Summary of Local Cohomology Theory

The following material and more was discussed in seminar but not in class. We give a summary here.

Let $I$ be an ideal of a Noetherian ring $R$ and let $M$ be any $R$-module, not necessarily finitely generated. We define

$$H^i_I(M) = \lim_{\rightarrow} \text{Ext}^i_R(R/I^t, M).$$

This is called the $i$th local cohomology module of $M$ with support in $I$.

$$H^0_I(M) = \lim_{\rightarrow} \text{Hom}_R(R/I^t, M)$$

which may be identified with $\bigcup_t \text{Ann}_R M/I^t \subseteq M$. Every element of $H^j_I(M)$ is killed by a power of $I$. Evidently, if $M$ is injective then $H^j_I(M) = 0$ for $j \geq 1$. By a taking a direct limit over $t$ of long exact sequences for Ext, we see that if

$$0 \to M' \to M \to M'' \to 0$$

is exact there is a functorial long exact sequence for local cohomology:

$$0 \to H^0_I(M') \to H^0_I(M) \to H^0_I(M'') \to \cdots \to H^j_I(M') \to H^j_I(M) \to H^j_I(M'') \to \cdots.$$ 

It follows that $H^j_I(M)$ is the $j$th right derived functor of $H^0_I(M)$. In the definition we may use instead of the ideals $I^t$ any decreasing sequence of ideals cofinal with the powers of $I$. It follows that if $I$ and $J$ have the same radical, then $H^i_I(M) \cong H^i_J(M)$ for all $i$.

**Theorem.** Let $M$ be a finitely generated module over the Noetherian ring $R$, and $I$ and ideal of $R$. Then $H^i_I(M) \neq 0$ for some $i$ if and only if $IM \neq M$, in which case the least integer $i$ such that $H^i_I(M) \neq 0$ is $\text{depth}_IM$.

**Proof.** $IM = M$ iff $I + \text{Ann}_R M = R$, and every element of every $H^j_I(M)$ is killed by some power $I^N$ of $I$ and by $\text{Ann}_R M$: their sum must be the unit ideal, and so all the local cohomology vanishes in this case.
Now suppose that $IM \neq M$, so that the depth $d$ is a well-defined integer in $\mathbb{N}$. We use induction on $d$. If $d = 0$, some nonzero element of $M$ is killed by $I$, and so $H^0_I(M) \neq 0$. If $d > 0$ choose an element $x \in I$ that is not a zerodivisor on $M$, and consider the long exact sequence for local cohomology arising from the short exact sequence

$$0 \to M \overset{x}{\to} M \to M/\langle x \rangle M \to 0.$$ 

From the induction hypothesis, $H^j_M(M/\langle x \rangle M) = 0$ for $j < d - 1$ and $H^{d-1}_{M}(M/\langle x \rangle M) \neq 0$. The long exact sequence therefore yields the injectivity of the map

$$H^{j+1}_I(M) \xrightarrow{x} H^{j+1}_I(M)$$

for $j < d - 1$. But every element of $H^{j+1}_I(M)$ is killed by a power of $I$ and, in particular, by a power of $x$. This implies that $H^{j+1}_I(M) = 0$ for $j < d - 1$. Since $0 = H^{d-1}_I(M) \to H^{d-1}_I(M/\langle x \rangle M) \to H^d_I(M)$ is exact, $H^{d-1}_I(M/\langle x \rangle M)$, which we know from the induction hypothesis is not 0, injects into $H^d_I(M)$. □

If $A^\bullet$ and $B^\bullet$ are two right complexes of $R$-modules with differentials $d$ and $d'$, the total tensor product is the right complex whose $n$th term is

$$\bigoplus_{i+j=n} A^i \otimes_R B^j$$

and whose differential $d''$ is such that $d''(a_i \otimes b_j) = d(a_i) \otimes b_j + (-1)^ia_i \otimes d'(b_j)$.

