Math 711: Lecture of October 3, 2007

Koszul homology

We define the i th *Koszul homology module* $H_i(x_1, \dots, x_n; M)$ of M with respect to $x_1, \dots, x_n$ as the i th homology module $H_i\bigl(\mathcal{K}_\bullet(x_1, \dots, x_n; M)$ of the Koszul complex.

We note the following properties of Koszul homology.

Proposition. *Let R be a ring and $\underline{x} = x_1, \dots, x_n \in R$. Let $I = (\underline{x})R$. Let M be an R-module.*

(a) $H_i(\underline{x}; M) = 0$ *if* $i < 0$ *or if* $i > n$.

(b) $H_0(\underline{x}; M) \cong M/IM$.

(c) $H_n(\underline{x}; M) = \mathrm{Ann}_M I$.

(d) $\mathrm{Ann}_R M$ *kills every* $H_i(x_1, \dots, x_n; M)$.

(e) *If M is Noetherian, so is its Koszul homology* $H_i(\underline{x}; M)$.

(f) *For every i, $H_i(\underline{x};\ \_\ )$ is a covariant functor from R-modules to R-modules.*

(g) *If*

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of R-modules, there is a long exact sequence of Koszul homology

$$\cdots \to H_i(\underline{x}; M') \to H_i(\underline{x}; M) \to H_i(\underline{x}; M'') \to H_{i-1}(\underline{x}; M) \to \cdots$$

(h) *If $x_1, \dots, x_n$ is a possibly improper regular sequence on M, then $H_i(\underline{x}; M) = 0$, $i \geq 1$.*

Proof. Part (a) is immediate from the definition. Part (b) follows from the fact that last map in the Koszul complex from $\mathcal{K}_1(\underline{x}; M) \to \mathcal{K}_0(\underline{x}; M)$ may be identified with the map $M^n \to M$ such that $(v_1, \dots, v_n) \mapsto x_1 v + \cdots + x_n v_n$. Part (c) follows from the fact that the map $\mathcal{K}_n(\underline{x}; M) \to \mathcal{K}_{n-1}(\underline{x}; M)$ may be identified with the map $M \to M^n$ such that $v \mapsto (x_1 v, -x_2 v, \cdots, (-1)^{n-1} x_n v)$.

Parts (d) and (e) are clear, since every term in the Koszul complex is itself a direct sum of copies of M.

To prove (f), note that if we are given a map $M \to M'$, there is an induced map of complexes

$$\mathcal{K}_\bullet(\underline{x}; R) \otimes M \to \mathcal{K}_\bullet(\underline{x}; R) \otimes M'.$$

This map induces a map $H_i(\underline{x};\, M) \to H_i\underline{x};\, M')$. Checking that this construction gives a functor is straightforward.

For part (g), we note that

$$(*) \quad 0 \to \mathcal{K}_\bullet(\underline{x};\, R) \otimes_R M' \to \mathcal{K}_\bullet(\underline{x};\, R) \otimes_R M \to \mathcal{K}_\bullet(\underline{x};\, R) \otimes_R M'' \to 0$$

is a short exact sequence of complexes, because each $\mathcal{K}_j(\underline{x};\, R)$ is R-free, so that the functor $\mathcal{K}_j(\underline{x};\, R) \otimes_R \_$ is exact. The long exact sequence is simply the result of applying the snake lemma to $(*)$. (This sequence can also be constructed by interpreting Koszul homology as a special case of Tor: we return to this point later.)

Finally, part (h) is immediate by induction from the iterative construction of the Koszul complex as a mapping cone and the Proposition at the top of p. 6 of the Lecture Notes of October 1. The map of augmentations is the map given by multiplication by x_n from $M/(x_1, \ldots, x_{n-1})M$ to itself, which is injective because $x_1, \ldots, x_n$ is a possibly improper regular sequence. $\square$

Corollary. *Let $\underline{x} = x_1, \ldots, x_n$ be a regular sequence on R and let $I = (\underline{x})R$. Then R/I has a finite free resolution of length n over R, and does not have any projective resolution of length shorter than n. Moreover, for every R-module M,*

$$\mathrm{Tor}_i^R(R/I,\, M) \cong H_i(\underline{x};\, M).$$

Proof. By part (f) of the preceding Proposition, $\mathcal{K}_\bullet(\underline{x};\, R)$ is acyclic. Since this is a free complex of finitely generated free modules whose augmentation is R/I, we see that R/I has the required resolution. Then, by definition of Tor, we may calculate $\mathrm{Tor}_i^R(R/I,\, M)$ as $H_i\big(\mathcal{K}_\bullet(\underline{x};\, R) \otimes_R M\big)$, which is precisely $H_i(\underline{x};\, M)$. To see that there is no shorter projective resolution of R/I, take $M = R/I$. Then

$$\mathrm{Tor}_n(R/I,\, R/I) = H_n(\underline{x};\, R/I) = \mathrm{Ann}_{R/I} I = R/I,$$

by part (c) of the preceding Proposition. If there were a shorter projective resolution, we would have $\mathrm{Tor}_n(R/I,\, R/I) = 0$. $\square$

