Math 711: Lecture of December 7, 2007

We assume that we have the situation of the displayed paragraph near the bottom of p. 6 of the Lecture Notes from December 5. Specifically, let

$$R \to T_0 \to T_1 \to \cdots \to T_r$$

be a sequence of algebras obtained from a complete local domain R of prime characteristic $p > 0$ by successive forcing with respect to $\sigma_0, \sigma_1, \ldots, \sigma_r$. Here σ_0 forces a column $u \in R^h$ that is in the tight closure of the column space of an $h \times k$ matrix α into the column space of α, and for $i \geq 1$, σ_i yields an algebra modification with repsect to a relation on part of a system of parameters in R. Moreover, we assume that $1 \in (x_1, \ldots, x_d)T_r$, where $x_1, \ldots, x_d$ is a system of parameters for R. We want to obtain a contradiction.

Each T_i is presented as the quotient of polynomial ring over T_{i-1} in finitely many variables Z_{ij} by an ideal generated by polynomials that are linear in the new variables. Putting these presentations together gives a presentation of every T_i as a polynomial ring over R. Thus, we have an increasing sequence of polynomial rings $R \subseteq \mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \cdots \subseteq \mathcal{T}_r$ and in each $\mathcal{T}_i$ an ideal $\mathfrak{B}_i$ such that $T_i \cong \mathcal{T}_i/\mathfrak{B}_i$.

For any polynomial ring $\mathcal{T}$ in finitely many variables over R, let $\mathcal{T}_{\leq N}$ denote the span of the monomials of total degree at most N over R. Evidently, $\mathcal{T}_{\leq N}$ is a finitely generated R-module. Let $M_i^{(N)}$ be the image of $(\mathcal{T}_i)_N$ in T_i. Thus, we have a commutative diagram:

$$\begin{array}{ccccccccc}
R & \longrightarrow & T_0 & \longrightarrow & T_1 & \longrightarrow & \cdots & \longrightarrow & T_r \\
{\scriptstyle \mathbf{1}}\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
R & \longrightarrow & M_0^{(N)} & \longrightarrow & M_1^{(N)} & \longrightarrow & \cdots & \longrightarrow & M_r^{(N)}
\end{array}$$

where the vertical maps are inclusions. The bottom row consists of finitely generated R-modules, and the top row is the ascending union over N of the bottom rows. We refer to the bottom row as a *sequence of partial forcing algebras* over R. Because T_r is the direct limit of the modules $M_r^{(N)}$, we have that the image of $1 \in R$ under the composite map from the bottom row is in $(x_1, \ldots, x_d)M_r^{(N)}$ for any sufficiently large choice of N. We also know that for every i, the elements of T_i occurring in σ_i are in $M_i^{(N)}$ for N sufficiently large. For the rest of the argument, we fix a choice of N sufficiently large that both these conditions hold.

Also fix a $\mathbb{Z}$-valued valuation ord on R that is nonnegative on R and positive on m. In particular, $\mathrm{ord}\,(x_i) \geq 1$ for every x_i. Extend ord to a $\mathbb{Q}$-valued valuation on R^+ that is nonnegative on R^+. We can do this by the argument at the bottom of p. 1 and top of p. 2 of the Lecture Notes from November 12.

We now use characteristic p techniques to obtain a contradiction. Everything that we have done so far is independent of the characteristic, but the final part of the argument depends heavily on the fact that we are in characteristic p.

Here is the key fact:

Lemma. *Let* $N_0 = N$ *and let* $N_{i+1} = N\big((\sum_{j=0}^{i} N_i) + N_i\big)$ *for* $i \geq 0$*. For every test element* $c \in R$ *and every* ${}^{[q]} = p^e$ *there is a commutative diagram of* R*-linear maps:*

$$\begin{array}{ccccccccc}
R^{+}[1/c^{1/q}] & \xrightarrow{\mathbf{1}} & R^{+}[1/c^{1/q}] & \xrightarrow{\mathbf{1}} & R^{+}[1/c^{1/q}] & \xrightarrow{\mathbf{1}} & \cdots & \xrightarrow{\mathbf{1}} & R^{+}[1/c^{1/q}] \\
\uparrow & & \phi_0\uparrow & & \phi_1\uparrow & & & & \phi_r\uparrow \\
R & \longrightarrow & T_0 & \longrightarrow & T_1 & \longrightarrow & \cdots & \longrightarrow & T_r \\
\mathbf{1}\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
R & \longrightarrow & M_0^{(N)} & \longrightarrow & M_1^{(N)} & \longrightarrow & \cdots & \longrightarrow & M_r^{(N)}
\end{array}$$

such that the maps from the middle row to the top row are R*-algebra homomorphisms and such that for every* i*,* $0 \leq i \leq r$*, the image of* M_i *is contained in the cyclic* R^{+} *module spanned by* $\dfrac{1}{c^{N_i/q}}$.

Before proving the Lemma, we state for emphasis that the values of the integers N_i are independent of the choice of q.

Proof of the Lemma. We construct the ϕ_i recursively. T_0 is formed from R by forcing (u, α), where u is in the tight closure of the column space of α. This implies that $c^{1/q}u$ is an $R^{1/q}$-linear combination of the the columns of α, and, hence, an R^{+}-linear combination of the columns of α. This enables us to write u as a linear combination of the columns of α with coefficients in $R^{+}[1/c^{1/q}]$ and so we obtain a map $T_0 \to R^{+}[1/c^{1/q}]$. The elements Z_i are sent into the cyclic R^{+}-module spanned by $1/c^{1/q}$. Hence, $M_0^{(N)}$ maps into the cyclic R^{+}-module spanned by $1/c^{N/q} = 1/c^{N_0/q}$, and since $N_0 = N$.