Now let $f = f_1, \ldots, f_n$ generate an ideal with the same radical as $I$. Let $C^\bullet(f^\infty; R)$ denote the total tensor product of the complexes $0 \to R \to R_{f_j} \to 0$, which gives a complex of flat $R$-modules:

$$0 \to R \to \bigoplus_j R_{f_j} \to \bigoplus_{j_1 < j_2} R_{f_{j_1},f_{j_2}} \to \cdots \to \bigoplus_{j_1 < \cdots < j_t} R_{f_{j_1},\ldots,f_{j_t}} \to \cdots \to R_{f_1,\ldots,f_n} \to 0.$$ 

The differential restricted to $R_g$, where $g = f_{j_1} \cdots f_{j_t}$, takes $u$ to the direct sum of its images, each with a certain sign, in the rings $R_{f_{j_{t+1}}}$, where $j_{t+1}$ is distinct from $j_1, \ldots, j_t$.

Let

$$C^\bullet(f^\infty; M) = C^\bullet(f^\infty; R) \otimes_R M,$$

which looks like this:

$$0 \to M \to \bigoplus_j M_{f_j} \to \bigoplus_{j_1 < j_2} M_{f_{j_1},f_{j_2}} \to \cdots \to \bigoplus_{j_1 < \cdots < j_t} M_{f_{j_1},\ldots,f_{j_t}} \to \cdots \to M_{f_1,\ldots,f_n} \to 0.$$
We temporarily denote the cohomology of this complex as \( \mathcal{H}^\bullet_2(M) \). It turns out to be the same, functorially, as \( H^\bullet_1(M) \). We shall not give a complete argument here but we note several key points. First,

\[
\mathcal{H}^0_2(M) = \text{Ker} (M \to \bigoplus_j M_{f_j})
\]

is the same as the submodule of \( M \) consisting of all elements killed by a power of \( f_j \) for every \( j \), and this is easily seen to be the same as \( H^0_0(M) \). Second, by tensoring a short exact sequence of modules

\[
0 \to M' \to M \to M'' \to 0
\]

with the complex \( C^\bullet(f^\infty; R) \) we get a short exact sequence of complexes. This leads to a functorial long exact sequence for \( H^\bullet_1(M) \). These two facts imply an isomorphism of the functors \( H^\bullet_1(M) \) and \( H^\bullet_2(M) \) provided that we can show that \( H^j_1(M) = 0 \) for \( j \geq 1 \) when \( M \) is injective. We indicate how the argument goes, but we shall assume some basic facts about the structure of injective modules over Noetherian rings.

First note that if one has a map \( R \to S \) and an \( S \)-module \( M \), then if \( g \) is the image of \( f \) in \( S \), we have \( H^\bullet_1(M) = H^\bullet_2(M) \). This has an important consequence for local cohomology once we establish that the two theories are the same: see the Corollary below.

Every injective module over a Noetherian ring \( R \) is a direct sum of injective hulls \( E(R/P) \) for various primes \( P \). \( E(R/P) \) is the same as the injective hull of the residue class field of the local ring \( R_P \). This, we may assume without loss of generality that \( (R, m, K) \) is local and that \( M \) is the injective hull of \( K \). This enables to reduce to the case where \( M \) has finite length over \( R \), and then, using the long exact sequence, to the case where \( M = K \), since \( M \) has a finite filtration such that all the factors are \( K \). Thus, we may assume that \( M = K \). The complex \( C^\bullet(f^\infty; R) \) is then a tensor product of complexes of the form

\[
0 \to R \to R \to 0
\]

and

\[
0 \to R \to 0
\]

with the complex \( C^\bullet(f^\infty; R) \) we get a short exact sequence of complexes. This leads to a functorial long exact sequence for \( H^\bullet_1(M) \). These two facts imply an isomorphism of the functors \( H^\bullet_1(M) \) and \( H^\bullet_2(M) \) provided that we can show that \( H^j_1(M) = 0 \) for \( j \geq 1 \) when \( M \) is injective. We indicate how the argument goes, but we shall assume some basic facts about the structure of injective modules over Noetherian rings.

