1. Let $K$ be a field of characteristic $p > 0$ with $p \neq 3$. Let $X, Y,$ and $Z$ be indeterminates over $K$, and let $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3) = K[x, y, z]$.

(a) Show that $R$ is module-finite, torsion-free, and generically étale over $A = K[x, y]$, and find the discriminant $D$ of $R$ over $A$ with respect to the basis $1, z, z^2$.

(b) Use the Jacobian ideal to show that $(x^2, y^2, z^2) \subseteq \tau_b(R)$.

2. Continue the notation of problem 1. Show that $xyz^2$ is a socle generator modulo $I = (x^2, y^2)R$. Show that $I^* = (x^2, y^2, xyz^2)R$. Conclude using problem 4. of Problem Set #3 that the test ideal of $R$ is $m$.

3. Let $x_1, \ldots, x_k$ be part of a system of parameters in an excellent local domain $(R, m, K)$ of prime characteristic $p > 0$, and let $I = (x_1, \ldots, x_k)R$. Prove that for every multiplicative system $W$ in $R$, $(IW^{-1})^* = I^*W^{-1}R$. That is, tight closure commutes with localization for such an ideal $I$.

4. Let $R$ be a locally excellent domain of prime characteristic $p > 0$ that is a direct summand of every module-finite extension domain. Prove that $R$ is F-rational. In particular, it follows that $R$ is Cohen-Macaulay.

5. Let $(R, m, K)$ be a complete weakly F-regular local ring of Krull dimension $d$ of prime characteristic $p > 0$. Let $S$ be a Noetherian $R$-algebra such that the height of $mS$ in $S$ is $d$. Prove that $R$ is a direct summand of $S$.

6. Let $R$ be an $\mathbb{N}$-graded domain over an algebraically closed field $K$ of prime characteristic $p > 0$, where $R_0 = K$, and let $I$ be an ideal generated by forms of degree $\geq d \geq 1$. Let $G$ be a form of degree $d$ that is not in $I$. Prove that $G \notin I^*$. 