Now suppose that we have constructed $\phi_0, \ldots, \phi_i$ such that $\phi_j(M_j^{(N)}) \subseteq R^{+}w_j$ for $1 \leq j \leq i$, where $w_j = 1/c^{N_j/q}$. We want to construct ϕ_{i+1}. Now, T_{i+1} is a modification of T_i with respect to a relation

$$y_{k+1}v_{k+1} = y_1v_1 + \cdots + y_kv_k,$$

where $y_1, \ldots, y_k$ are part of a system of parameters for R. By our choice of N, the elements $v_1, \ldots, v_{k+1}$ are in $M_i^{(N)}$. Hence, their images under ϕ_i are expressible in the form s_jw with the $s_j \in R^{+}$ and $w = w_i = 1/c^{N_i/q}$. This means that we can write

$$y_{k+1}\phi_i(v_{k+1}) \in (y_1, \ldots, y_k)R^{+}w.$$

Since R^{+} is a big Cohen-Macaulay algebra for R, this implies that

$$\phi_i(v_{k+1}) \in (y_1, \ldots, y_k)R^{+}w,$$

say

$$\phi_i(v_{k+1}) = \sum_{j=1}^{k} y_js_jw$$

where the $s_j \in R^+$. We can now define ϕ_{i+1} by letting its values on the variables be the elements $s_j w$. Its values on any monomial of degree N in all of the variables that occur up to and including the $i+1$ spot will involve, at worst, $(c^{1/q})^{N_{i+1}}$, where

$$N_{i+1} = NN_0 + NN_1 + \cdots + NN_i + NN_i,$$

as claimed. $\square$

We are now ready for the dénoument, i..e, we can complete the proof of the Theorem stated on p. 1 of the Lecture Notes from December 5.

The final step in the proof of the existence of big Cohen-Macaulay algebras that capture tight closure. We keep the notation of the Lemma above. We have that $1 \in (x_1, \ldots, x_d)M_r^{(N)}$, and that if $w_i = 1/c^{N_i/q}$, $1 \leq i \leq r$, for every q there is a commutative diagram

$$\begin{array}{ccccccccc}
R^+ & \longrightarrow & R^+w_0 & \longrightarrow & R^+w_1 & \longrightarrow & \cdots & \longrightarrow & R^+w_r \\
\uparrow & & \phi_0\uparrow & & \phi_1\uparrow & & & & \phi_r\uparrow \\
R & \longrightarrow & M_0^{(N)} & \longrightarrow & M_1^{(N)} & \longrightarrow & \cdots & \longrightarrow & M_r^{(N)}
\end{array}$$

where the vertical maps are the restrictions of the ϕ_i and the horizontal maps in the first row are inclusion maps. By can consider the composite map from $R \to R^+w_r$ obtained by iterated composition by traversing two edges of the rectaqngle in two different ways. If we use the leftmost vertical arrow and the top row, we see that the image is simply $1 \in R^+w_r \subseteq R^+[1/c^{1/q}]$. By using the bottom row and the rightmost vertical arrow, as well as the fact that the image of $1 \in R$ is in $(x_1, \ldots, x_d)M_r$, we obtain that $1 \in (x_1, \ldots, x_d)R^+w_r$ for all q. It follows that $c^{N_r/q} \in (x_1, \ldots, x_d)R^+$ for all q, which leads to the conclusion that

$$\frac{1}{q}\mathrm{ord}\,(c^{N_r}) = \mathrm{ord}\,(c^{N_r/q}) \geq \min_j \mathrm{ord}\,(x_j) \geq 1$$

for all q. This is a contradiction, since we can choose $q > \mathrm{ord}\,(c^{N_r})$. $\square$

Note that if S and T are R-algebras, where R is a domain, and there is a map $S \to T$, then S is solid if T is solid. For if T is solid there is an R-linear map $T \to R$ such that $1 \mapsto c \in R^\circ$, and the composition $S \to T \to R$ will also be such a map. We obtain at once:

Corollary. *Let (R, m, K) be a complete local domain of prime characteristic $p > 0$. Let $u \in R$, and let $I = (f_1, \ldots, f_k)R$ be an ideal of R. Then $u \in I^*$ if and only if*

$$S = R[Z_1, \ldots, Z_k]/(u - \sum_{i=1}^{k} f_i Z_i)$$

is a solid R-algebra, i.e., if and only if $H_m^d(S) \neq 0$.

Proof. We know that $u \in I^*$ if and only if there is homomorphism from $R \to T$ such that $u \in IT$ and T is solid. Hence, if S is solid, $u \in I^*$, for $u \in IS$. On the other hand, if $u \in IT$ with T solid, we can map the forcing algebra S to T as an R-algebra, and it follows that S is solid. $\square$

This characterization is the starting point for the work of H. Brenner on tight closure.

Remark. There is an entirely similar criterion for when an element is in the tight closure of a submodule of a finitely generated free module over R.

It appears to be a very difficult problem to determine whether a given finitely generated R-algebra is solid. Any non-trivial result in this direction would be of great interest.

It is an open question whether every solid algebra over a complete local domain R can be mapped as an R-algebra to a big Cohen-Macaulay algebra. (If it can be mapped to a big Cohen-Macaulay algebra, it is solid.) G. Dietz has examples of finitely generated algebras that he conjectures are solid, but which cannot be mapped to a big Cohen-Macaulay algebra. However, it appears difficult to prove that these algebras are solid.

Or next objective will be to study the notion of *phantom homology.* In part of a complex of finitely generated modules (the maps may raise or lower degree), say $G' \to G \to G''$, an element of the homology at the middle spot is called *phantom* if it is represented by a cycle that is in the tight closure of the module of boundaries in G. This notion turns out to have many applications.