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**Corollary.** If \( R \to S \) is a homomorphism of Noetherian rings, \( M \) is an \( S \)-module, and \( rM \) denotes \( M \) viewed as an \( R \)-module via restriction of scalars, then for every ideal \( I \) of \( R \), \( H^\bullet_I(rM) \cong H^\bullet_I(S)(M) \).

**Proof.** Let \( f_1, \ldots, f_n \) generate \( I \), and let \( g_1, \ldots, g_n \) be the images of these elements in \( S \): they generate \( IS \). Then we have

\[
H^\bullet_I(rM) \cong \mathcal{H}^\bullet_2(rM) \cong \mathcal{H}^\bullet_2(S)(M) \cong H^\bullet_I(S)(M).
\]

\[ \square \]
We note that the complex $0 \to R \to R_f \to 0$ is isomorphic to the direct limit of the cohomological Koszul complexes $K^•(f^t; R)$, where the maps between consecutive complexes are given by the identity on the degree 0 copy of $R$ and by multiplication by $f$ on the degree 1 copy of $R$ — note the commutativity of the diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & R \\
 \uparrow \text{id} & & \uparrow f \\
0 & \longrightarrow & R \\
\end{array}
\]

Tensoring these Koszul complexes together as $f$ runs through $f_1, \ldots, f_n$, we see that

\[
C^•(f^\infty; M) = \lim_{\to} K^•(f_1^t, \ldots, f_n^t; M).
\]

Hence, whenever $f_1, \ldots, f_n$ generate $I$ up to radicals, taking cohomology yields

\[
H_t^i(M) \cong \lim_{\to} H^•(f_1^t, \ldots, f_n^t; M).
\]

When $R$ is a local ring of Krull dimension $d$ and $x_1, \ldots, x_d$ is a system of parameters, this yields

\[
H^d_m(R) = \lim_{\to} R/(x_1^t, \ldots, x_d^t)R.
\]

Likewise, for every $R$-module $M$,

\[
H^d_m(M) = \lim_{\to} M/(x_1^t, \ldots, x_d^t)M \cong H^d_m(R) \otimes_R M.
\]

We next recall that when $(R, m, K)$ is a complete local ring and $E = E_R(K)$ is an injective hull of the residue class field (this means that $K \subseteq E$, where $E$ is injective, and every nonzero submodule of $E$ meets $K$), there is duality between modules with ACC over $R$ and modules with DCC: if $M$ satisfies one of the chain conditions then $M^\vee = \text{Hom}_R(M, E)$ satisfies the other, and the canonical map $M \to M^{\vee\vee}$ is an isomorphism in either case. In particular, when $R$ is complete local, the obvious map $R \to \text{Hom}_R(E, E)$ is an isomorphism. An Artin local ring $R$ with a one-dimensional socle is injective as a module over itself, and, in this case, $E_R(K) = R$. If $R$ is Gorenstein and $x_1, \ldots, x_d$ is a system of parameters, one has that each $R_t = R/(x_1^t, \ldots, x_d^t)R$ is Artin with a one-dimensional socle, and one can show that in this case $E_R(K) \cong H^d_M(R)$. When $R$ is local but not complete, if $M$ has ACC then $M^\vee$ has DCC, and $M^{\vee\vee}$ is canonically isomorphic with $\hat{M}$. If $M$ has DCC, $M^\vee$ is a module with ACC over $\hat{R}$, and $M^{\vee\vee}$ is canonically isomorphic with $M$.

We can make use of this duality theory to gain a deeper understanding of the behavior of local cohomology over a Gorenstein local ring.
Theorem (local duality over Gorenstein rings). Let \((R, m, K)\) be a Gorenstein local ring of Krull dimension \(d\), and let \(E = H^d_m(R)\), which is also an injective hull for \(K\). Let \(M\) be a finitely generated \(R\)-module. Then for every integer \(j\), \(H^j_m(M) = \text{Ext}^{d-j}_R(M, R)^\vee\).

