1. Let $\theta(1) = c \neq 0$. If $r \in IS \cap R$, we have $r = f_1s_1 + \cdots + f_ns_n$ with $r \in R$, the $f_i \in I$, and $s_1, \ldots, s_n \in S$. Then for all $q$, $r^q = f_1^q s_1^q + \cdots + f_n^q s_n^q$, and applying the $R$-linear map $\theta$ yields $cr^q = r\theta(1) = f_1^q\theta(s_1^q) + \cdots + f_n^q\theta(s_n^q) \in I[a]$. Hence, $r \in I^s$. 

2. Since $S$ is weakly $F$-regular, it is normal, and, hence, a finite product of weakly $F$-regular domains. It follows that $R$ is reduced. We use induction first on the number of factors of $R$, if $R$ is a product, and second on the number of factors of $S$. If $R$ is not a domain, we can partition the minimal primes into two nonempty sets $M_1$ and $M_2$. We can construct $a$ in all of the primes that are in $M_1$ and not in any of the primes that are in $M_2$, and $b$ in all of the primes in $M_2$ and in none of the primes in $M_1$. Then $ab = 0$ and $a + b$ is not a zerodivisor in $R$. If we kill any minimal prime of $S$, either $a$ or $b$ becomes 0, and, in either case, $a$ is in the ideal $(a + b)S$. Hence, $a$ is in its tight closure and therefore in the ideal in $S$. Then $a \in (a + b)S \cap R = (a + b)R$, and so we can find $e \in R$ such that $a = e(a + b)$. Modulo every prime in $M_1$, we must have $e \equiv 0$, and modulo every prime in $M_2$ we must have $e \equiv 1$. It follows that $e \equiv e^2$ mod every minimal prime, and, hence, that $e$ is a nontrivial idempotent in $R$. It is immediate that $R$ is a product $Re \times Rf$ with $f = 1 - e$, and $Re \twoheadrightarrow Se$ and $Rf \twoheadrightarrow SF$ inherit the hypothesis. Hence, by induction on the number of factors of $R$, both $Re$ and $Rf$ are weakly $F$-regular: consequently, so is $R$. Thus, we may reduce to the case where $R$ is a domain. If $R^s$ maps into $S^s$, which is automatic if $S$ is a domain, then whenever $cr^q \in I[a]$ for all $q \gg 0$ in $R$, we have that $cr^q \in I[a]S = (IS)^[a]$ for all $q \gg 0$ in $S$, and then $r \in (IS)^s$ in $S$, i.e., $r \in IS$, since $S$ is weakly $F$-regular. But then $r \in IS \cap R = I$, and so every ideal $I$ of $R$ is tightly closed. Now suppose that $S = S_1 \times \cdots \times S_n$ where $n \geq 2$. We proceed by induction on $n$. Every $S_i$ is an $R$-algebra. If $R \to S_i$ is injective for every $i$, then $R^s$ maps into $S^s = S_1^s \times \cdots \times S_n^s$. If not, we may assume by renumbering that $R \to S_n$ has a nonzero kernel $P$. Let $T = S_1 \times \cdots \times S_{n-1}$. We shall show that $R \to T$ still has the property that $IT \cap R = I$ for all $I \subset R$, and then the result follows by induction on $n$. Suppose not, and choose $I \subset R$ and $u \in R - I$ such that $u \in IT \cap R$. Let $a$ be a nonzero element of $P$. Then $au \in aIT \cap R$, but $au \notin aI$ in $R$. Since $au$ maps to 0 in $S_n$ and $aIS = (0)$ in $S_n$, we have that $au \in aIT \times S_n = aIS$ as well, and so $au \in aIS \cap R - aI$, a contradiction. 

3. If $P \in \text{Ass}(M)$ then $R/P \twoheadrightarrow M$. Applying $F^c$, which is faithfully flat, we have $R/P[a] \twoheadrightarrow F^c(M)$. Since $\text{Rad}(P[a]) = P$, $P$ is a minimal prime of $P[a]$, and so $R/P \twoheadrightarrow R/P[a] \twoheadrightarrow F^c(M)$. Hence, $P \in \text{Ass}(F^c(M))$. Now suppose that $P \notin \text{Ass}(M)$. Whether $P \in \text{Ass}(M)$ or $P \in \text{Ass}(F^c(M))$ is unaffected by localization at $P$. (Note that if $T = R_P$, $F^c_T(M_P) \cong F^c(R)(M_P)$.) Therefore, we may assume that $(R, P, K)$ is local, and that $P \notin \text{Ass}(M)$. Then there exists $x \in P$ that is not a zerodivisor on $M$. It follows that $x^q$ is a nonzerodivisor on $F^c(M)$, and so $P \notin \text{Ass}(F^c(M))$, as required. 

4. For every generator $u_i$ of $J$ we can choose $c_i \in R^s$ such that $cu_i^q \in I[a]$ for all $q \gg 0$. Let $c \in R^s$ be the product of the $c_i$. Then $c \in R^s$ is such that $cJ[a] \subseteq I[a]$ for all $q \gg 0$. Now $0 \leq \ell(R/I[a]) - \ell(R/J[a])$ (since $I[a] \subseteq J[a]$), and this is the same as $\ell(J[a]/I[a])$. Since $J$ has $k$ generators, $J[a]$ has at most $k$ generators, and the same holds for $N_q = J[a]/I[a]$. 

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Problem Set #1 Solutions
Since $c$ and $I^{[q]}$ both kill $N_q$, we can map $R/(I^{[q]} + cR)^{\oplus k}$ onto $N_q$, which bounds its length by $k\ell(R/(I^{[q]} + cR)) = k\ell(\overline{R}/\mathfrak{A}^{[q]}) \leq k\ell(\overline{R}/\mathfrak{A}^{[q]})$. Since $\overline{R}$ has dimension $d - 1$ and $\mathfrak{A}$ is primary to its maximal ideal, this length is bounded by $C_1(qh)^{d-1}$, using the ordinary Hilbert function. This yields the upper bound $kC_1h^{d-1}q^{d-1}$, so that we may take $C = kC_1h^{d-1}$. □

5. Let $p_1, \ldots, p_n$ be the minimal primes of $R$, and let $c_i \in R - p_i$ represent a test element in $R/p_i$. We may choose $d_i$ not in $p_i$ but in all other minimal primes of $R$. Let $c = c_1d_1 + \cdots + c_nd_n$. Then $c \in R^{\circ}$, and if $u \in N^{\star}_M$, this is true modulo every $p_i$, and so $c_iu \in N + p_iM$ for all $i$. Then $c_i d_i u \in N$, since $d_i$ kills $p_i$, and adding shows that $cu \in N$, as required. (The argument is valid both for test elements and for big test elements.) □

6. This result is proved in the Lecture Notes of October 8.