Discussion: local cohomology. Let \( y_1, \ldots, y_d \) be a sequence of elements of a Noetherian ring \( S \) and let \( N \) be an \( S \)-module, which need not be finitely generated. Let \( J \) be an ideal whose radical is the same as the radical of \( (y_1, \ldots, y_d)S \). Then the \( d \)th local cohomology module of \( N \) with supports in \( J \), denoted \( H_d^J(N) \), may be obtained as

\[
\lim_{k \to \infty} \frac{N}{(y_1^k, \ldots, y_d^k)N}
\]

where the map from

\[
N_k = \frac{N}{(y_1^k, \ldots, y_d^k)N}
\]

to \( N_{k+h} \) is induced by multiplication by \( z^h \), where \( z = y_1 \cdots y_d \), on the numerators. If \( u \in N \), \( \langle u; y_1^k, \ldots, y_d^k \rangle \) denotes the image of the class of \( u \) in \( N_k \) in \( H_d^J(N) \). With this notation, we have that

\[
\langle u; y_1^k, \ldots, y_d^k \rangle = \langle z^h u; y_1^{k+h}, \ldots, y_d^{k+h} \rangle
\]

for every \( h \in \mathbb{N} \).

If \( y_1, \ldots, y_d \) is a regular sequence on \( N \), these maps are injective. We also know from the seminar that if \( (S, \mathfrak{n}, L) \) is a Gorenstein local ring and \( y = y_1, \ldots, y_d \) is a system of parameters for \( S \), then \( H_d^J(S) = H_d^L(S) \) is an injective hull for the residue class field \( L = S/\mathfrak{n} \) of \( S \) over \( S \). In the sequel, we want to prove a relative form of this result when \( R \to S \) is a flat local homomorphism whose closed fiber is Gorenstein.

**Theorem.** Let \( (R, m, K) \to (S, \mathfrak{n}, L) \) be a flat local homomorphism such that the closed fiber \( S/mS \) is Gorenstein. Let \( \dim(R) = n \) and let \( \dim(S/mS) = d \). Let \( y = y_1, \ldots, y_d \in \mathfrak{n} \) be elements whose images in \( S/mS \) are a system of parameters. Let \( E = E_R(K) \) be an injective hull for the residue class field \( K = R/m \) of \( R \) over \( R \). Then \( E \otimes_R H_d^J(S) \) is an injective hull for \( L = S/\mathfrak{n} \) over \( S \).

In the case where the rings are of prime characteristic \( p > 0 \),

\[
\mathcal{F}_S^R(E \otimes_R H_d^J(S)) \cong \mathcal{F}_R(E) \otimes_R H_d^J(S),
\]

and if \( u \in E \) and \( s \in S \), then

\[
(u \otimes (s; y_1^k, \ldots, y_d^k))^q = u^q \otimes (s^q; y_1^{qk}, \ldots, y_d^{qk}).
\]

**Proof.** We first give an argument for the case where \( R \) is approximately Gorenstein, which is somewhat simpler. We then treat the general case. Suppose that \( \{I_t\} \) is a descending
sequence of \(m\)-primary ideals of \(R\) cofinal with the powers of \(M\). We know that \(E = \lim_i R/I_i\) for any choice of injective maps \(R/I_i \to R/I_{i+1}\). Let \(\mathfrak{A}_{t,k} = I_t S + J_k\), where \(J_k = (y_1^k, \ldots, y_d^k)S\). For every \(k\) we may tensor with the faithfully flat \(R\)-algebra \(S/J_k\) to obtain an injective map \(S/\mathfrak{A}_{t,k} \to S/\mathfrak{A}_{t+1,k}\). Since \(y_1, \ldots, y_d\) is a regular sequence on \(S/I_t S\) for every \(I_t\), we also have an injective map \(S/\mathfrak{A}_{t,k} \to S/\mathfrak{A}_{t+1,k}\) induced by multiplication by \(z = y_1 \cdots y_d\) on the numerators. The ideals \(\mathfrak{A}_{t,k}\) are \(n\)-primary irreducible ideals and as \(t, k\) both become large, are contained in arbitrarily large powers of \(n\). (Once \(I_t \subseteq m^n\) and \(k \geq s\), we have that \(\mathfrak{A}_{t,k} \subseteq m^n S + n^s \subseteq n^s\).) Thus, we have

\[
E_{S}(L) \cong \lim_{\to \, t,k} \frac{S}{\mathfrak{A}_{t,k}} = \lim_{\to \, t,k} \left( \frac{R}{I_t} \otimes \frac{S}{J_k} \right) \cong \lim_k \left( \lim_{\to \, t} \left( \frac{R}{I_t} \otimes \frac{S}{J_k} \right) \right) \cong \lim_k \left( \left( \lim_{\to \, t} \frac{R}{I_t} \right) \otimes \frac{S}{J_k} \right) \cong E \otimes_R H^d(y)(S).
\]

