THE EISENBUD-GREEN-HARRIS CONJECTURE FOR DEFECT TWO QUADRATIC IDEALS

SEMA GÜNTÜRKÜN AND MELVIN HOCHSTER

ABSTRACT. The Eisenbud-Green-Harris (EGH) conjecture states that a homogeneous ideal in a polynomial ring $K[x_1, \ldots, x_n]$ over a field $K$ that contains a regular sequence $f_1, \ldots, f_n$ with degrees $a_i$, $i = 1, \ldots, n$ has the same Hilbert function as a lex-plus-powers ideal containing the powers $x_i^{a_i}$, $i = 1, \ldots, n$. In this paper, we discuss a case of the EGH conjecture for homogeneous ideals generated by $n + 2$ quadrics containing a regular sequence $f_1, \ldots, f_n$ and give a complete proof for EGH when $n = 5$ and $a_1 = \cdots = a_5 = 2$.

1. Introduction

Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over a field $K$ with the homogeneous lexicographic order in which $x_1 > \cdots > x_n$ and with the standard grading $R = \bigoplus_{i \geq 0} R_i$. We denote the Hilbert function of a $\mathbb{Z}$-graded $R$-module $M$ by Hilb$_M(i) := \dim_K M_i$, where $M_i$ is the homogeneous component of $M$ in degree $i$. When $I$ is a homogeneous ideal of $R$ and $M$ is $R$, or $I$, or $R/I$, the Hilbert function has value 0 when $i < 0$. When the Hilbert function of $M$ is 0 in negative degree, we may discuss the Hilbert function of $M$ by giving the sequence of its values, and we refer to this sequence of integers as the $O$-sequence of $M$.

In 1927, Macaulay [13] showed that the Hilbert function of any homogeneous ideal of $R$ is attained by a lexicographic ideal in $R$. Later, in Kruskal-Katona’s theorem [11, 12], it is shown that the polynomial ring $R$ in Macaulay’s result can be replaced with the quotient $R/(x_1^2, \ldots, x_n^2)$. After this result, Clement and Lindström, in [5], generalized the result to $R/(x_1^{a_1}, \ldots, x_n^{a_n})$ if $a_1 \leq \cdots \leq a_n < \infty$.

In [7] Eisenbud, Green and Harris conjectured a generalization of the Clement-Lindström result. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, where $2 \leq a_1 \leq \cdots \leq a_n$.

Conjecture 1.1 (Eisenbud-Green-Harris (EGH$_{\mathbf{a},n}$) Conjecture [7]). If $I$ is a homogeneous ideal in $R = K[x_1, \ldots, x_n]$ containing a regular sequence $f_1, f_2, \ldots, f_n$ with degrees $\deg f_i = a_i$, then there is a monomial ideal $\mathcal{L} = (x_1^{a_1}, \ldots, x_n^{a_n}) + J$, where $J$ is a lexicographic ideal in $R$, such that $R/\mathcal{L}$ and $R/I$ have the same Hilbert function.

Although there has been some progress on the conjecture, it remains open. The conjecture is shown to be true for $n = 2$ by Richert in [14]. Francisco [8] shows the conjecture for almost complete intersections. Caviglia and Maclagan in [2] prove the result if $a_i > \frac{i-1}{\sum_{j=1}^{i-1}}(a_j - 1)$ for $2 \leq i \leq n$. The rapid growth required for the degrees does not yield much insight into cases like the one in which the regular sequence consists of quadratic forms. When $n = 3$, Cooper in [6] proves the EGH conjecture for the cases where $(a_1, a_2, a_3) = (2, a_2, a_3)$ and $(a_1, a_2, a_3) = (3, a_2, a_3)$ with $a_2 \leq a_3 \leq a_2 + 1$.

One of the most intriguing cases is when $a_1 = \cdots = a_n = 2$ for any $n \geq 2$, which is the case for which Eisenbud, Green and Harris originally stated their conjecture. It is known that the conjecture holds for homogeneous ideals minimally generated by generic quadrics: the case where char $K = 0$ was proved by Herzog and Popescu [10] and the case of arbitrary characteristic.