Proof. Let \(x_1, \ldots, x_d\) be a system of parameters for \(R\). In the Cohen-Macaulay case, the local cohomology of \(R\) vanishes for \(i < d\), and so \(C^\bullet(x^\infty; R)\), numbered backwards, is a flat resolution of \(E\). Thus,

\[ H^j_m(M) \cong \text{Tor}^{R}_{d-j}(M, E). \]

Let \(G\) be a projective resoloution of \(M\) by finitely generated projective \(R\)-modules. Then

\[ \text{Ext}^{d-j}_R(M, R)^\vee \cong H^{d-j}(\text{Hom}_R(G, R), E) \]

(since \(E\) is injective, \(\text{Hom}_R(\_, E)\) commutes with the calculation of cohomology). The functor \(\text{Hom}_R(\text{Hom}_R(\_, R), E)\) is isomorphic with the functor \(\_ \otimes E\) when restricted to finitely generated projective modules \(G\). To see this, observe that for every \(G\) there is an \(R\)-bilinear map \(G \times E \to \text{Hom}_R(\text{Hom}_R(G, R), E)\) that sends \((g, u)\) (where \(g \in G\) and \(u \in E\)) to the map whose value on \(f : G \to R\) is \(f(g)u\). This map is an isomorphism when \(G = R\), and commutes with direct sum, so that it is also an isomorphism when \(G\) is finitely generated and free, and, likewise, when \(G\) is a direct summand of a finitely generated free module. But then

\[ \text{Ext}^{d-j}_R(M, R)^\vee \cong H^{d-j}(G, \otimes E) \cong \text{Tor}^{R}_{d-j}(M, E), \]

which is \(\cong H^j_m(M)\), as already observed. \(\square\)

Corollary. Let \(M\) be a finitely generated module over a local ring \((R, m, K)\). Then the modules \(H^i_m(M)\) have DCC.

Proof. The issues are unchanged if we complete \(R\) and \(M\). Then \(R\) is a homomorphic image of a complete regular local ring, which is Gorenstein. The problem therefore reduces to the case where the ring is Gorenstein. By local duality, \(H^i_m(M)\) is the dual of the Noetherian module \(\text{Ext}^{n-i}_R(M, R)\), where \(n = \text{dim}(R)\). \(\square\)

The action of the Frobenius endomorphism on local cohomolgy

Let \(R\) be a ring of prime characteristic \(p > 0\), and let \(I = (f_1, \ldots, f_n)R\). Consider the complex \(C^\bullet = C^\bullet(f^\infty; R)\), which is

\[ 0 \to R \to \bigoplus_j R_{f_j} \to \bigoplus_{j_1 < j_2} R_{f_{j_1}f_{j_2}} \to \cdots \to \bigoplus_{j_1 < \cdots < j_t} R_{f_{j_1} \cdots f_{j_t}} \to \cdots \to R_{f_1 \cdots f_n} \to 0. \]
This complex is a direct sum of rings of the form \( R_g \) each of which has a Frobenius endomorphism \( F_{R_g} : R_g \to R_g \). Given any homomorphism \( h : S \to T \) of rings of prime characteristic \( p > 0 \), there is a commutative diagram:

\[
\begin{array}{ccc}
S & \xrightarrow{h} & T \\
F_g & \uparrow & \uparrow F_T \\
S & \xrightarrow{h} & T
\end{array}
\]

The commutativity of the diagram follows simply because \( h(s)^p = h(s^p) \) for all \( s \in S \).

Since every \( C^i \) is a direct sum of \( R \)-algebras, each of which has a Frobenius endomorphism, collectively these endomorphisms yield an endomorphism of \( C^i \) that stabilizes every summand and is, at least, \( \mathbb{Z} \)-linear. This gives an endomorphism of \( C^* \) that commutes with differentials \( \delta^i : C^i \to C^{i+1} \) in the complex. The point is that the restriction of the differential to a term \( R_g \) may be viewed as a map to a product of rings of the form \( R_{gf} \).

