Math 711: Lecture of November 26, 2007

Étale and pointed étale homomorphisms and a generalization of Artin approximation

Let R be a Noetherian ring. We shall say that $R \to S$ is *étale* if S is essentially of finite type over R, flat, and the fibers are étale field extensions, so that for all $P \in \mathrm{Spec}\,(R)$, $\kappa_P \otimes_R S$ is a finite product of finite separable field extensions of κ_P. An equivalent conditions is that S is essentially of finite type over R and flat with 0-dimensional geometrically regular fibers. There are many other characterizations: a detailed treatment is given in the Lecture Notes from Math 711, Fall 2004.

A local homomorphism if local rings $(R,\, m,\, K) \to (S,\, \mathfrak{n},\, L)$ is called *pointed étale* if it is étale and the induced map of residue class fields $K \to L$ is an isomorphism. There is a structure theorem for pointed étale extensions:

Theorem. *Let $(R,\, m,\, K)$ be a local ring. S is a pointed étale extension of R if and only if there is a monic polynomial $f \in R[X]$ in one variable such that the image $\overline{f}$ of f in $K[X]$ has a simple root $\lambda \in K$, and*

$$S \cong \frac{R[X]_{\mathcal{M}}}{(f)},$$

where $\mathcal{M}$ is the kernel of the composite map $R[X] \twoheadrightarrow K[X] \twoheadrightarrow K$ and the right hand surjection is the evaluation map $g(X) \mapsto g(\lambda)$.

If r in R is such that $r \equiv \lambda$ modulo m, then the maximal ideal $\mathcal{M}$ may alternatively be described as $mR[X] + (x-r)R[X]$. Note that the image of the formal derivative f' of the polynomial f with respect to X is inveritble in S: because λ is a simple root of $\overline{f}$, $\overline{f}'(\lambda) \neq 0$, i.e., $f'(r) \in R - m$.

The Theorem above and many other results about the structure of étale, smooth, and umramified homomorphisms may be found in the Lecture Notes from Math 711, Fall 2004.

Here, we shall not use much more than the fact that a pointed étale extension is a localization at a maximal ideal of a module-finite extension. But we do mention a few properties to help give the reader some feeling for what may be expected from such an extension.

Let $(R,\, m,\, K)$ be local and let $(S, \mathfrak{n},\, L)$ be a pointed étale extension.

(1) S is fatihfully flat over R.

(2) $\mathfrak{n} = mS$, so that the closed fiber is $S/mS = S/\mathfrak{n} = L \cong K$.

(3) There is a unique local R-algebra embedding of S into $\widehat{R}$, and $\widehat{R}$ is also the completion of S.

(4) If R is excellent and normal, so is S.

(5) If $R \to R'$ is local, then $S' = R' \to R' \otimes_R S$ is a pointed étale extension of R'. In particular, for every proper ideal I of R, S/IS is a pointed étale extension of S/I.

Propertiers (1), (2), and (5) are immediate from the Theorem above. The proof of (3) uses Hensel's Lemma, which implies that the factorization $\overline{f} = (X - \lambda)g$, where $x - \lambda$ and g are relatively prime, lifts uniquely to a factorization $f = (x - \rho)G$ over $\widehat{R}$; here, $\rho \in \widehat{R}$. The embedding maps the image of X in S to ρ. Since S is essentially of finite type over R it is excellent, while the normality of S follows, for example, from the normality of $\widehat{R}$, which is faithfully flat over S.

We shall need the following generalization of the Artin approximation theorem.