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was proved by Gasharov [9] around the same time. There have been several other results on the EGH conjecture. More recently, the case when every $f_i$, $i = 1, \ldots, n$, in the regular sequence is a product of linear forms is settled by Abedelfatah in [1], and results on the EGH conjecture using linkage theory are given by Chong [4].

In this section we recall some definitions and state some known results that are used throughout the paper.

**Definition 2.1.** Let $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$ be monomials in $R$ of the same degree. We say that $u$ is greater than $v$ with respect to the lexicographic (or lex) order if there exists an $i$ such that $a_i > b_i$ and $a_j = b_j$ for all $j < i$.

A monomial ideal $J \subseteq R$ is called a lexicographic ideal (or lex ideal) if, for all degrees $d$, the $d$-th degree component of $J$, denoted by $J_d$, is spanned over the base field $K$ by an initial segment of the degree $d$ monomials in the lexicographic order.

**Definition 2.2.** Given $2 \leq a_1 \leq \cdots \leq a_n$, a lex-plus-powers ideal (LPP ideal) $\mathcal{L}$ is a monomial ideal in $R$ that can be written as $\mathcal{L} = (x_1^{a_1}, \ldots, x_n^{a_n}) + J$ where $J$ is a lex ideal in $R$.

This definition agrees with the one in [2]. Some authors require that the $x_i^{a_i}$ be minimal generators of $\mathcal{L}$, which we do not. However, since we consider only nondegenerate homogeneous ideals in this paper, i.e., ideals contained in $(x_1, \ldots, x_n)^2$, in the case where $a_1 = \cdots = a_n = 2$ it is automatic that the $x_i^2$ are minimal generators of the ideal under consideration.

In [8] Francisco showed the following for almost complete intersections.

**Theorem 2.3** (Francisco [8]). Let integers $2 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and $d \geq a_1$ be given. Let the ideal $I$ have minimal generators $f_1, \ldots, f_n$, $g$ where $f_1, \ldots, f_n$ form a regular sequence with $\deg f_i = a_i$ and $g$ has degree $d$. Let $\mathcal{L} = (x_1^{a_1}, \ldots, x_n^{a_n}, m)$ be the lex-plus-powers ideal where $m$ is the greatest monomial in lex order in degree $d$ that is not in $(x_1^{a_1}, \ldots, x_n^{a_n})$. Then $\text{Hilb}_{R/I}(d + 1) \leq \text{Hilb}_{R/\mathcal{L}}(d + 1)$.

Note that, necessarily, $d \leq \sum_{i=1}^{n} (a_i - 1)$, since $(f_1, \ldots, f_n)$ contains all forms of degree larger than that. If $a_1 = \cdots = a_n = 2$, then $d \leq n$.

The following corollary is an immediate consequence of Theorem 2.3 above. If $g \in R$ is a nonzero form of degree $i$ we write $g R_j$ for the vector space $\{gh : h \in R_j\} \subseteq R_{i+j}$.

**Corollary 2.4.** Let $I = (f_1, \ldots, f_n, g)$ be an almost complete intersection as in Theorem 2.3 above such that $a_1 = \cdots = a_n = 2$. Then

$$\dim_K ((f_1, \ldots, f_n)_{d+1} \cap g R_1) \leq d.$$  

**Proof.** We can write

$$\dim_K I_{d+1} = \dim_K (f_1, \ldots, f_n)_{d+1} + \dim_K g R_1 - \dim_K ((f_1, \ldots, f_n)_{d+1} \cap g R_1),$$
where \( \dim_K gR_1 = n \). Then by Theorem 2.3, we have

\[
\dim_K I_{d+1} \geq \dim_K (x_1^2, \ldots, x_n^2, x_1 \cdots x_d)_{d+1} = \dim_K (x_1^2, \ldots, x_n^2)_{d+1} + n - d
\]

Since \( \text{Hilb}_R/(f_1, \ldots, f_n) (i) = \text{Hilb}_R/(x_1^2, \ldots, x_n^2) (i) \) for all \( i \geq 0 \), we can conclude that

\[
\dim_K ((f_1, \ldots, f_n)_{d+1} \cap gR_1) \leq d.
\]

The next statement is a weaker version of the EGH conjecture. It focuses on the Hilbert function of the given homogeneous ideal only at the two consecutive degrees \( d \) and \( d + 1 \) for some non-negative integer \( d \).

