, x_n)R$. By the definition of a regular sequence, we have that $\mathfrak{A}M \neq M$. We want to point out that this condition implies the *a priori* stronger condition that $mM \neq M$. The reason is that m is nilpotent modulo $\mathfrak{A}$. Thus, we can choose s such that $m^s \subseteq \mathfrak{A}$. If $M = mM$, we can multiply by m^t to conclude that $m^tM = m^t(mM) = m^{t+1}M$. Thus

$$M = mM = m^2M = \cdots = m^tM = \cdots .$$

Then

$$M = m^sM \subseteq \mathfrak{A}M \subseteq M,$$

and we find that $\mathfrak{A}M = M$, a contradiction.

If M is faithfully flat over R, we have that $(R/m) \otimes M = M/mM \neq 0$, so that $mM \neq M$. Moreover, whenever $x_1, \ldots, x_n$ is a system of parameters for R, it is a regular sequence on R, and the fact that M is faithfully flat over R implies that $x_1, \ldots, x_n$ is a regular sequence on M. This shows that a faithfully flat R-module is a big Cohen-Macaulay module over R. The converse remains to be proved.

We next sketch a completely different proof that F is flat for a regular ring R. As noted above, this comes down to the local case. We use the fact that a local map $R \to S$ of local rings is flat if and only if the induced map $\widehat{R} \to \widehat{S}$ is flat. Hence, by the structure theory of complete local rings, we may assume that $R = K[[x_1, \ldots, x_n]]$ is a formal power series ring over a field. Since this ring is the completion of $K[x_1, \ldots, x_n]_P$ where $P = (x_1, \ldots, x_n)$, it suffices to prove the result for the localized polynomial ring $R = K[x_1, \ldots, x_n]$ itself. But $F(R) = K^p[x_1^p, \ldots, x_n^p]$. Thus, all we need to show is that $K^p[x_1, \ldots, x_n] \subseteq K[x_1, \ldots, x_n]$ is flat. We prove a stronger result: R is free over $F(R)$ in this case. Since K is free over K^p, $K[x_1^p, \ldots, x_n^p]$ is free on the same basis over $K^p[x_1^p, \ldots, x_n^p]$. Thus, we need only see that $K[x_1, \ldots, x_n]$ is free over $K[x_1^p, \ldots, x_n^p]$. It is easy to check that the monomials $x_1^{a_1} \cdots x_n^{a_n}$ such that $0 \leq a_i \leq p-1$ are free basis. $\square$

We now fill in the missing details of the argument sketched above.

Proposition. *Let $\theta : (R, m, K) \to (S, \mathfrak{n}, L)$ be a homomorphism of local rings that is* local, *i.e., $\theta(m) \subseteq \mathfrak{n}$. Let Q be a finitely generated S-module. Then Q is flat over R if and only if for every injective map $N \hookrightarrow M$ of finite length R-modules, $Q \otimes_R N \to Q \otimes_R M$ is injective.*

Proof. The condition is obviously necessary. We shall show that it is sufficient. Since tensor commutes with direct limits and every injection $N \hookrightarrow M$ is a direct limit of injections of finitely generated R-modules, it suffices to consider the case where $N \subseteq M$ are finitely generated. Suppose that some $u \in S \otimes_R N$ is such that $u \mapsto 0$ in $S \otimes_R M$. It will suffice to show that there is also such an example in which M and N have finite length. Fix any integer $t > 0$. Then we have an injection

$$N/(m^t M \cap N) \hookrightarrow M/m^t M$$

and there is a commutative diagram

$$\begin{array}{ccc} Q \otimes_R N & \xrightarrow{\iota} & Q \otimes_R M \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\ Q \otimes_R \big(N/(m^t M \cap N)\big) & \xrightarrow{\iota'} & Q \otimes_R \big(M/m^t M\big) \end{array}.$$

The image $f(u)$ of u in $Q \otimes_R \bigl(N/(m^t M \cap N)\bigr)$ maps to 0 under ι', by the commutativity of the diagram. Therefore, we have the required example provided that $f(u) \neq 0$. However, for all $h > 0$, we have from the Artin-Rees Lemma that for every sufficiently large integer t, $m^t M \cap N \subseteq m^h N$. Hence, the proof will be complete provided that we can show that the image of u is nonzero in

$$Q \otimes_R (N/m^h N) \cong Q \otimes_R \bigl((R/m^h) \otimes_R N\bigr) \cong (R/m^h) \otimes_R (Q \otimes_R N) \cong (Q \otimes_R N)/m^h (Q \otimes_R N)$$

for $h \gg 0$. But

$$m^h (Q \otimes_R N) \subseteq \mathfrak{n}^h (Q \otimes_R N),$$

and the result follows from the fact that the finitely generated S-module $Q \otimes_R N$ is $\mathfrak{n}$-adically separated. $\square$

We can now prove the following result, which is the only missing ingredient needed to fill in the details of our proof that F is flat.

