TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 374, Number 11, November 2021, Pages 8307–8308
https://doi.org/10.1090/tran/8300
Article electronically published on August 30, 2021

CORRIGENDUM TO "STRENGTH CONDITIONS, SMALL SUBALGEBRAS, AND STILLMAN BOUNDS IN DEGREE ≤ 4"

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Abstract. The statement and proof of a proposition, which appeared in Trans. Amer. Math. Soc. 373 (2020), no. 7, 4757–4806, about the locus where strength of a form is at most k are corrected: the locus is constructible but not known to be closed. Needed corrections are made in the proof of another result about strength: the statement of that result is not changed.

The proof of Proposition 3.3 of [1] is incorrect, and there is a counter-example in [3].

It is correct that the the loci described are constructible sets. No other results are affected, although the proof of Theorem 5.7 needs to be modified slightly, as described below. Some remarks related to Proposition 3.3 should be deleted. In particular, the sentence preceding Definition 1.13, the sentence preceding Proposition 3.3, and the short paragraph preceding Corollary 4.4 should be omitted. Remark 3.4 should read "See also Corollary 4.4 for a strengthening of 3.3(c) in the case of quadratic forms." In the statement of Proposition 3.3 the word "closed" should be replaced by "constructible," and the first paragraph of the proof should be simplified as shown immediately below. Note that the way in which parts (b) and (c) are deduced from part (a) is unchanged, but the word "closed" should be replaced by "constructible."

Proposition 3.3. *Let K be an algebraically closed field, and let R be a polynomial ring in finitely many variables over K. Let $d \geq 1$ be an integer and let R_d denote the K-vector space of d-forms in R. Let k be a positive integer, and let $d_1, \ldots, d_k$ be positive integers that are $\leq d$.*

(a) *The set of elements f of R_d that are contained in an ideal I with k generators $f_1, \ldots, f_k$ such that $f_i \in R_{d_i}$ is constructible in R_d. The set of points of $\mathbb{P}(R_d)$ represented by such a nonzero f is constructible in $\mathbb{P}(R_d)$.*
(b) *Let $h \in \mathbb{N}$. The set of elements f of R_d that have a (k,h)-collapse is constructible in R_d. The set of points of $\mathbb{P}(R_d)$ represented by such a nonzero f is constructible in $\mathbb{P}(R_d)$.*
(c) *The set of elements f of R_d having a strict k-collapse is constructible in R_d. The set of points of $\mathbb{P}(R_d)$ represented by such a nonzero f is constructible in $\mathbb{P}(R_d)$.*

Received by the editors June 3, 2020, and, in revised form, August 5, 2020.

2020 *Mathematics Subject Classification.* Primary 13D05, 13F20.

Key words and phrases. Polynomial ring, ideal, quartic form, cubic form, quadratic form, projective dimension, regular sequence.

The second author was partially supported by grants from the National Science Foundation (DMS–0901145 and DMS–1401384).

Proof. (a) The specified set of elements is the image of the map of affine algebraic sets $\prod_{i=1}^{k}(R_{d_i} \times R_{d-d_i}) \to R_d$ sending the element whose i th entry is (f_i, g_i), $1 \le i \le k$, to $\sum_{i=1}^{k} f_i g_i$, and the image of a map of affine algebraic sets is constructible. The corresponding statement for projective case follows. $\square$

The only other corrections are to the proof of Theorem (5.7). The argument works with only minor changes, using the fact that one has Proposition (3.3) with being constructible as the conclusion rather than being closed, but some rewording is needed. There are unrelated corrections to the first paragraph of the proof: in the fifth line from the bottom of that paragraph, "element" should be "entry" and in that same line, "wv" should be "vw." The second and third paragraphs of the proof of Theorem (5.7) should be reworded as shown just below — the rest of the argument is unchanged.

"Next, consider the subset $\mathcal{Z} \subseteq W_b \times \mathcal{V}^b$ consisting of pairs (w, v) such that $w * v$ has a strict k-collapse or is 0. The set of forms of degree d with a strict k-collapse together with 0 form a constructible set by Proposition (3.3)(c). $\mathcal{Z}$ is the inverse image of this set under the regular morphism μ, and so $\mathcal{Z}$ is constructible in $W_b \times \mathcal{V}^b$. Since $\mathcal{Z}$ is closed under multiplying either coordinate by a nonzero scalar, it determines a constructible set $X \subseteq \mathbb{P}(W_b) \times \mathbb{P}(\mathcal{V}^b)$. If the projection map $\pi_2 : X \to \mathbb{P}(\mathcal{V}^b)$ does not have dense image, the set of matrices representing points of the complement of the closure of its image is a dense open set U in $\mathcal{V}^b$ such that every nonzero linear combination of the entries of any element of U is k-strong. Therefore we may assume that the projection map has dense, constructible image in $\mathcal{V}^b$, and so the image has nonempty interior.

Since the image of the projection map contains a dense open subset of $\mathbb{P}(\mathcal{V}^b)$, the dimension of X is at least $bM - 1$. It follows that there is a point of $\mathbb{P}(W_b)$ such that the fiber of $\pi_1 : X \to \mathbb{P}(W_b)$ is a constructible subset of $\mathbb{P}(W_b)$ that has dimension at least $(bM - 1) - (b - 1) = bM - b$. Fix a nonzero $b \times 1$ matrix w representing an element where the fiber has dimension at least $bM - b$, and so has codimension at most $bM - 1 - (bM - b) = b - 1$ in $\mathbb{P}(\mathcal{V}^b)$. Choose an irreducible locally closed subset X_1 of the fiber over w of codimension at most $b - 1$ in $\mathbb{P}(\mathcal{V}^b)$."

Acknowledgments

The authors want to thank Arthur Bik and Andrew Snowden for pointing out the error. [2] is a corrected version of the full manuscript.

References

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