**Definition 2.5 (EGH_{a,n}(d)).** Following Caviglia-Maclagan [2], we say that “EGH_{a,n}(d) holds” if for any homogeneous ideal \( I \subset K[x_1, \ldots, x_n] \) containing a regular sequence of degrees \( a = (a_1, \ldots, a_n) \), where \( 2 \leq a_1 \leq \cdots \leq a_n \), there exists a lex-plus-powers ideal \( \mathcal{L} \) containing \( \{x_i^{a_i} : 1 \leq i \leq n\} \) such that

\[
\dim_K I_d = \dim_K \mathcal{L}_d \quad \text{and} \quad \dim_K I_{d+1} = \dim_K \mathcal{L}_{d+1}.
\]

**Lemma 2.6.** The condition EGH\(_{(d,\ldots,d),n}(d)\) on a polynomial ring \( R = K[x_1, \ldots, x_n] \) is equivalent to the statement that for the ideal \( I \) generated by \( n + \delta \) \( K \)-linearly independent forms of degree \( d \) containing a regular sequence, one has that \( \dim_K I_{d+1} \geq \dim_K \mathcal{L}_{d+1} \), where \( \mathcal{L} = (x_1^d, \ldots, x_n^d) + J' \) and \( J' \) is minimally generated by the greatest in lex order \( \delta \) forms of degree \( d \) not already in \( (x_1^d, \ldots, x_n^d) \).

**Proof.** If there is an LPP ideal \( (x_1^d, \ldots, x_n^d) + J \), where \( J \) is a lex ideal, with the same Hilbert function as \( I \) in degrees \( d \) and \( d + 1 \), it is clear that \( J_d \) must be spanned over \( K \) by the specified generators of \( J' \), so that \( (x_1^d, \ldots, x_n^d) + J' \subseteq (x_1^d, \ldots, x_n^d) + J \), which implies the specified inequality on the Hilbert functions. Moreover, when that inequality holds we may increase \( \mathcal{L} := (x_1^d, \ldots, x_n^d) + J' \) to an LPP ideal with the same Hilbert function as \( I \) in degrees \( d \) and \( d + 1 \): if \( \Delta = \text{Hilb}_I(d+1) - \text{Hilb}_\mathcal{L}(d+1) \), we may simply include the greatest (in lex order) \( \Delta \) forms of degree \( d + 1 \) not already in \( \mathcal{L} \). \( \square \)

**Remark 2.7.** We shall eventually be focused on EGH\(_{a,n}(d)\) in the case where \( a_1 = \cdots = a_n = d = 2 \), simply referred as EGH\(_{(2,\ldots,2),n}(2)\) or EGH\(_{2,n}(2)\). We shall routinely make use of this lemma in this case of quadratic regular sequence and \( d = 2 \).

**Lemma 2.8 (Caviglia-Maclagan [2]).** Fix \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \) where \( 2 \leq a_1 \leq a_2 \leq \cdots \leq a_n \) and set \( s = \sum_{i=1}^{n} (a_i - 1) \). Then for any \( 0 \leq d \leq s - 1 \), EGH\(_{a,n}(d)\) holds if and only if EGH\(_{a,n}(s-1-d)\) holds.

Furthermore, the EGH\(_{a,n}\) conjecture holds if and only if EGH\(_{a,n}(d)\) holds for all degrees \( d \geq 0 \).

From now on, we always assume \( a = 2 = (2, \ldots, 2) \) for \( n \geq 2 \), unless it is stated otherwise.

**Remark 2.9.** For any \( n \geq 2 \), EGH\(_{2,n}(0)\) holds trivially. In [3, Proposition 2.1], Chen showed that EGH\(_{2,n}(1)\) is true for any \( n \geq 2 \).

Chen proved the following.