Each component map is either \( h \) or \( -h \), where \( h : R_g \to R_{gf} \) is the natural localization map, and is a ring homomorphism. The homomorphism \( h \) commutes with the actions of the Frobenius endomorphisms, and it follows that \( -h \) does as well.

This yields an action of \( F \) on the complex and, consequently, on its cohomology, i.e., an action of \( F \) on the local cohomology modules \( H_j^i(R) \). It is not difficult to verify that this action is independent of the choice of generators for \( I \). This action of \( F \) is more than \( \mathbb{Z} \)-linear. It is easy to check that for all \( r \in R \), \( F(ru) = r^p F(u) \). This is, in fact, true for the action on \( C^* \) as well as for the action on \( H_j^i(R) \).

If \( R \to S \) is any ring homomorphism, there is an induced map of complexes

\[
C^*(f^\infty; R) \to C^*(f^\infty; S).
\]

It is immediate that the actions of \( F \) are compatible with the induced maps of local cohomology, i.e., that the diagrams

\[
\begin{array}{ccc}
H_j^i(R) & \xrightarrow{F} & H_j^i(S) \\
\uparrow F & & \uparrow F \\
H_j^i(R) & \to & H_j^i(S)
\end{array}
\]

commute.

We now want to use our understanding of local cohomology to prove the Theorem of Huneke and Lyubeznik.

**Discussion.** We are primarily interested in studying \( R^+ \) when \( (R, m, K) \) is a local domain that is a homomorphic image of a Gorenstein ring \( A \). If \( M \) is the inverse image of \( m \) in \( A \), we may replace \( A \) by \( A_M \) and so assume that \( A \) is local.
Note, however, that when we take a module-finite extension domain of $R$, the ring that we obtain is no longer local; it is only semilocal. Therefore, we shall frequently have the hypothesis that $R$ is a semilocal domain that is a module-finite extension of a homomorphic image of a Gorenstein local ring.

Let $(A, m, K)$ denote a Gorenstein local ring, $p$ a prime ideal of this ring, and $R$ a local domain that is a module-finite extension of $B = A/p$. $R$ is semilocal in this situation. The maximal ideals of $R$ are the same as the prime ideals $m$ that lie over $m/p$, since $R/m$ is a module finite extension of $B/(m \cap B)$, and so $R/m$ has dimension 0 if and only if $B/(m \cap B)$ has dimension 0, which occurs only when $m \cap B$ is the maximal ideal $m/p$ of $B$. Note that since $A$ is Gorenstein, it is Cohen-Macaulay, and therefore universally catenary. Hence, so is $R$. The Jacobson radical $\mathfrak{A}$ of $R$ will be the same as the radical of $mR$.

By the dimension formula, which is stated on p. 3 of the Lecture Notes from September 18 for Math 711, Fall 2006, and proved in the Lecture Notes from September 20 from the same course on pp. 3–5, we have that $\text{height}(m) = \text{height}(m/p)$ for every maximal ideal $m$ of $R$: thus, the height of every maximal ideal is the same as $\dim(R) = \dim(A/p)$.

We next observe the following fact:

**Proposition.** Let $R$ be a domain and let $W$ be a multiplicative system of $R$ that does not contain 0.

(a) If $T$ is an extension domain of $W^{-1}R$ and $u \in T$ is integral over $W^{-1}R$, then there exists $w \in W$ such that $wu$ is integral over $R$.

(b) If $T$ is module-finite (respectively, integral) extension domain of $R$ then there exists a module-finite (respectively, integral) extension domain $S$ of $R$ within $T$ such that $T = W^{-1}S$.

(c) If $W^{-1}(R^+/\mathfrak{A})$ is an absolute integral closure for $W^{-1}R$, i.e., we may write $W^{-1}(R^+/\mathfrak{A}) \cong (W^{-1}R)^+$. That is, plus closure commutes with localization.