We now give an alternative argument that works more generally. In particular, we do not assume that \(R\) is approximately Gorenstein. Let \(E_t\) denote \(\text{Ann}_{K^{m^t}}\). We first claim that that \(E_{t,k}\), which we define as \(E_t \otimes_R (S/J_k)\), is an injective hull of \(L\) over \(S_{t,k} = (R/m^{t}) \otimes_R (S/J_k)\). By part (f) of the Theorem on p. 2 of the Lecture Notes from October 29, it is Cohen-Macaulay of type 1, since that is true for \(E_t\) and for the closed fiber of \(S/J_k\), since \(S/m S\) is Gorenstein. Hence, \(E_{t,k}\) is an essential extension of \(L\), and it is killed by \(\mathfrak{A}_{t,k} = m^{t} S + J_k\). To complete the proof, it suffices to show that it has the same length as \(S_{t,k}\). Let \(M\) denote either \(R/m^t\) or \(E_t\). Note that \(M\) has a filtration with \(\ell(M)\) factors, each of which is \(\cong K = R/m\). Since \(S/J_k\) is \(R\)-flat, this gives a filtration of \(M \otimes_R S/J_k\) with \(\ell(M)\) factors each of which is isomorphic with \(K \otimes_R S/J_k = S/(m S + J_k)\). Since \(\ell(R/m^t) = \ell(E_t)\), it follows that \(S_{t,k}\) and \(E_{t,k}\) have the same length, as required.

If \(t \leq t'\) we have an inclusion \(E_t \hookrightarrow E_{t'}\), and if \(k \leq k'\), we have an injection \(S/J_k \to S/J_{k'}\) induced by multiplication by \(z^{k'-k}\) acting on the numerators. This gives injections \(E_{t} \otimes_R S_{k} \to E_{t'} \otimes_R S_{k}\) (since \(S_{k}\) is \(R\)-flat) and \(E_{t'} \otimes_R S_{k} \to E_{t'} \otimes_R S_{k}\) (since \(y_1, \ldots, y_d\) is a regular sequence on \(E_{t'} \otimes_R S\)). The composites give injections \(E_{t,k} \hookrightarrow E_{t',k}\) and the direct limit over \(t, k\) is evidently \(E \otimes K H^d(y)(S)\). The resulting module is clearly an essential extension of \(L\), since it is a directed union of essential extensions. Hence, it is contained in a maximal essential extension \(E_{S}(L)\) of \(L\) over \(S\). We claim that this inclusion is an equality. To see this, suppose that \(u \in E_{S}(L)\) is any element. Then \(u\) is killed by \(\mathfrak{A} = \mathfrak{A}_{t,k} = m^n S + J_k\) for any sufficiently large choices of \(t\) and \(k\). Hence \(u \in \text{Ann}_{E_{S}(L)}\mathfrak{A} = \mathfrak{A}\), which we know is an injective hull for \(L\) over \(S\mathfrak{A}\). But \(E_t \otimes_R S/J_k\) is a submodule of \(N\) contained in \(E \otimes_R H^d(y)(S)\), and is already an injective hull for \(L\) over \(S/\mathfrak{A}\). It follows, since they have the same length, that we must have that \(E_t \otimes_R S/J_k \subseteq N\) is all of \(N\), and so \(u \in E_t \otimes_R S/J_k \subseteq E \otimes_R H^d(y)(S)\).

To prove the final statement about the Frobenius functor, we note that by the first problem of Problem Set #4, one need only calculate \(F^d_y H^d(y)(S)\), and this calculation is the precisely the same as in third paragraph of p. 1 of the Lecture Notes from October 24. □
We are now ready to prove the analogue for strong F-regularity of the Theorem at the top of p. 5 of the Lecture Notes from October 29, which treated the weakly F-regular case.