**Theorem 2.10 (Chen [3]).** The EGH\(_{2,n}\) conjecture holds when \( 2 \leq n \leq 4 \).

Chen’s proof of this uses Lemma 2.8 above, and the observation that, when \( n = 4 \), to demonstrate that the EGH\(_{2,4}\) conjecture is true, it suffices to show that EGH\(_{2,4}(0)\) and EGH\(_{2,4}(1)\) are true.
3. \( \text{EGH}_{2,n}(2) \) for defect two ideals

In this section, we focus on the homogeneous ideals in \( K[x_1, \ldots, x_n] \) for \( n \geq 5 \) that are generated by \( n + 2 \) quadratic forms containing a regular sequence. In particular, we study their Hilbert functions in degree 3.

**Definition 3.1.** If \( I \) is a homogeneous ideal minimally generated by \( n + \delta \) forms that contain a regular sequence of length \( n \), then \( I \) is said to be a defect \( \delta \) ideal.

Clearly, when \( \delta = 0 \) then \( I \) is generated by a regular sequence, it is a complete intersection, and we understand the Hilbert function completely. If \( \delta = 1 \), then \( I \) is an almost complete intersection.

**Definition 3.2.** We call a homogeneous ideal a quadratic ideal if it is generated by quadratic forms.

Let \( I = (f_1, \ldots, f_n, g, h) \) be a homogeneous ideal minimally generated by \( n + 2 \) quadrics where \( f_1, \ldots, f_n \) form a regular sequence. We call such an ideal a defect two ideal generated by quadrics or simply a defect two quadratic ideal. More generally, if a quadratic ideal is a defect \( \delta \) ideal, then we call it defect \( \delta \) quadratic ideal.

**Example 3.3.** The lex-plus-powers ideal \( L = (x_1^2, \ldots, x_n^2, x_1x_2, x_1x_3) \) in \( R \) is also a defect two quadratic ideal.

Further, for any homogeneous defect two quadratic ideal \( I \), we have the equality

\[
\dim_K I_2 = n + 2 = \dim_K L_2.
\]

**Main Question 3.4 (\( \text{EGH}_{2,n}(2) \) for defect two quadratic ideals).** For any \( n \geq 5 \), is it true that

\[
\dim_K I_3 \geq n^2 + 2n - 5 = \dim_K L_3?
\]

An affirmative answer for this question is proved completely in Theorem 3.17 below.

**Notation 3.5.** Throughout the rest of this paper we write \( \mathcal{J} \) for the ideal \( (f_1, \ldots, f_n)R \) when \( f_1, \ldots, f_n \) is a regular sequence of quadratic forms, and in the defect \( \delta \) quadratic ideal case we write \( g \) for the additional generators \( g_1, \ldots, g_\delta \) of the quadratic ideal. Here, \( f_1, \ldots, f_n, g_1, \ldots, g_\delta \) are assumed to be linearly independent over \( K \). Moreover, henceforth, we write \( J \) for the ideal \( \mathcal{J} + (g_1, \ldots, g_{\delta-1}) \). However, when \( \delta = 1 \) or 2 we may write \( g, h \) for \( g_1, g_2 \), so that whenever \( \delta = 2 \) we henceforth write \( J \) for the ideal \( \mathcal{J} + (g_1) = \mathcal{J} + (g) \). We denote the graded Gorenstein Artin \( K \)-algebra \( R/\mathcal{J} \) by \( A \).

We know that, if \( a_1 = \cdots = a_n = \deg g = 2 \), Theorem 2.3 shows that

\[
\dim_K J_3 \geq n^2 + n - 2
\]

and then Corollary 2.4 gives \( \dim_K (\mathcal{J}_3 \cap gR_1) \leq 2 \).

**Remark 3.6.** In [3, Proposition 3.7] Chen gave a positive answer to the Question 3.4 for defect two quadratic ideals \( I = \mathcal{J} + (g, h) \) if \( \dim_K (\mathcal{J}_3 \cap gR_1) = 2 \). We shall make repeated use of this fact in the sequel.