Theorem. *Let $F_i(Z_1, \dots, Z_n) = 0$, $1 \le i \le k$, be a finite system of polynomial equations with coefficients in an excellent local ring (R, m, K). Let N be a positive integer. Suppose that the equations have a solution $\widehat{z}_1, \dots, \widehat{z}_n$ in $\widehat{R}$. Then they also have a solution $s_1, \dots, s_n$ in a pointed étale extension S of R such that $s_i \equiv \widehat{z}_i$ mod $m^N\widehat{R}$, $0 \le i \le n$.*

If one states the theorem without the congruence condition mod $m^N\widehat{R}$, one can still deduce that condition easily. Let $u_1, \dots, u_t$ be generators of m^N in R. Choose $r_i \in R$ such that $r_i \cong \widehat{z}_i$ mod $m^N\widehat{R}$ for every i. Then the congruence condition can be expressed equationally by introducing auxiliary variables Z_{ij} and auxiliary equations

$$Z_i - r_i - \sum_{j=1}^{t} u_j Z_{ij} = 0$$

for $1 \le i \le n$.

The theorem was first proved by M. Artin in the case of a local ring essentially of finite type over a field or over an excellent DVR, and also in the case of an analytic local ring (i.e., for a quotient of the ring of convergent power series in n variables over the complex numbers — analytic local rings are Henselian). The more general result follows from a theorem of D. Popescu called General Néron Desingularization: for a considerable time there was some disagreement about whether Popescu's argument, which certainly had gaps, was essentially correct. The conflict was resolved in an expository paper by R. Swan which gave all details of a complete and correct proof. Swan's version was based in turn on a paper by T. Ogoma which also gave an exposition of the proof of Popescu's theorem. We refer the reader to [M. Artin, *On the solutions of analytic equations*, Invent. Math. **5** (1968) 277–291], [M. Artin, *Algebraic approximation of structures over complete local rings*, Publ. Math. I.H.E.S. (Paris) 36 (1969) 23–56], [T. Ogoma,*General Néron desingularization based on the idea of Popescu*, J. Algebra **167** (1994) 57–84], [D. Popescu, *General Néron desingularization* Nagoya Math. J. **100** (1985) 97–126], [D. Popescu, *General Néron desingularization and* *approximation*, Nagoya Math. J. **104** (1986) 85–115], and, especially, [Swan, R. G. *Néron-Popescu desingularization*, in Algebra and geometry (Taipei, 1995), 135–192, Lect. Algebra Geom. **2** Int. Press, Cambridge, MA, 1998].

We next want to observe the following:

Lemma. *Let R be a normal Noetherian ring, and let S be the integral closure of R in a finite normal algebraic extension $\mathcal{L}$ of the fraction field $\mathcal{K}$ of R. Then the group $G = Aut(\mathcal{L}/\mathcal{K})$ stabilizes S, fixes R, and acts transitively on the set of maximal ideals of S.*

Proof. If we replace R by $R^{1/q}$, which is a normal local ring isomorphic with R, for sufficiently large q, and $\mathcal{L}$ by $\mathcal{L}[\mathcal{K}^{1/q}]$, then S is replaced by a ring purely inseparable over S, which will have the same spectrum. Now the extension of fraction fields is separable, and so Galois, and the group of field automorphisms is unaffected. Hence, we may assume without loss of generality that the extension of fraction fields is Galois, and then S is module-finite over R.

G fixes R and therefore stabilizes the set of elements of $\mathcal{L}$ integral over R, which is S. The fixed ring S^G must consist of elements of $\mathcal{K}$ that are integral over R. Since R is normal, $S^G = R$. Let $m_1, \dots, m_k$ be the maximal ideals of S. They all lie over m. Choose $s \in m_1 - \bigcup_{j=2}^k m_j$. Consider $r = \prod_{g\in G} g(s)$. Since it is fixed by G, it is in R. In fact, it is in $m_1 \cap R = m$. But then, for every i, $r \in m_i$. It follows that at leat one of the elements $g(s)$ is in m_i, since their product is. Since s is contained in exactly one maximal ideal, m_1, of S, and g is an automorphism of S, $g(s)$ is contained in exactly one maximal ideal, $g(m_1)$, of s. But $g(s) \in m_i$, and so we must have that $g(m_1) = m_i$, as required. □

The following result will enable us to reduce the problem of proving that tight closure is the same as plus closure for parameter ideals to the complete case.