**Theorem.** Let \((R, m, K) \to (S, n, L)\) be a local homomorphism of local rings of prime characteristic \(p > 0\) such that the closed fiber \(S/m\) is regular. Suppose that \(c \in R^o\) is a big test element for both \(R\) and \(S\). If \(R\) is strongly F-regular, then \(S\) is strongly F-regular.

**Proof.** Let \(u\) be a socle generator in \(E = E_R(K)\), and let \(y = y_1, \ldots, y_d \in n\) be elements whose images in the closed fiber \(S/mS\) form a minimal set of generators of the maximal ideal \(n/mS\). Let \(z = y_1 \cdots y_d\). Then the image of \(1\) in \(S/(mS + (y_1, \ldots, y_d)S)\) is a socle generator, and it follows that \(v = u \otimes (1; y_1, \ldots, y_d)\) generates the socle in \(E_S(L) \cong E \otimes_R H^d_{(y)}(S)\). Since \(c\) is a big test element for \(S\), it can be used to test whether \(v\) is in the tight closure of \(0\) in \(E \otimes_R H^d_{(y)}(S)\).

This occurs if and only if for all \(q \gg 0\), \(c(u \otimes (1; y_1, \ldots, y_d))^q = 0\) in \(F^c_R(E) \otimes_R H^d_{(y)}(S)\), and this means that \(cu^q \otimes (1; y_1^q, \ldots, y_d^q) = 0\) in \(F^c_R(E) \otimes_R H^d_{(y)}(S)\). By part (c) of the Theorem on p. 2 of the Lecture Notes from October 29, \(y_1, \ldots, y_d\) is a regular sequence on \(E \otimes_R S\), from which it follows that the module \(F^c_R(E) \otimes_R (S/(y_1^q, \ldots, y_d^q))\) injects into \(F^c_R(E) \otimes_R H^d_{(y)}(S)\). Since \(S = S/(y_1^q, \ldots, y_d^q)\) is faithfully flat over \(R\), the map \(F^c_R(E) \to F^c_R(E) \otimes_R S/(y_1^q, \ldots, y_d^q)\) sending \(w \mapsto w \otimes 1\) is injective. The fact that \(cu^q \otimes (1; y_1^q, \ldots, y_d^q) = 0\) implies that \(cu^q \otimes 1_S = 0\) in \(F^c_R(E) \otimes_R S\), and hence that \(cu^q = 0\) in \(R\). Since this holds for all \(q \gg 0\), we have that \(u \in 0^e_L\), a contradiction. \(\square\)

The following result will be useful in studying algebras essentially of finite type over an excellent semilocal ring that are not F-finite but are strongly F-regular: in many instances, it permits reductions to the F-finite case.

**Theorem.** Let \(R\) be a reduced Noetherian ring of prime characteristic \(p > 0\) that is essentially of finite type over an excellent semilocal ring \(B\).

(a) Let \(\hat{B}\) denote the completion of \(B\) with respect to its Jacobson radical. Suppose that \(R\) is strongly F-regular. Then \(\hat{B} \otimes_B R\) is essentially of finite type over \(\hat{B}\) and is strongly F-regular and faithfully flat over \(R\).

(b) Suppose that \(B = A\) is a complete local ring with coefficient field \(K\). Fix a \(p\)-base \(\Lambda\) for \(K\). For all \(\Gamma \ll \Lambda\), let \(R^\Gamma = A^\Gamma \otimes_A R\). We may identify \(\text{Spec}(R^\Gamma)\) with \(X = \text{Spec}(R)\) as topological spaces, and we let \(Z^\Gamma\) denote the closed set in \(\text{Spec}(R)\) of points corresponding to primes \(P\) such that \(R^\Gamma_P\) is not strongly F-regular. Then \(Z^\Gamma\) is the same for all sufficiently small \(\Gamma \ll \Lambda\), and this closed set is the locus in \(X\) consisting of primes \(P\) such that \(R_P^\Gamma\) is not strongly F-regular.

In particular, if \(R\) is strongly F-regular, then for all \(\Gamma \ll \Lambda\), \(R^\Gamma\) is strongly F-regular.