In this section we show \( \text{EGH}_{2,n}(2) \) for a defect two quadratic ideal \( I = \mathcal{J} + (g, h) \) under the condition that \( \dim_K (\mathcal{J}_3 \cap gR_1) \leq 1 \) for all \( g' \in Kg + Kh - \{0\} \); this covers all the cases for which Chen’s result in Proposition 3.6 is not applicable.

**Lemma 3.7.** As in Notation 3.5, \( J \) is the defect 1 quadratic ideal \( \mathcal{J} + gR \). Then:

\[
\dim_K I_3 = n^2 + 2n - \dim_K (\mathcal{J}_3 \cap gR_1) - \dim_K (J_3 \cap hR_1).
\]

Consequently, for the cases that are not covered by the Proposition 3.6 we have:
We know that $Ann\left( f_3 \cap gR_1 \right) = 1$ then $\dim_K I_3 = n^2 + 2n - 1 - \dim_K \left( J_3 \cap hR_1 \right)$, and $EGH_{2,n}(2)$ holds for a defect two quadratic ideal $I$ if and only if $\dim_K \left( J_3 \cap hR_1 \right) \leq 4$.

(ii) If $\dim_K \left( f_3 \cap gR_1 \right) = 0$ then $\dim_K I_3 = n^2 + 2n - \dim_K \left( J_3 \cap hR_1 \right)$, and $EGH_{2,n}(2)$ holds for $I$ if and only if $\dim_K \left( J_3 \cap hR_1 \right) \leq 5$.

**Proof.** We have:

$$\dim_K I_3 = \dim_K J_3 + \dim_K (hR_1) - \dim_K \left( J_3 \cap hR_1 \right)$$

$$= \dim_K f_3 + \dim_K (gR_1) - \dim_K \left( f_3 \cap gR_1 \right) + \dim_K (hR_1) - \dim_K \left( J_3 \cap hR_1 \right)$$

$$= n^2 + 2n - \dim_K \left( f_3 \cap gR_1 \right) - \dim_K \left( J_3 \cap hR_1 \right),$$

and then (i) and (ii) are immediate. \(\square\)

**Remark 3.8.** Let $n = 5$, so that $f = (f_1, \ldots, f_5)$. For a defect two quadratic ideal $I = (f, g, h) \subseteq K[x_1, \ldots, x_5]$, if $\dim_K \left( f_3 \cap gR_1 \right) = 0$ then clearly $\dim_K \left( f, g, h \right) \leq 5$, therefore $EGH_{2,5}(2)$ holds for such an ideal $I$. However, we must give an argument to cover all possible cases, that is, when $\dim_K \left( f_3 \cap gR_1 \right) = 1$, to be able to confirm $EGH_{2,5}(2)$ for every defect two quadratic ideal. In the last section, we discuss the EGH conjecture for $n = 5$ and $a_1 = \cdots = a_5 = 2$ in detail.

Next, we proceed with two useful lemmas.

**Lemma 3.9.** Let $A$ be the graded Gorenstein Artin $K$-algebra $R/f$ with $\dim_K A_1 = n$. Let $g, h$ be two quadratic forms such that $gA_1 = hA_1$. Then $Ann_{A_1} g = Ann_{A_1} h$.

Moreover, $Ann_{A_1}(g) = Ann_{A_1}(h)$ if $i \neq n - 2$.

**Proof.** Suppose that the linear annihilator space of $g$, $Ann_{A_1} g$, has dimension $a$ and $gA_1 = hA_1$. Thus $gA_1$ has dimension $n - a$ and clearly $hA_1$ and $Ann_{A_1} h$ have dimensions $n - a$ and $a$, respectively.

