We want to use the theory of strongly F-regular F-finite rings to prove the existence of test elements.

We first prove two preliminary results:

**Lemma.** Let $R$ be an F-finite reduced ring and $c \in R^\circ$ be such that $R_c$ is F-split (which is automatic if $R_c$ is strongly F-regular). Then there exists an $R$-linear map $\theta : R^{1/p} \to R$ such that the value on 1 is a power of $c$.

**Proof.** We can choose an $R_c$-linear map $(R_c)^{1/p} \to R_c$ such that $1 \mapsto 1$, and $(R_c)^{1/p} \simeq (R^{1/p})_c$.

Then $\text{Hom}_{R_c}(R_c^{1/p}, R_c)$ is the localization of $\text{Hom}_R(R^{1/p}, R)$ at $c$, and so we can write $\theta = \frac{1}{c^N} \alpha$, where $N \in \mathbb{N}$ and $\alpha : R^{1/p} \to R$ is $R$-linear. But then $\alpha = c^N \beta$ and so $\alpha(1) = c^N \beta(1) = c^N$, as required. \qed

**Lemma.** Let $R$ be a reduced F-finite ring and suppose that there exists an $R$-linear map $\theta : R^{1/p} \to R$ such that $\theta(1) = c \in R^\circ$. Then for every $q = p^e$, there exists an $R$-linear map $\eta_q : R^{1/q} \to R$ such that $\eta_q(1) = c^2$.

**Proof.** We use induction on $q$. If $q = 1$ we may take $\eta_1 = c^2 1_R$, and if $q = p$ we may take $\eta_p = c \theta$. Now suppose that $\eta_q$ has been constructed for $q \geq p$. Then $\eta_q^{1/p} : R^{1/pq} \to R^{1/p}$, it is $R^{1/p}$-linear, hence, $R$-linear, and its value on 1 is $c^{2/p}$. Define

$$\eta_{pq}(u) = \theta(c^{(p-2)/p} \eta_q(u)).$$

Consequently, we have, as required, that

$$\eta_{pq}(1) = \theta(c^{(p-2)/p} \eta_q(1)) = \theta(c^{(p-2)/p} c^{2/p}) = \theta(c) = c \theta(1) = c^2. \quad \square$$

We can now prove the following:

**Theorem (existence of big test elements).** Let $R$ be F-finite and reduced. If $c \in R^\circ$ and $R_c$ is strongly F-regular, then $c$ has a power that is a big test element. If $R_c$ is strongly F-regular and there exists an $R$-linear map $\theta : R^{1/p} \to R$ such that $\theta(1) = c$, then $c^3$ is a big test element.

**Proof.** Since $R_c$ is strongly F-regular it is F-split. By the first Lemma on p. 1 there exist an integer $N$ and an $R$-linear map $\theta : R^{1/p} \to R$ such that $\theta(1) = c^N$. By the second
Suppose that \( c \) satisfies the hypothesis of the second statement. By part (a) of the Proposition at the bottom of p. 8 of the Lecture Notes of September 17, it suffices to show that if \( N \subseteq M \) are arbitrary modules and \( u \in N^1_M \), then \( c^3 u \in N \). We may map a free module \( G \) onto \( M \), let \( H \) be the inverse image of \( N \) in \( G \), and let \( v \in G \) be an element that maps to \( u \in N \). Then we have \( v \in H^*_G \), and it suffices to prove that \( c^3 v \in H \). Since \( v \in H^*_G \) there exists \( d \in R^2 \) such that \( dv^q \in H^{[q]} \) for all \( q \geq q_1 \). Since \( R_c \) is strongly F-regular, there exist \( q_d \) and an \( R_c \)-linear map \( \beta : (R_c)^{1/q_d} \to R_c \) that sends \( d^{1/q_d} \to 1 \): we may take \( q_d \) larger, if necessary, and so we may assume that \( q_d \geq q_1 \). As usual, we may assume that \( \beta = \frac{1}{c^q} \alpha \) where \( \alpha : R^{1/q_d} \to R \) is \( R \)-linear. Hence, \( \alpha = c^q \beta \), and \( \alpha(d^{1/q_d}) = c^q \). It follows that \( \alpha^{1/q} : R^{1/q_d} \to R^{1/q} \) is \( R^{1/q} \)-linear, hence, \( R \)-linear, and its value on 1 is \( c \). By the preceding Lemma we have an \( R \)-linear map \( \eta_q : R^{1/q} \to R \) whose value on 1 is \( c^2 \), so that \( \eta_q(c) = c \eta^q(1) = c^3 \). Let \( \gamma = \eta_q \circ \alpha^{1/q} \), which is an \( R \)-linear map \( R^{1/q_d} \to R \) sending \( d^{1/q_d} \) to \( \eta_q(c) = c^3 \). Since \( q_d q \geq q_1 \), we have \( dv^{q_d} \in H^{[q_d]} \), i.e.,

