Math 711, Fall 2007 **Problem Set #4**
Due: Wednesday, November 28

1. Let $R \to S$ be a homomorphism of rings of prime characteristic $p > 0$, let M be an R-module, and let N be an S-module. Prove that $\mathcal{F}_S^e(M \otimes_R N) \cong \mathcal{F}_R^e(M) \otimes_R \mathcal{F}_S^e(N)$, and that with this identification, $(u \otimes v)^q = u^q \otimes v^q$ for $u \in M$ and $v \in N$.

2. Let R be a Noetherian ring and I a nilpotent ideal. Suppose that I has a filtration $I = I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n = 0$ such that I_t/I_{t+1} is (R/I)-free for $1 \le i \le n-1$.

(a) Show that $x \in R$ is a nonzerodivisor if and only if it is a nonzerodivisor on R/I, in which case $I/xI \hookrightarrow R/xR$ and there is a corresponding filtration of I/xI given by the images if the I_t in which the factors are the modules $(R/xR) \otimes_R (I_t/I_{t+1})$.

(b) Prove that R is Cohen-Macaulay if and only if R/I is Cohen-Macaulay.

3. (a) Let R be a Noetherian ring and P a height 0 prime ideal of R. Show that we can localize at one element $c \in R - P$ such that R_c is Cohen-Macaulay if and only if $(R/P)_c$ is Cohen-Macaulay. [Show that we may reduce to the case where P is the only minimal prime. Consider a filtration of P by ideals P_i such that $P = P_1$, P_i/P_{i+1} is killed by P, and $P_n = 0$ for some n. (If $P^n = 0$, we may take $P_i = P^i$. Alternatively, we may take $P_i = \mathrm{Ann}_P P^{n-i}$.) Localize at one element c such that all of the factors P_i/P_{i+1} are R_c free. The "if" direction was done in class.]

(b) Let R be a local ring that is a homorphic image of a Cohen-Macaulay ring. Prove that the Cohen-Macaulay locus in R is open.

4. Let $R \hookrightarrow S$ be a module-finite extension of Noetherian domains of prime characteristic $p > 0$. Let $N \subseteq M$ be R-modules and let $u \in M$. Show that if $1 \otimes u \in \langle S \otimes_R N \rangle^*_{S \otimes_R M}$ then $u \in N^*_M$.

5. Let (R, m, K) be a complete one-dimensional local domain with $\mathrm{frac}\, R = \mathcal{L}$, and let M be a finitely generated torsion free R-module. Show that the type of $\mathcal{L} \otimes_R M$, which is the same as its vector space dimension over $\mathcal{L}$ (this is also the torsion-free rank of M) is at most the type of M. (The latter may be described as the K-vectcor space dimension of M/xM, where $x \in m$ is a parameter, or as the K-vector space dimension of $\mathrm{Ext}^1_R(K, M)$.)

6. Let R be either an excellent ring or a local ring that is a homomorphic image of a Gorenstein ring. Prove that the locus where R is Cohen-Macaulay of type at most t is open. In particular, prove that the locus where R is Gorenstein is open.