Notice that $gA(-2) \cong A/Ann_{A}(g)$, hence it is Gorenstein and it has a symmetric O-sequence

$$(0, 0, 1, n - a, e_4, e_5, \ldots, e_5, e_4, n - a, 1),$$

where $e_i$ denotes the dimension of $[gA]_i$ and $e_i = e_{n - i + 2}$ for $2 \leq i \leq n$. Then the Hilbert function of $A/gA$ is

$$\left( n \atop 0 \right) - 1, \left( n \atop 1 \right) - n + a, \left( n \atop 2 \right) - e_4, \ldots, \left( n \atop 3 \right) - e_5, \left( n \atop 4 \right) - e_4, a, 0).$$

Since $Ann_{A}(g) \cong \text{Hom}_K(A/gA, A) \cong (A/gA)\lor$, the Hilbert function of $Ann_{A}(g)$ is

$$(0, a, \left( n \atop 2 \right) - e_4, \ldots, \left( n \atop 3 \right) - n + a, \left( n \atop 4 \right) - 1, n, 1).$$

Recall that $gA_1 = hA_1$, $gA_i = hA_i$ for all $i \geq 2$, so $(g, h)A$ has the Hilbert function

$$(0, 0, 2, n - a, e_4, \ldots, e_4, n - a, 1).$$

Then the O-sequence of $A/(g, h)$ becomes

$$\left( n \atop 2 \right) - 2, \left( n \atop 3 \right) - n + a, \left( n \atop 4 \right) - e_4, \ldots, \left( n \atop 5 \right) - e_5, \left( n \atop 6 \right) - e_4, a, 0),$$

and it follows that $Ann_{A}(g, h)$ has the Hilbert function

$$(0, a, \left( n \atop 2 \right) - e_4, \ldots, \left( n \atop 3 \right) - e_4, \left( n \atop 4 \right) - n + a, \left( n \atop 5 \right) - 2, n, 1).$$

We know that $Ann_{A}(g, h) = Ann_{A}(g) \cap Ann_{A}(h)$, and in degree 1, $Ann_{A}(g, h)$ has dimension $a$, so $Ann_{A}(g, h) = Ann_{A_1}(g) = Ann_{A_1}(h)$. Further, $Ann_{A}(g)$ and $Ann_{A}(h)$ are the same in every degrees except in degree $n - 2$. \(\square\)
Lemma 3.10. Let \( g, h \) be two quadratic forms in a graded Gorenstein Artin \( K \)-algebra \( A \) such that \( gA_i = hA_i \) and \( g, h \) have the same annihilator space \( V \) in \( A_i \) for some \( i \geq 1 \). Then there exists \( g' \in Kg + Kh - \{0\} \) such that

\[
\dim_K \text{Ann}_A(g') \geq \dim_K V + 1.
\]

Proof. Consider the multiplication maps by \( g \) and \( h \),

\[
\phi_g : A_i/V \to gA_i \quad \text{and} \quad \phi_h : A_i/V \to hA_i
\]

whose images \( gA_i, hA_i \) are subspaces in \( A_{i+2} \) and \( gA_i = hA_i \) by assumption. Then there is an automorphism

\[
T : A_i/V \to A_i/V
\]
such that \( g\ell = hT(\ell) \) for any \( \ell \in A_i/V \). However, \( T \) has at least one nonzero eigenvector \( u \) with \( T(u) = cu \) for some \( c \in K \). Say \( \ell_u \) be a form in degree \( i \) represented by this eigenvector \( u \) in \( A_i \) and not in the annihilator space \( V \), thus \( g\ell_u = h\ell_u \). Then there is a quadratic form \( g' := g - ch \in Kg + Kh - \{0\} \) such that \( g' \) annihilated by the space \( V \) and also by \( \ell_u \in A_i \setminus V \). Hence \( \dim_K \text{Ann}_A(g') \geq \dim_K V + 1 \).

From now on, \( I = (f_1, \ldots, f_n, g, h) = \delta + (g, h) \) is a homogeneous ideal where \( \dim_K \big( \langle f_3 \cap g'R_1 \rangle \big) \neq 2 \) for a quadratic form \( g' \in Kg + Kh - \{0\} \), which means that \( \dim_K g'A_1 \neq n - 2 \). Therefore \( \dim_K g'A_1 \) is either \( n \) or \( n - 1 \).

Proposition 3.11. For the graded Gorenstein Artin \( K \)-algebra \( A \), if \( gA_1 = hA_1 \) with \( \dim_K gA_1 = n - 1 = \dim_K hA_1 \), that is \( \dim_K (\langle f_3 \cap gR_1 \rangle) = \dim_K (\langle f_3 \cap hR_1 \rangle) = 1 \), then \( \text{EGH}_{2,n}(2) \) holds for the homogeneous defect two quadratic ideal \( I = \delta + (g, h) \).