Independence of Koszul homology of the base ring

The following observation is immensely useful. Suppose that we have a ring homomorphism $R \to S$ and an S-module M. By restriction of scalars, M is an R-module. Let $\underline{x} = x_1, \ldots, x_n \in R$ and let $\underline{y} = y_1, \ldots, y_n$ be the images of the x_i in S. Note that the actions of x_i and y_i on M are the same for every i. This means that the complexes $\mathcal{K}_\bullet(\underline{x};\, M)$ and $\mathcal{K}_\bullet(\underline{y};\, M)$ are the same. In consequence, $H_j(\underline{x};\, M) \cong H_j(\underline{y};\, M)$ for all j, as S-modules. Note that even if we treat M as an R-module initially in caclulating $H_j(\underline{x};\, M)$, we can recover the S-module structure on the Koszul homoolgy from the S-module structure of M. For every $s \in S$, multiplication by s is an R-linear map from M to M, and since $H_i(\underline{x};\ \_)$ is a covariant functor, we recover the action of s on $H_i(\underline{x};\ M)$.

Koszul homology and Tor

Let R be a ring and let $\underline{x} = x_1, \dots, x_n \in R$. Let M be an R-module. We have already seen that if $x_1, \dots, x_n$ is a regular sequence in R, then we may interpret $H_i(x_1, \dots, x_n;\ M)$ as a Tor over R.

In general, we may interpret $H_i(\underline{x};\ M)$ as a Tor over an auxiliary ring. Let A be any ring such R is an A-algebra. We may always take $A = \mathbb{Z}$ or $A = R$. If R contains a field K, we may choose $A = K$. Let $\underline{X} = X_1, \dots, X_n$ be indeterminates over A, and map $B = A[X_1, \dots, X_n] \to R$ by sending $X_j \mapsto x_j$ for all j. Then M is also a B-module, as in the section above, and $X_1, \dots, X_n$ is a regular sequence in B.

Hence:

Proposition. *With notation as in the preceding paragraph,*

$$H_i(\underline{x}_1, \dots, \underline{x}_; M) \cong \mathrm{Tor}_j^B\big(B/(\underline{X})B,\ M\big).$$

Corollary. *Let $\underline{x} = x_1, \dots, x_n \in R$, let $I = (\underline{x})R$, and let M be an R-module. Then I kills $H_i(\underline{x};\ M)$ for all i.*

Proof. We use the idea of the discussion preceding the Proposition above, taking $A = R$, so that with $\underline{X} = X_1, \dots, X_n$ we have an R-algebra map $B = R[\underline{X}] \to R$ such that $X_i \mapsto x_i$, $1 \le i \le n$. Then

$$(*) \quad H_i(\underline{x};\ M) \cong \mathrm{Tor}_i^B(B/(\underline{X})B,\ M).$$

When M is viewed as a B-module, every $X_i - x_i$ kiils M. But $\underline{X}$ kills $B/(\underline{X})B$, and so for every i, both $X_i - x_i$ and X_i kill $\mathrm{Tor}_i^B(B/(\underline{X})B,\ M)$. It follows that every $x_i = X_i - (X_i - x_i)$ kills it as well, and the result now follows from $(*)$. $\square$

An application to the study of regular local rings

Let M be a finitely generated R-module over a local ring $(R,\ m,\ K)$. A *minimal* free resolution of M may be constructed as follows. Let b_0 be the least number of generators of M, and begin by mapping R^{b_0} onto M using these generators. If

$$R^{b_i} \xrightarrow{\alpha_i} \cdots \xrightarrow{\alpha_1} R^{b_0} \xrightarrow{\alpha_0} M \longrightarrow 0$$

has already been constructed, let b_{i+1} be the least number of generators of $Z_i = \mathrm{Ker}\,(\alpha_i)$, and construct $\alpha_{i+1} : R^{b_{i+1}} \to R^{b_i}$ by mapping the free generators of $R^{b_{i+1}}$ to a minimal set of generators of $Z_i \subseteq R^{b_i}$. Think of the linear maps α_i , $i \ge 1$, as given by matrices. Then it is easy to see that a free resolution for M is minimal if and only if all of the matrices α_i for $i \ge 1$ have entries in m. We have the following consequence:

Proposition. *Let (R, m, K) be local, let M be a finitely generated R-modules, and let*

$$\cdots \to R^{b_i} \xrightarrow{\alpha_i} \cdots \to R^{b_0} \to M \to 0$$

be a minimal resolution of M. Then for all i, $\mathrm{Tor}_i^R(M, K) \cong K^{b_i}$.