Theorem. *Let (R, m, K) be an excellent, normal local ring. Let $u \in R$, and let $x_1, \dots, x_d$ be generators of an ideal I of R. Suppose that there is a module-finite extension domain T of $\widehat{R}$ such that $u \in IT$. Then there is a module-finite extension domain S of R such that $u \in IS$.*

Proof. It suffices to construct a module-finite extension S_1 of R such that $u \in IS_1$. We may then get a domain S by killing a minimal prime of S_1 disjoint from $R - \{\{0\}$.

The idea of the argument is to encode the problem of giving a module-finite extension T of a ring R_1 with $R \subseteq R_1 \subseteq \widehat{R}$ such that $u \in IT$ into solving a finite system of polynomial equations in finitely many variables with coefficients in R (but whose solutions we think of as in R_1). Once we have such a system, from the fact that the equations have a solution in $\widehat{R}$, we can deduce that they have a solution in a pointed étale extension of R. We then need to do some further work to show that R itself has a module-finite extension S such that $u \in IS$.

Fix generators for the module-finite extension T of R_1: call them $\theta_1, \dots, \theta_h$. We choose h from the module-finite extension domain of $\widehat{R}$ such that u is in the expansion of I. We may assume without loss of generality that θ_1 is taken to be 1. The structure of T as a module-finite R_1-algebra, the fact that $u \in IT$, and the condition that T is an extension of R_1 provided that R_1 is a domain are consequences of the equational conditions described in the following seven paragraphs.

(1) We keep track of the relations on the θ_k over R_1: think of these as given by an $h \times s$ unknowns $\bigl(Z_{ij}\bigr)$. The values z_{ij} of these coefficients determine the structure of T as an R_1-module: we assume that $\sum_i z_{ij}\theta_i = 0$ for every j, so that the module is the cokernel of the matrix $\bigl(z_{ij}\bigr)$.

(2) For each θ_i and θ_j, we keep track of the coefficients needed to write $\theta_i\theta_j$ as an R_1-linear combination of $\theta_1, \dots, \theta_h$. Specifically,

$$(\#) \quad \theta_i\theta_j = \sum_{k=1}^{h} z_{ijk}\theta_k.$$

Think of the Z_{ijk} as h^3 additional unknowns whose values z_{ijk} determine the multiplication.

(3) We require that $Z_{ijk} = Z_{jik}$ for all i, j, so that the multiplication will be commutative, and that $Z_{1ik} = Z_{1ki}$ be 1 if $i = k$ and 0 otherwise, which will imply the θ_1 will be the identity element.

(4) In order for the multiplication to be well-defined, we also need equations which assert that for every column of the matrix of relations on $\theta_1, \dots, \theta_h$, when we multiply the relation the column gives by θ_ν, and then rewrite each $\theta_\nu\theta_k$ as a linear combination of $\theta_1, \dots, \theta_1 k$ using the equations (#) displayed in (2), the resulting linear combination of $\theta_1, \dots, \theta_h$ is 0, i.e., the vector of coefficients is in the column space of the matrix of relations. It should be clear that we can formulate a system of equations in auxiliary variables which does exactly this. Solving the equations we have specified so far in an R-algebra R_1 gives just the information we need to construct a symmetric R_1-bilinear form (i.e., a multiplication) from the cokernel of the relation matrix, call it T, to T, such that $\theta_1, \dots, \theta_h$ is a set of generators for T and θ_1 is an identity element.

(5) We next want to address the problem of using equational conditions to guarantee that we have an associative multiplication. We can use the values of the Z_{ijk} in (2) to work out $(\theta_\mu\theta_\nu)\theta_\rho$. We first use the equations given in (2) to write $\theta_\mu\theta_\nu$ as a linear combination of the θ_k. We then multiply by θ_ρ and use several applications of the equations in (2) to write this as a linear combination of the θ_k. We proceed similarly to expand $\theta_\mu(\theta_\nu\theta_\rho)$. We get two different expressions, each of which is a linear combination of the θ_k, whose coefficients are polynomials in the variables Z_{ijk}. The difference should be a relation on the θ_k. The vector of coefficients from such a difference must therefore be a linear combination of the relations on the θ_k specified in (1). Thus, for each choice of μ, ν, and ρ, we write down equations that represent the difference vector of coefficients described above as a linear combination, using new unknowns for coefficients, of the column vectors given in (1).