Lemma. *Let $(R,\, m,\, K) \to (S,\, \mathfrak{n}, L)$ be a local homomorphism of local rings. Then S is flat over R if and only if $\widehat{S}$ is flat over $\widehat{R}$, and this holds iff $\widehat{S}$ is flat over R.*

Proof. If S is flat over R then, since $\widehat{S}$ is flat over S, we have that $\widehat{S}$ is flat over R. Conversely, if $\widehat{S}$ is flat over R, then S is flat over R because $\widehat{S}$ is faithfully flat over S: if $N \hookrightarrow M$ is injective but $S \otimes_R N \to S \otimes_R M$ has a nonzero kernel, the kernel remains nonzero when we apply $\widehat{S} \otimes_S \_$, and this has the same effect as applying $\widehat{S} \otimes_R \_$ to $N \hookrightarrow M$, a contradiction.

We have shown that $R \to S$ is flat if and only $R \to \widehat{S}$ is flat. If $\widehat{R} \to \widehat{S}$ is flat then since $R \to \widehat{R}$ is flat, we have that $R \to \widehat{S}$ is flat, and we are done. It remains only to show that if $R \to S$ is flat, then $\widehat{R} \to \widehat{S}$ is flat. By the Proposition, it suffices to show that if $N \subseteq M$ have finite length, then $\widehat{S} \otimes N \to \widehat{S} \otimes M$ is injective. Suppose that both modules are killed by m^t. Since $S/m^t S$ is flat over R/m^t, if Q is either M or N we have that

$$\widehat{S} \otimes_{\widehat{R}} Q \cong \widehat{S}/m^t \widehat{S} \otimes_{\widehat{R}/m^t \widehat{R}} Q \cong \widehat{S}/m^t \widehat{S} \otimes_{R/m^t} Q \cong \widehat{S} \otimes_R Q,$$

and the result now follow because $\widehat{S}$ is flat over R. $\square$

The following result on behavior of the colon operation on ideals under flat base change, while quite easy and elementary, plays a very important role in tight closure theory. Recall that when $I \subseteq R$ and $R \to S$ is a flat homomorphism, the map $I \otimes_R S \to R \otimes_R S = S$ is injective. Its image is clearly IS, the expansion of I to S. Thus, $I \otimes_R S$ may be naturally identified with IS when S is flat over R. Recall that if I and J are ideals of R, then

$$I :_R J = \{r \in R : rJ \subseteq I\},$$

which is an ideal of R. If $J = fR$ is principal, we may write $I :_R f$ for $I_R : fR$.

Proposition. *Let $R \to S$ be flat and let I and J be ideals of R such that J is finitely generated. Then $IS :_R JS = (I :_R J)S$.*

Proof. Let $J = (f_1, \dots, f_n)R$. We have an exact sequence

$$0 \to I :_R J \hookrightarrow R \to (R/I)^{\oplus n}$$

where the rightmost map sends $r \mapsto (\overline{rf_1}, \dots, \overline{rf_n})$; here, $\overline{g}$ denotes the image of g modulo I. The exactness is preserved when we apply $S \otimes_R \_$, which yields an exact sequence

$$(*) \quad 0 \to (I :_R J)S \hookrightarrow S \to (S/IS)^{\oplus n}$$

where the rightmost map sends $s \mapsto (\widetilde{sf_1}, \dots, \widetilde{sf_n})$ and $\widetilde{g}$ denotes the image of g modulo IS. From the definition of this map, the kernel is $IS :_S JS$, while from the exact sequence $(*)$ just above, the kernel is $(I :_R J)S$. $\square$

The result is false without the hypothesis that J be finitely generated. Let K be a field, and let $R = K[y, x_1, x_2, x_3, \dots]$ be a polynomial ring in infinitely many variables over K. Let $I = (x_1 y, x_2 y^2, \dots, x_n y^n, \dots)$ and let $J = (x_1, x_2, \dots, x_n, \dots)$. Then $I :_R J = I$, but if $S = R_y$, $IS = JS$ and $IS :_S JS = S$.

The proposition above has the following very important consequence:

Corollary. *Let R be a regular Noetherian ring of prime characteristic $p > 0$. Let I and J be any two ideals of R. Then for every $q = p^e$, we have that $I^{[q]} :_R J^{[q]} = (I :_R J)^{[q]}$.*

Proof. Take $R \to S$ to be the map $F^e : R \to R$, which is flat. Then

$$I^{[q]} :_R J^{[q]} = IS :_S JS = (I :_R J)S = (I :_R J)^{[q]}. \qquad \square$$

We can now prove that every ideal of a regular ring is tightly closed.

Theorem. *Let R be a regular Noetherian ring of prime characteristic $p > 0$. Let $I \subseteq R$ be any ideal. Then $I = I^*$.*

Proof. Suppose that we have a counterexample with $u \in I^* - I$. Choose a prime P in the support of $(I + Ru)/I$. In R_P, the image of u is still in $(IR_P)^*$ working over R_P, while it is not in IR_P by our choice of P. Therefore, it suffices to prove the result for a regular local ring (R, m, K). Since $u \in I^* - I$, we have that $I :_R u$ is a proper ideal of R. Hence, $I :_R u \subseteq m$. We know that there exists $c \in R^\circ$ such that for all $q \gg 0$, $cu^q \in I^{[q]}$. Hence, for all $q \geq q_0$ we have

$$c \in I^{[q]} :_R u^q = (I :_R u)^{[q]} \subseteq m^{[q]} \subseteq m^q,$$

i.e., $c \in \bigcap_{q \geq q_0} m^q = (0)$, contradicting that $c \in R^\circ$. $\square$