Math 711: Lecture of October 26, 2007

It still remains to prove the final assertion of the Theorem from p. 3 of the Lecture Notes of October 22: that if $R$ is F-finite and weakly F-regular, then $R$ is strongly F-regular. Before doing so, we want to note some consequences of the theory of test elements, and also of the theory of approximately Gorenstein rings.

**Theorem.** Let $(R, m, K)$ be a local ring of prime characteristic $p > 0$.

(a) If $R$ has a completely stable test element, then $\hat{R}$ is weakly F-regular if and only if $R$ is weakly F-regular.

(b) If $R$ has a completely stable big test element, then $\hat{R}$ has the property that every submodule of every module is tightly closed if and only if $R$ does.

**Proof.** We already know that if a faithfully flat extension has the relevant property, then $R$ does. For the converse, it suffices to check that 0 is tightly closed in every finite length module over $\hat{R}$ (respectively, in the injective hull $E$ of the residue class field over $\hat{R}$, which is the same as the injective hull of the residue class field over $R$). A finite length $\hat{R}$-module is the same as a finite length $R$-module. We can use the completely stable (big, for part (b)) test element $c \in R$ in both tests, which are then bound to have the same outcome for each element of the modules. For a module $M$ supported only at $m$,

$$\mathcal{F}_R^c(M) \cong \mathcal{F}_\hat{R}^c(\hat{R} \otimes_R M) \cong \hat{R} \otimes_R \mathcal{F}_R^c(M) \cong \mathcal{F}_R^c(M).$$

□

**Proposition.** Let $R$ have a test element (respectively, a big test element) $c$ and let $N \subseteq M$ be finitely generated (respectively, arbitrary) $R$-modules. Let $d \in R^\circ$ and suppose $u \in M$ is such that $cu^q \in N[3]_q$ for infinitely many values of $q$. Then $u \in N^*_M$.

**Proof.** Suppose that $du^q \in N[3]_q$ and that $p^{\epsilon_1} = q_1 < q$, so that $q = q_1q_2$. Then $(du^q)^{q_2} = d^{q_2-1}du^q \in (N[q_1])^{q_2} = N[q_1]$, and it follows that for all $q_3$, $(du^{q_3})^{q_2q_3} \in (N[q_1])^{q_2q_3}$. Hence, $du^{q_3} \in (N[q_1])^* \in \mathcal{F}_{\epsilon_1}(M)$ whenever $q_1 \leq q$. Hence, if $du^q \in N[3]_q$ for arbitrarily large values of $q$, then $du^q \in (N[3])^* \in \mathcal{F}_\epsilon(M)$ for all $q$ and it follows that $cd\epsilon^{q_1} \in N[3]_q$ for all $q$, so that $u \in N^*_M$. □

**Theorem.** Let $R$ be a Noetherian ring of prime characteristic $p > 0$.

(a) If every ideal of $R$ is tightly closed, then $R$ is weakly F-regular.

(b) If $R$ is local and $\{I_t\}$ is a descending sequence of irreducible $m$-primary ideals cofinal with the powers of $m$, then $R$ is weakly F-regular if and only if $I_t$ is tightly closed for all $t \geq 1$.  

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Proof. (a) We already know that every ideal is tightly closed if and only if every ideal primary to a maximal ideal is tightly closed, and this is not affected by localization at a maximal ideal. Therefore, we may reduce to the case where \( R \) is local. The condition that every ideal is tightly closed implies that \( R \) is normal and, hence, approximately Gorenstein. Therefore, it suffices to prove (b). For (b), we already know that \( R \) is weakly F-regular if and only if 0 is tightly closed in every finitely generated \( R \)-module that is an essential extension of \( K \). Such a module is killed by \( I_t \) for some \( t > 0 \), and so embeds in \( E_{R/I_t}(K) \cong R/I_t \) for some \( t \). Since \( I_t \) is tightly closed in \( R \), 0 is tightly closed in \( R/I_t \), and the result follows. \( \square \)

We next want to establish a result that will enable us to prove the final assertion of the Theorem from p. 3 of the Lecture Notes of October 22.

**Theorem.** Let \((R, m, K)\) be a complete local ring of prime characteristic \( p > 0 \). If \( R \) is reduced and \( c \in R^o \), let \( \theta_{q,c} : R \to R^{1/q} \) denote the \( R \)-linear map such that \( 1 \mapsto c^{1/q} \). Then the following conditions are equivalent:

1. Every submodule of every module is tightly closed.
2. 0 is tightly closed in the injective hull \( E = E_R(K) \) of the residue class field \( K = R/m \) of \( R \).
3. \( R \) is reduced, and for every \( c \in R^o \), there exists \( q \) such that the \( \theta_{q,c} \) splits.
4. \( R \) is reduced, and for some \( c \) that has a power which is a big test element for \( R \), there exists \( q \) such that \( \theta_{q,c} \) splits.
5. \( R \) is reduced, and for some \( c \) such that \( R_c \) is regular, there exists \( q \) such that \( \theta_{q,c} \) splits.