\[
(\#) \quad dv^{q_d} = \sum_{i=1}^n r_i h_i^q,
\]

for some integer \( n > 0 \) and elements \( r_1, \ldots, r_n \in R \) and \( h_1, \ldots, h_n \in H \).

Consider \( G' = R^{1/q_d} \otimes_R G \). We identify \( G \) with its image under the map \( G \to G' \) that sends \( g \mapsto 1 \otimes g \). Thus, if \( s \in R^{1/q_d} \), we may write \( sg \) instead of \( s \otimes g \). Note that \( G' \) is free over \( R^{1/q_d} \), and the \( R \)-linear map \( \gamma : R^{1/q_d} \to R \) induces an \( R \)-linear map

\[
\gamma' : G' = R^{q_d} \otimes_R G \to R \otimes_R G \cong G
\]

that sends \( sg \mapsto \gamma(s)g \) for all \( s \in R^{1/q_d} \) and all \( g \in G \). Note that by taking \( q_d q \) th roots in the displayed equation (\#) above, we obtain

\[
(\dagger) \quad d^{1/q_d} v = \sum_{i=1}^n r_i^{1/q_d} h_i.
\]

We may now apply \( \gamma' \) to both sides of (\dagger): we have

\[
c^3 v = \sum_{i=1}^n \gamma(r_i^{1/q_d}) h_i \in H,
\]

exactly as required. \( \square \)

**Discussion.** As noted on the bottom of p. 2 and top of p. 3 of the Lecture Notes of September 21, it follows that every F-finite reduced ring has a big test element: one can choose \( c \in R^o \) such that \( R_c \) is regular. This is a consequence of the fact that F-finite
rings are excellent. But one can give a proof of the existence of such elements $c$ in $F$-finite rings of characteristic $p$ very easily if one assumes that a Noetherian ring is regular if and only if the Frobenius endomorphism is flat (we proved the “only if” direction earlier). See [E. Kunz, Characterizations of regular local rings of characteristic $p$, Amer. J. Math. 91 (1969) 772–784]. Assuming the “if” direction, we may argue as follows. First note that one can localize at one such element $c$ so that the idempotent elements of the total quotient ring of $R$ are in the localization. Therefore, there is no loss of generality in assuming that $R$ is a domain. Then $R^{1/p}$ is a finitely generated torsion-free $R$-module. Choose a maximal set $s_1, \ldots, s_n$ of $R$-linearly independent elements in $R^{1/p}$. This gives an inclusion

$$R^n \cong R s_1 + \cdots + R s_n \subseteq R^{1/p}.$$ 

Call the cokernel $C$. Then $C$ is finitely generated, and $C$ must be a torsion module over $R$: if $s_{n+1} \in R^{1/p}$ represents an element of $C$ that is not a torsion element, then $s_1, \ldots, s_{n+1}$ are linearly independent over $R$, a contradiction. Hence, there exists $c \in R^e$ that kills $C$, and so $c R^{1/p} \subseteq R^n$. It follows that $(R^{1/p})_c \cong R^n_c$, and so $(R_c)^{1/p}$ is free over $R_c$. But this implies that $F_{R_c}$ is flat, and so $R_c$ is regular, as required. □

In any case, we have proved:

**Corollary.** If $R$ is reduced and $F$-finite, then $R$ has a big test element. Hence, $\tau_b(R)$ is generated by the big test elements of $R$, and $\tau(R)$ is generated by the test elements of $R$. □

Our next objective is to show that the big test elements produced by the Theorem on p. 1 are actually completely stable. In fact, we shall prove something more: they remain test elements after any geometrically regular base change, i.e., their images under a flat map $R \to S$ with geometrically regular fibers are again test elements.