(6) Recall that $x_1, \ldots, x_d$ generate I. The fact that $u \in IT$ can be expressed as follows. Consider $u\theta_1$ minus a linear combination of the θ_k, where each of the coefficients in the subtrahend is a linear combination of $x_1, \ldots, x_d$ with new unknown coefficients. Write the vector of coefficients of the θ_k arising in this difference as an unknown linear combination of the relation vectors from (1) using new unknowns again.

(7) One can express the condition that the map $R_1 \to T$ be injective (in the case where, as here, circumstances are forcing R_1 to be a domain) by requiring that there exist a map $T \to R_1$ whose value on θ_1 (the identity) is not 0. The map is determined by its values on the θ_k. Use new unknowns Z'_k for those values. The Z'_k must satisfy the same relations imposed on the θ_k in (1), which leads to a new system of equations. In the case we are considering, the original solution of the equations will be in $\widehat{R}$ and the new solution in a pointed étale extension of R. We get a linear map from T to $\widehat{R}$ that is nonzero on 1 because T is a finitely generated torsion-free module over $\widehat{R}$: it can be embedded in a finitely generated free $\widehat{R}$-module, the value of 1 will have a nonzero coordinate, and the projection map to $\widehat{R}$ corresponding to this coordinate will be the required map. Suppose that this map has nonzero value z'_1 on 1. Then we can choose N so large that $z'_1 \notin m^N\widehat{R}$. When we apply the generalized Artin approximation theorem, we can require that the value of Z'_1 be congruent to z'_1 modulo $m^N\widehat{R}$.

A module-finite extension domain T of $\widehat{R}$ such that $u \in IT$ gives a solution of the equations and congruence condition specified in (1) – (7) in $\widehat{R}$. By the Artin-Popescu approximation theorem, there is a solution in a pointed étale extension R_1 of R. Then $R_1 \subseteq \widehat{R}$. Since R is excellent and normal, R_1 is excellent and normal. In particular, R_1 is a domain. The solution in R_1 gives a ring S_1 module-finite over R_1 in which $u \in IS_1$. It is an extension ring because there is an R_1-linear map f to R_1 whose value on 1 is not 0. If $c \in R_1^\circ$ were mapped to 0 in S_1, i.e., if $c\theta_1 = 0$, then we would have

$$0 = f(0) = f(c \cdot \theta_1) = cf(\theta_1) = cz'_1 \neq 0,$$

a contradiction.

It is not clear that S_1 is a domain. But we may kill a minimal prime disjoint from $S_1 - \{0\}$ and so we get a module-finite extension domain S_2 of R_1 such that $u \in IS_2$.

Now, S_2 is a domain module-finite over a localization at a maximal ideal of a module-finite extension domain of R. We can replace S_2 by its localization at a maximal ideal. The resulting ring can be obtained as the localization at a maximal ideal m_3 of a module-finite extension domain S_3 of R. There is no harm in replacing S_3 by a larger domain that is integral over it. Choose a finite algebraic field extension $\mathcal{L}$ of $\mathcal{K} = \operatorname{frac}(R)$ containing $\operatorname{frac}(S_3)$ that is normal. Let S be the integral closure of R in this field. We know that $u \in IS_{\mathcal{M}}$ for a maximal ideal $\mathcal{M}$ of S lying over m_3 in S_3. We claim that $u \in IS$.

To prove this, it suffices to show that it is true after localization at every maximal ideal of S. By the Lemma near the bottom of p. 2, the Galois group G of the extension of fraction fields between R and S acts transitively on the maximal ideals of S, while fixing R. But for every $g \in G$, we have that $u \in IS_{g(\mathcal{M})}$, since g fixes u and I. $\square$

In consequence, we can extend the Theorem that R^+ is a big Cohen-Macaulay algebra to the excellent case.