**Proof.** Note that all of the conditions imply that \( R \) is reduced.

We already know that conditions (1) and (2) are equivalent. Let \( u \) denote a socle generator in \( E \). Then we have an injection \( K \to E \) that sends 1 \( \mapsto u \), and we know that 0 is tightly closed in \( E \) if and only if \( u \) is in the tight closure of 0 in \( E \). This is the case if and only if for some \( c \in R^o \) (respectively, for a single big test element \( c \in R^o \)), \( cu^q = 0 \) in \( \mathcal{F}^c(E) \) for all \( q \gg 0 \). We may view \( \mathcal{F}^c(E) \) is identified with \( R^{1/q} \otimes_R E \), and \( u \) acts via the isomorphism \( R \cong R^{1/q} \) such that \( r \mapsto r^{1/q} \). Then \( u^q \) corresponds to \( 1 \otimes u \), and \( cu^q \) corresponds to \( c^{1/q} \otimes u \).

Then \( u \in 0^c_E \) if and only if for every \( c \in R^o \) (respectively, for a single big test element \( c \in R^o \)), the map \( K \to R^{1/q} \otimes_R E \) that sends \( 1 \mapsto c^{1/q} \otimes u \) is 0 for all \( q \gg 0 \). We may now apply the functor \( \text{Hom}_R(\_ \otimes_R E, E) \) to obtain a dual condition. Namely, \( u \in 0^c_E \) if and only if for every \( c \in R^o \) (respectively, for a single big test element \( c \in R^o \)), the map

\[
\text{Hom}_R(R^{1/q} \otimes_R E, E) \to \text{Hom}_R(K, E)
\]

is 0 for all \( q \gg 0 \). The map is induced by composition with \( K \to R^{1/q} \otimes_R E \). By the adjointness of tensor and \( \text{Hom} \), we may identify this map with

\[
\text{Hom}_R(R^{1/q}, \text{Hom}_R(E, E)) \to \text{Hom}_R(K, E).
\]
This map sends $f$ to the composition of $K \to R^{1/q} \otimes_R E$ with the map such that $s \otimes v \mapsto f(s)(v)$. Since $\text{Hom}_R(E, E) \cong R$ by Matlis duality and $\text{Hom}_R(K, E) \cong K$, we obtain the map

$$\text{Hom}_R(R^{1/q}, R) \to K$$

that sends $f$ to the image of $f(c^{1/q})$ in $R/m$.

Thus, $u \in 0_E^*$ if and only if for every $c \in R^0$ (respectively, for a single big test element $c \in R^0$), every $f : R^{1/q} \to R$ sends $c^{1/q}$ into $m$ for every $q \gg 0$. This is equivalent to the statement that $\theta_{q,c} : R \to R^{1/q}$ sending $1 \mapsto c^{1/q}$ does not split for every $q \gg 0$, since if $f(c^{1/q}) = a$ is a unit of $R$, $a^{-1}f$ is a splitting.

Note that if $R \to R^{1/q}$ sending $1 \mapsto c^{1/q}$ splits, then $R \to R^{1/q}$ splits as well: the argument in the Lecture Notes from September 21 (see pages 4 and 5) applies without any modification whatsoever. Moreover, the second Proposition on p. 5 of those notes shows that if one has the splitting for a given $q$, one also has it for every larger $q$.

We have now shown that $u \in 0_E^*$ if and only if for every $c \in R^0$ (respectively, for a single big test element $c \in R^0$), $\theta_{q,c} : R \to R^{1/q}$ sending $1 \mapsto c^{1/q}$ does not split for every $q$.

Hence, $0$ is tightly closed in $E$ if and only if for every $c \in R^0$ (respectively, for a single big test element $c \in R^0$) the map $\theta_{q,c}$ splits for some $q$.

We have now shown that conditions (1), (2), and (3) are equivalent, and that (4) is equivalent as well provided that $c$ is a big test element.

Now suppose that we only know that $c$ has a power that is a big test element. Then this is also true for any larger power, and so we can choose $q_1 = p^{e_1}$ such that $c^{e_1}$ is a test element. If the equivalent conditions (1), (2), and (3) hold, then we also know that the map $R \to R^{1/q_1}$ sending $1 \mapsto (c^{q_1})^{1/q_1} = c^{1/q}$ splits for all $q \gg 0$, and we may restrict this splitting to $R^{1/q}$. Thus, (1) through (4) are equivalent.