Proof. (a) Consider an equation of integral dependence for $u$ on $T$. We may multiply by a common denominator $w \in W$ for the coefficients that occur to obtain an equation

$$wu^k + r_1u^{k-1} + \cdots + r_iu^{k-i} + \cdots + r_{k-1}u + r_k = 0,$$

where the $r_i \in R$. Multiply by $w^{k-1}$. The resulting equation can be rewritten as

$$(wu)^k + r_1(wu)^{k-1} + \cdots + w^{i-1}r_i(wu)^{k-i} + \cdots + w^{k-2}r_{k-1}(wu) + w^{k-1}r_k = 0,$$

which shows that $wu$ is integral over $R$, as required.

(b) If $T$ is module-finite over $R$, choose a finite set of generators for $T$ over $R$. In the case where $T$ is integral, choose an arbitrary set of generators for $T$ over $R$. For each
generator \( t_i \), choose \( w_i \in W \) such that \( w_it_i \) is integral over \( R \). Let \( S \) be the extension of \( R \) generated by all the \( w_it_i \).

(c) We have that \( W^{-1}R^+ \) is integral over \( W^{-1}R \) and so can be enlarged to a plus closure \( T \). But each element \( u \in T \) is integral over \( W^{-1}R \), and so there exists \( w \in W \) such that \( wu \) is integral over \( R \), which means that \( wu \in R^+ \). But then \( u = w^{-1}(wu) \in W^{-1}R^+ \), and it follows that \( T = W^{-1}R^+ \).

(d) Note that \( \supseteq \) is obvious. Now suppose that \( u \in I(W^{-1}R)^+ = IW^{-1}(R^+) \) by part (c). Then we can choose \( w \in W \) such that \( wu \) is integral over \( R \), which means that \( wu \in R^+ \). But then \( u = w^{-1}(wu) \in W^{-1}R^+ \), and it follows that \( T = W^{-1}R^+ \).

Remark. Part (d) may be paraphrased as asserting that plus closure for ideals commutes with arbitrary localization. I.e., \( (IW^{-1}R)^+ = IW^{-1}(R^+) \). Here, whenever \( J \) is an ideal of a domain \( S \), \( J^+ = (JS^+) \cap S \).

Remark: plus closure for modules. If \( R \) is a domain we can define the plus closure of \( N \subseteq M \) as the set of elements of \( M \) that are in \( \langle R^+ \otimes R N \rangle \) in \( R^+ \otimes R M \). It is easy to check that the analogue of (d) holds for modules as well.

We are now ready to begin the proof of the following result.

Theorem (Huneke-Lyubeznik). Let \( R \) be a semilocal domain of prime characteristic \( p > 0 \) that is a module-finite extension of a homomorphic image of a Gorenstein local ring \((A, m, K)\). Let \( \mathfrak{A} \) denote the Jacobson radical in \( R \), which is the same as the radical of \( mR \). Let \( d \) be the Krull dimension of \( R \). Then there is a module-finite extension domain \( S \) of \( R \) such that for all \( i < d \), the map \( H^i_m(R) \to H^i_m(S) \) is 0. If \( \mathfrak{B} \) denotes the Jacobson radical of \( S \), we may rephrase this by saying that \( H^i_A(R) \to H^i_B(S) \) is 0 for all \( i < d \).

Proof. Let \( n \) denote the Krull dimension of the local Gorenstein ring \((A, m, K)\). Since \( R \) is a module-finite extension of \( A/p \), we have that the height of \( p \) is \( n - d \).

Recall from the discussion above that \( \mathfrak{A} \) (respectively, \( \mathfrak{B} \)) is the radical of \( mR \) (respectively, \( mS \)). This justifies the rephrasing. We may think of the local cohomology modules as \( H^i_m(R) \) and \( H^i_m(S) \).

It suffices to solve the problem for one value of \( i \). The new ring \( S \) satisfies the same hypotheses as \( R \). We may therefore repeat the process \( d \) times, if needed, to obtain a module-finite extension such that all local cohomology maps to 0: once it maps to 0 for a given \( S \), it also maps to 0 for any further module-finite extension. In the remainder of the proof, \( i \) is fixed.