Theorem (Hochster-Huneke). *Let R be an excellent semi-local domain of prime characteristic $p > 0$ such that all maximal ideals of R have height $d = \dim(R)$. Let $\mathfrak{A}$ be the Jacobson radical of R, and $x_1, \ldots, x_d$ elements of $\mathfrak{A}$ such that $\mathfrak{A} = \mathrm{Rad}\,(x_1, \ldots, x_d)$. Then $x_1, \ldots, x_d$ is a regular sequence in R^+.*

In particular, if R is an excellent local domain of prime characteristic $p > 0$, then R^+ is a big Cohen-Macaulay algebra for R.

Proof. Suppose that $ux_{k+1} \in (x_1, \ldots, x_k)R^+$. We must show that $u \in (x_1, \ldots, x_k)R^+$. We may replace R by a module-finite extension domain. Thus, we may assume that $u \in (x_1, \ldots, x_k)R$ and that R is normal. If $u \notin (x_1, \ldots, x_k)R^+$, this remains true when we localize at a suitable maximal ideal $\mathcal{M}$ of R^+: suppose that $\mathcal{M}$ lies over m in R. Then R_m is a normal excellent local domain, $x_1, \ldots, x_d$ is a system of parameters, $ux_{k+1} \in (x_1, \ldots, x_k)R$, and $u \notin (x_1, \ldots, x_k)R^+$. The completion whR is again a normal local domain in which $x_1, \ldots, x_d$ is a system of parameters. Since whR is a homomorphic image of a regular local ring, we know that $\widehat{R}^+$ is a big Cohen-Macaulay algebra for $\widehat{R}$. It follows that there is a module-finite extension domain T of $\widehat{R}$ such that $u \in (x_1, \ldots, x_d)T$. We may now apply the preceding Theorem to replace T by a module-finite extension domain S of R. $\square$

We can now begin our treatment of the proof the Theorem stated at the bottom of p. 4 of the Lecture Notes of November 21. We follows the steps listed on p. 5 of those Lecture Notes.

Step 1.. Choose a counterexample in which d is minimum.

Step 2.: reduction to the case where R is excellent, local, normal, of Krull dimension d, and $x_1, \ldots, x_d$ is a system of parameters. First, we may replace R by its normalization. $(x_1, \ldots, x_d)$ still has height height d, and R^+ does not change. The element u remains in $(x_1, \ldots, x_d)^*$ and it is still true that $u \notin (x_1, \ldots, x_d)R^+ = \mathfrak{A}$. Choose a minimal prime P in R of the support of $(\mathfrak{A} + Ru)/\mathfrak{A}$. We can replace u by a multiple ru in Ru whose image in $(\mathfrak{A} + Ru)/\mathfrak{A}$ has annihilator P. We still have that $ru \notin \bigl((x_1, \ldots, x_d)R_P\bigr)^+$, since plus closure commutes with localization. Now R_P is excellent, local, and normal. We have, moreover, $Pu \subseteq (x_1, \ldots, x_d)^+$. We claim that P must be a minimal prime of $x_1, \ldots, x_d)$. If not, P is not contained in the union of the minimal primes of $x_1, \ldots, x_d$. Then we can choose $x_{d+1} \in P$ such that $x_1, \ldots, x_d, x_{d+1}$ is part of a system of parameters for R_P. Then $x_{d+1}ru \in (x_1, \ldots, x_d)R^+ = f\mathfrak{A}$. Since R^+ is a big Cohen-Macaulay algebra, we have that $ru \in \mathfrak{A}$ after all, a contradiction. Thus, P is a minimal prime of $(x_1, \ldots, x_d)R$. We have now reduced to the case where R is excellent, local, normal, of Krull dimension d, and $x_1, \ldots, x_d$ is a system of parameters.