Finally, (5) is equivalent as well, because we know that if $c \in R^0$ is such that $R_c$ is regular, then $c$ has a power that is a big test element. □

**Remark.** It is not really necessary to assume that $R$ is reduced in the last three conditions. We can work with $R^{(c)}$ instead of $R^{1/q}$, where $R^{(c)}$ denotes $R$ viewed as an $R$-algebra via the structural homomorphism $F_c$. We may then define $\theta_{q,c}$ to be the $R$-linear map $R \to R^{(c)}$ such that $1 \mapsto c$. The fact that this map is split for some some $c \in R^0$ and some $q$ implies that $R$ is reduced: if $r$ is a nonzero nilpotent, we can replace it by a power which is nonzero but whose square is 0. But then the image of $r$ is $r^q c = 0$, and the map is not even injective, a contradiction. Once we know that $R$ is reduced, we can identify $R^{(c)}$ with $R^{1/q}$ and $c$ is identified with $c^{1/q}$.

We want to apply the preceding Theorem to the F-finite case. We first observe:

**Lemma.** Let $(R, m, K)$ be an F-finite reduce local ring. Then $\hat{R}^{1/q} \cong \hat{R}^{1/q} \cong \hat{R} \otimes_R R^{1/q}$ for all $q = p^e$. 

Proof. $R^{1/q}$ is a local ring module-finite over $R$. Hence, the maximal ideal of $R$ expands to an ideal primary to the maximal ideal of $R^{1/q}$, and it follows that $\widehat{R^{1/q}}$ is the $mR^{1/q}$-adic completion of $R^{1/q}$. Thus, we have an isomorphism $\alpha : R^{1/q} \cong \widehat{R} \otimes_R R^{1/q}$. Since $R$ is reduced, so is $R^{1/q}$. Since $R$ is F-finite, so is $R^{1/q}$, and $R^{1/q}$ is consequently excellent. Hence, the completion $\widehat{R^{1/q}}$ is reduced. If we use the identification $\alpha$ to write a typical element of $u \in \widehat{R^{1/q}}$ as a sum of terms of the form $s \otimes r^{1/q}$, where $s \in \widehat{R}$ and $r \in R$, we see that $u^q \in \widehat{R}$. This shows that we have $\widehat{R^{1/q}} \subseteq \widehat{R}^{1/q}$. On the other hand, if $r_0, r_1, \ldots, r_k, \ldots$ is a Cauchy sequence in $R$ with limit $s$, then $r_0^{1/q}, r_1^{1/q}, \ldots, r_k^{1/q}, \ldots$ is a Cauchy sequence in $R^{1/q}$, and its limit is $s^{1/q}$. This shows that $\widehat{R^{1/q}} \subseteq \widehat{R}^{1/q}$. □

From the preceding Theorem we then have:

Corollary. If $R$ is F-finite, then $R$ is strongly F-regular if and only if every submodule of every module is tightly closed.

Proof. We need only show that if every submodule of every module is tightly closed, then $R$ is strongly F-regular. We know that both conditions are local on the maximal ideals of $R$ (cf. problem 6. of Problem Set #3). Thus, we may assume that $(R, m, K)$ is local. We know that $R$ has a completely stable big test element $c$. By part (b) of the Theorem on the first page, $\widehat{R}$ has the property that every submodule of every module is tightly closed: in particular, $0$ is tightly closed in $E = E_\mathcal{R}(K) \cong E_R(K)$. By the equivalence of (2) and (4) in the preceding Theorem, we have that the $\widehat{R}$-linear map $\widehat{\theta} : \widehat{R} \rightarrow \widehat{R^{1/q}}$ that sends $1 \mapsto c^{1/q}$ splits for some $q$. This map arises from the $R$-linear map $\theta : R \rightarrow R^{1/q}$ that sends $1 \mapsto c^{1/q}$ by applying $\widehat{R} \otimes_R \_$. Since $\widehat{R}$ is faithfully flat over $R$, the map $\theta$ is split if and only if $\widehat{\theta}$ is split, and so $\theta$ is split as well. □

Finally, we can prove the final statement in the Theorem on p. 4 of the Lecture Notes from October 22.

Corollary. If $R$ is Gorenstein and F-finite, then $R$ is weakly F-regular if and only if $R$ is strongly F-regular.

Proof. The issue is local on the maximal ideals of $R$. We have already shown that in the local Gorenstein case, $(R, m, K)$ is weakly F-regular if and only if $0$ is tightly closed in $E_R(K)$. By the Corollary just above, this implies that $R$ is strongly F-regular in the F-finite case. □

This justifies extending the notion of strongly F-regular ring as follows: the definition agrees with the one given earlier if the ring is F-finite.

Definition. Let $R$ be a Noetherian ring of prime characteristic $p > 0$. We define $R$ to be strongly F-regular if every submodule of every module (whether finitely generated or not) is tightly closed.