It follows from local duality over \( A \) that is suffices to choose a module-finite extension \( S \) of \( R \) such that the map

\[
\text{Ext}^{n-i}_A(S, A) \to \text{Ext}^{n-i}_A(R, A)
\]

is 0, since the map of local cohomology is the dual of this map. Note that both of the modules in \( (\ast) \) are finitely generated as \( A \)-modules. We shall use induction on \( \dim(R) \) to
reduce to the case where the image of the map has finite length over \( A \): we then prove a theorem to handle that case. Let \( V_S \) denote the image of the map.

Let \( P_1, \ldots, P_h \) denote the associated primes over \( A \) of the image of this map that are not the maximal ideal of \( A \). Note that as \( S \) is taken successively larger, the image \( V_S \) cannot increase. Also note that since \( V_S \) is a submodule of \( N = \text{Ext}_A^{n-i}(R, A) \), any associated prime of \( V_S \) is an associated prime of \( N \). We show that for each \( P_i \), we can choose a module-finite extension \( S_i \) of \( R \) such that \( P \) is not an associated prime of \( V_{S_i} \). This will remain true when we enlarge \( S_i \) further. By taking \( S \) so large that it contains all the \( S_i \), we obtain \( V_S \) which, if it is not 0, can only have the associated prime \( m \). This implies that \( V_S \) has finite length over \( A \), as required.

We write \( P \) for \( P_i \). Let \( W = A - P \). Then \( W^{-1}R = R_P \) is module-finite over the Gorenstein local ring \( A_P \). Let \( P \) have height \( s \) in \( A \), where \( s < n \). By local duality over \( A_P \), we have that the dual of \( \text{Ext}_A^{n-i}(M_P, A_P) \) is, functorially, \( H_{P \cdot A_P}^{s-(n-i)}(M_P) \) for every finitely generated \( A \)-module \( M \). Since \( i < d \),

\[
s - (n - i) < s - (n - d) = s - \text{height}(\mathfrak{p}) = s - \text{height}(\mathfrak{p} A_P) = \dim(A_P/\mathfrak{p} A_P).
\]

By the induction hypothesis we can choose a module-finite extension \( T \) of \( R_P \) such that

\[
H_{P \cdot A_P}^{s-(n-i)}(T) \to H_{P \cdot A_P}^{s-(n-i)}(R_P)
\]

is 0. By part (b) of the Proposition on p. 7, we can choose a module-finite extension \( S \) of \( R \) such that \( T = S_P \). Then we have the dual statement that

\[
\text{Ext}_{A_P}^{n-i}(S_P, A_P) \to N_P
\]

is 0, which shows that \( (V_S)_P = 0 \). But then \( P \) is not an associated prime of \( V_S \), as required.

Thus, we can choose a module-finite extension \( S \) of \( R \) such that \( V_S \) has finite length as an \( A \)-module. Taking duals, we find that the image of \( H_m^n(R) \to H_m^n(S) \) has finite length as an \( A \)-module. Since Frobenius acts on both of these local cohomology modules so that the action is compatible with this map, it follows that the image \( W \) of the map is stable under the action of Frobenius. Moreover, \( W \) is a finitely generated \( A \)-module, and, consequently, a finitely generated \( S \)-module. It suffices to show that we can take a further module-finite extension \( T \) of \( S \) so as to kill the image of \( W \) in \( H_m^n(T) \). This follows from the Theorem below. \( \square \)

**Theorem.** Let \( I \subseteq S \) be an ideal of a Noetherian domain \( S \) of prime characteristic \( p > 0 \), and let \( W \) be a finitely generated submodule of \( H_I^1(S) \) that is stable under the action of the Frobenius endomorphism \( F \). Then there is a module-finite extension \( T \) of \( S \) such that the image of \( W \) in \( H_I^1(T) \) is 0.

Notice that there is no restriction on \( i \) in this Theorem. We shall, in fact, prove a somewhat stronger fact of this type.