**Proof.** (a) Since \(B \to \hat{B}\) is faithfully flat with geometrically regular fibers, the same is true for \(R \to \hat{B} \otimes_B R\). Choose \(c \in R^o\) such that \(R_c\) is regular. Then we also have that
(\hat{B} \otimes_B R)_c$ is regular. Hence, $c$ has a power that is a completely stable big test element in both rings. Let $Q$ be any prime ideal of $S = \hat{B} \otimes_B R$ and let $P$ be its contraction to $R$. We may apply the preceding Theorem to the map $R_P \to S_Q$, and so $S_Q$ is strongly F-regular for all $Q$. It follows that $S$ is strongly F-regular.

(b) For all choices of $\Gamma' \subseteq \Gamma$ cofinite in $\Lambda$, we have that $R \subseteq R^{\Gamma'} \subseteq R^\Gamma$, and that the maps are faithfully flat and purely inseparable. Since every $R^\Gamma$ is F-finite, we know that every $Z_\Gamma$ is closed. Since the map $R^{\Gamma'} \subseteq R^\Gamma$ is faithfully flat, $Z_\Gamma$ decreases as $\Gamma$ decreases. We may choose $\Gamma$ so that $Z = Z_\Gamma$ is minimal, and, hence, minimum, since a finite intersection of cofinite subsets of $\Lambda$ is cofinite.

We shall show that $Z$ must be the set of primes $P$ in Spec $(R)$ such that $R_P$ is strongly F-regular. If $Q$ is a prime of $R^\Gamma$ not in $Z_\Gamma$ lying over $P$ in $R$, the fact that $R_P \to R^\Gamma_Q$ is faithfully flat implies that $P$ is not in $Z$. Thus, $Z \subseteq Z_\Gamma$. If they are not equal, then there is a prime $P$ of $R$ such that $R_P$ is strongly F-regular but $R^\Gamma_Q$ is not strongly F-regular, where $Q$ is the prime of $R^\Gamma$ corresponding to $P$. Choose $\Gamma' \subseteq \Gamma$ such that $Q' = PR^{\Gamma'}$ is prime. It will suffice to prove that $S = R^\Gamma_Q$ is strongly F-regular, for this shows that $Z_\Gamma \subseteq Z_\Gamma - \{P\}$ is strictly smaller than $Z_\Gamma$. Since $S$ is F-finite, we may choose a big test element $c_1$ for $S$. Then $c_1$ has a $q_1$th power $c$ in $R_P$ for some $q_1$, and $c$ is still a big test element for $S$. The closed fiber of $R_P \to S$ is $S/PS = S/Q'$, a field. Hence, by the preceding Theorem, $S$ is strongly F-regular. □

Using this result, we can now prove:

**Theorem.** Let $R$ be reduced and essentially of finite type over an excellent semilocal ring $B$. Then the strongly F-regular locus in $R$ is Zariski open.

**Proof.** We first consider the case where $B = A$ is a complete local ring. Choose a coefficient field $K$ for $A$ and a $p$-base $\Lambda$ for it. Then the result is immediate from part (b) of the preceding Theorem by comparison with $R^\Gamma$ for any $\Gamma \ll \Lambda$.

In the general case, let $S = \hat{B} \otimes_B R$. Since $\hat{B}$ is a finite product of complete local rings, $S$ is a finite product of algebras essentially of finite type over a complete local ring, and so the non-strongly F-regular locus is closed. Let $J$ denote an ideal of $S$ that defines this locus.

Now consider any prime ideal $P$ of $R$ such that $R_P$ is strongly F-regular. Let $W = R - P$. Then we may apply part (a) of the preceding Theorem to $R_P \to \hat{B} \otimes_B R_P$ to conclude that $\hat{B} \otimes_B R_P = W^{-1}S$ is strongly F-regular. It follows that $W$ must meet $J$; otherwise, we can choose a prime $Q$ of $S$ containing $J$ but disjoint from $W$, and it would follow that $S_Q$ is strongly F-regular even though $J \subseteq Q$, a contradiction. Choose $c \in W \cap J$. Then $S_c$ is strongly F-regular, and since $R_c \to S_c$ is faithfully flat, so is $R_c$. Thus, the set of primes of $R$ not containing $c$ is a Zariski open neighborhood of $P$ that is contained in the strongly F-regular locus. □