1. \( \subseteq \) is clear. To prove \( \supseteq \), if \( c \) is a test element and \( u \in \bigcap_n (I + m^n)^* \) then for all \( q \) and \( n \), \( cu^q \in (I + m^n)[q] = I[q] + (m^n)[q] \in I^q + m^n \). Fix \( q \). Then \( cu^q \in \bigcap_n (I[q] + m^n) = I^q \). Hence, \( u \in I^* \). \( \square \)

2. \( u \in I^* \) iff \( cu^q \in I^q \) for all \( q \) if \( cu^q \in I^q \) for all \( q \) (since \( \hat{R} \) is faithfully flat over \( R \), \( \hat{R} \cap R = J \) for all \( J \subseteq R \)) iff \( cu^q \in (I \hat{R})[q] \) for all \( q \) if \( u \in (I \hat{R})^* \). It is not necessary that \( I \) be \( m \)-primary. More generally, if \( R \subseteq S \), \( c \in R \) is a test element for both rings, and \( JS \cap R = J \) for all \( J \subseteq R \), then \( u \in I^*_R \) iff \( u \in (IS)^*_S \). \( \square \)

3. Since \( R \) and \( S \) are domains, \( R^\circ \subseteq S^\circ \) and \( I^* \subseteq (IS)^* = IS \). Hence \( I^* \subseteq IS \cap R \). The fact that \( IS \cap R \subseteq I^* \) was proved in class (see the Theorem on the first page of the Lecture Notes from October 12).

4. Frobenius closure of ideals commutes with localization: if \( W \) is a multiplicative system in \( R \) and \( (u/w)^q \in (W^{-1}R)[q] \), where \( u \in R \) and \( w \in W \), then for some \( w_1 \in W \) we have \( w_1u^q \in I^q \), and then \( (w_1u)^q \in I^q \) as well. But then \( w_1u \in IF \), and so \( u \in IF \), which shows that \( u/w \in IFW^{-1}R \) as well. Now suppose that \( u \in I^* \) but the \( cu \notin IF \). We want to obtain a contradiction. The latter condition can be preserved by localizing at a maximal ideal \( m \) in the support of the image of \( cu \) in \( R/IF \). We then have that \( u/1 \in (IR_m)^* \) in \( R_m \), but that \( u/1 \notin (IR_m)^F \). Choose \( q \geq N_m \). We also have that \( u^q/1 \in (IR_m)[q]^* \), and so \( c^qu^q/1 \in (IR_m)[q] \), since \( c^q \) is a multiple of \( cN_m \). But this says that \( (cu/1)^q \in (IR_m)[q] \), which shows that \( cu/1 \in (IR_m)^F \), a contradiction. \( \square \)

5. Let \( c_S \) be a test element for \( S \). Then \( c_S \) satisfies an equation of integral dependence on \( R \) whose constant term is not 0 (or factor out a power of \( x \)). Hence, \( c_S \) has a multiple \( c \) in \( R^\circ \). Now suppose that \( u \in H^*_G \), where \( H \) is a submodule of the module \( G \) over \( R \). Fix an \( R \)-linear map \( \theta : S \to R \) whose value on 1 is nonzero: call the value \( d \). \( \theta \) induces an \( R \)-linear map \( \eta : S \otimes G \to G \) such that \( s \otimes g \mapsto \theta(s)g \): hence, if \( g \in G \), \( \eta(1 \otimes g) = dg \). We have \( 1 \otimes u \in \langle S \otimes_R H \rangle^*_S \otimes_R G \) and so \( 1 \otimes cu \in \langle S \otimes_R H \rangle \), i.e., \( cu = \sum_{j=1}^h s_j \otimes h_j, s_j \in S, h_j \in H \). Apply \( \eta \) to obtain \( cd = \sum_{j=1}^n \eta(s_j)h_j \in H \). Thus, \( cd/u \) is a test element for \( R \). \( \square \)

6. Let \( x \in m \). It suffices to show that if \( N \subseteq M \) are finitely generated modules and \( u \in N_M^* \), then \( xu \subseteq N \), for then \( m \subseteq \tau(R) \) (if \( \tau(R) = R \), \( R \) is weakly F-regular). If not, choose \( N' \) with \( N \subseteq N' \subseteq M \) such that \( N' \) is maximal with respect to not containing \( xu \). Then \( M/N' \) is a finite length essential extension of \( Rxu \), which is killed by \( m \). We may replace \( u \) and \( N \subseteq M \) by \( u + N' \) and 0 \( \subseteq M/N' \) as a counterexample. For large \( t \), \( I_t = (x_1^t, \ldots, x_d^t)R \) kills \( M \), and since \( R/I_t \) is \((R/I_t)\)-injective, \( M \) embeds in \( R/I_t \). \( xu \) must map to a socle generator, which we may take to be the image, \( z \), of \( (x_1 \cdots x_d)^{t-1}y \). By hypothesis, \((I_t, z)R \) is tightly closed, so that \( Kz \) is tightly closed in \( R/I_t \). But since \( u \in 0_M^* \), its image \( v \in 0_{R/I_t}^* \subseteq (Kz)_{R/I_t}^* = Kz \). Since \( v \in Kz, xv = 0 \). Since \( M \subseteq R/I_t \), we also have \( xu = 0 \), a contradiction. \( \square \)