Proof. We may use the minimal resolution displayed to calculate the values of Tor. We drop the augmentation M and apply $K \otimes_R \_$. Since all of the matrices have entries in m, the maps are alll 0, and we have the complex

$$\cdots \xrightarrow{0} K^{b_i} \xrightarrow{0} \cdots \xrightarrow{0} K^{b_0} \xrightarrow{0} 0.$$

Since all the maps are zero, the result stated is immediate. $\square$

Theorem (Auslander-Buchsbaum). *Let (R, m, K) be a regular local ring. Then every finitely generated R-module has a finite projective resolution of length at most $n = \dim(R)$.*

Proof. Let $\underline{x} = x_1, \ldots, x_n$ be a regular system of parameters for R. These elements form a regular sequence. It follows that $K = R/(\underline{x})$ has a free resolution of length at most n. Hence, $\mathrm{Tor}_i(M, K) = 0$ for all $i > n$ and for every R-module M.

Now let M be a finitely generated R-module, and let

$$\cdots \to R^{b_i} \to \cdots \to R^{b_1} \to R^{b_0} \to R \to M \to 0$$

be a minimal free resolution of M. For $i > n$, $b_i = 0$ because $\mathrm{Tor}_i(M, K) = 0$, and so $R^{b_i} = 0$ for $i > n$, as required. $\square$

It is true that a local ring is regular if and only if its residue class field has finite projective dimension: the converse part was proved by J.-P. Serre. The argument may be found in the Lecture Notes of February 13 and 16, Math 615, Winter 2004.

It is an open question whether, if M is a module of finite length over a regular local ring (R, m, K) of Krull dimension n, one has that

$$\dim_K \mathrm{Tor}_i(M, K) \geq \binom{n}{i}.$$

The numbers $\beta_i = \dim_K \mathrm{Tor}_i^R(M, K)$ are called the *Betti numbers* of M. If $\underline{x} = x_1, \ldots, x_n$ is a minimal set of generators of m, these may also be characterized as the dimensions of the Koszul homology modules $H_i(\underline{x}; M)$. A third point of view is that they give the ranks of the free modules in a minimal free resolution of M.

The binomial coefficients are the Betti numbers of $K = R/m$: they are the ranks of the free modules in the Koszul complex resolution of K. The question as to whether these are the smallest possible Betti numbers for an R-module was raised by David Buchsbaum and David Eisenbud in the first reference listed below, and was reported by Harthshorne in a

1979 paper (again, see the list below) as a question raised by Horrocks. The question is open in dimension 5 and greater. An affirmative answer would imply that the sum of the Betti numbers is at least 2^n: this weaker form is also open. We refer the reader interested in learning more about this problem to the following selected references:

D. Buchsbaum and D Eisenbud, *Algebra structures for finite free resolutions and some structure theorems for ideals of codimension 3*, Amer. J. of Math. **99** (1977) 447–485.

S.-T. Chang, *Betti numbers of modules of exponent two over regular local rings*, J. of Alg. **193** (1997) 640–659.

H. Charalambous, *Lower Bounds for Betti Numbers of Multigraded Modules*, J. of Alg. **137** (1991) 491–500.

D. Dugger, *Betti Numbers of Almost Complete Intersections*, Illinois J. Math. **44** (2000) 531–541.

D. Eisenbud and C. Huneke, editors, *Free resolutions in commutative algebra and algebraic geometry,* Research Notes in Mathematics: Sundance **90**, A. K. Peters, Ltd., 1992.

E.G. Evans and P. Griffith, *Syzygies*, London Math. Soc. Lecture Note Series **106** Cambridge Univ. Press, Cambridge, 1985.

R. Hartshorne, *Algebraic Vector Bundles on Projective Spaces: a Problem List*, Topology, **18** (1979) 117–128.

M. Hochster and B. Richert, *Lower bounds for Betti numbers of special extensions*, J. Pure and Appl. Alg **201** (2005) 328–339.

C. Huneke and B. Ulrich, *The Structure of Linkage*, Ann. of Math. **126** (1987) 277–334.

L. Santoni, *Horrocks' question for monomially graded modules*, Pacific J. Math. **141** (1990) 105–124.