Proof. Since \( \dim_K \text{Ann}_A(g) = \dim_K \text{Ann}_A(h) = 1 \) there is some \( g' \in Kg + Kh - \{0\} \) with \( \dim_K \text{Ann}_A(g') = 2 \) by Lemma 3.10. In consequence, \( \dim_K \big( \langle f_3 \cap g'R_1 \rangle \big) = 2 \), and so we are done by Proposition 3.6.

Proposition 3.12. For the graded Gorenstein Artin \( K \)-algebra \( A \), if \( \dim_K gA_1 = \dim_K hA_1 = n \), then there exists a quadratic form \( g' \) in \( Kg + Kh \) with a nonzero linear annihilator in \( A \).

Proof. By assumption \( \dim_K A_1 = \dim_K gA_1 = \dim_K hA_1 = n \), and so we may consider again the multiplication maps \( \phi_g : A_1 \to gA_1 \) and \( \phi_h : A_1 \to hA_1 \). Then we obtain an automorphism \( T : A_1 \to A_1 \) and there exists an nonzero linear form \( \ell \in A_1 \) such that \( T(\ell) = c\ell \) for some \( c \in K \), that is \( g\ell = ch\ell \). Consider \( g' = g - ch \in Kg + Kh \). Clearly, \( \ell \in \text{Ann}_A(g') \).

Next we assume that there is a linear annihilator \( L \in A_1 \) of \( g \) where \( LH \neq 0 \) over the Gorenstein ring \( A = R/\delta \). This case may come up either when \( \dim_K gA_1 = \dim_K hA_1 = n - 1 \) and the linear annihilator spaces \( \text{Ann}_A(g) \) and \( \text{Ann}_A(h) \) are distinct, or when \( \dim_K gA_1 = n - 1 \) and \( \dim_K hA_1 = n \).

We shall make repeated use of the following result, which is Lemma 3.3 of Chen’s paper [3].

Lemma 3.13 (Chen [3]). If \( f_1, \ldots, f_n \) is a regular sequence of 2-forms in \( R \) and we have a relation \( u_1f_1 + u_2f_2 + \cdots + u_nf_n = 0 \) for some \( t \)-forms \( u_1, \ldots, u_n \), then \( u_1, \ldots, u_n \in (f_1, \ldots, f_n)_t \).

More precisely, we have that \( t \geq 2 \) and there exists a skew-symmetric \( n \times n \) matrix \( B \) of \( (t-2) \)-forms such that \( (u_1 u_2 \cdots u_n) = (f_1 f_2 \cdots f_n)B \).

Proposition 3.14. Let \( I = \delta + g \) be a defect \( \delta \), where \( 2 \leq \delta \leq n - 1 \), quadratic ideal of \( R \) as in Notation 3.5. If there is a linear form \( L \) in \( \text{Ann}_A(g_1, \ldots, g_{\delta - 1}) \) such that \( Lg_\delta \neq 0 \) in \( A \), then

\[
\dim_K \big( \langle f_1, \ldots, f_n, g_1, \ldots, g_{\delta - 1} \rangle \cap g_\delta R_1 \big) \leq 3
\]

Chen [3] used an argument involving the Koszul relations on \( (x_1, \ldots, x_r) \) for \( r \leq n \) while introducing another proof for Theorem 2.3. In the proof of this proposition we use a very similar argument.
Proof. As in Notation 3.5, let $J = J + (g_1, \ldots, g_{\delta-1})$, and denote the row vector of the regular sequence $f_1, \ldots, f_n$ by $\tilde{f}$ and the row vector of quadratic forms $g_1, \ldots, g_{\delta-1}$ by $\vec{g}$.

Suppose $\dim_K(J_3 \cap g_\delta R_1) \geq 4$, and without loss of generality we may assume that
\[
\begin{align*}
x_1g_\delta &= \vec{g} \cdot \vec{\ell}_1 + \tilde{f} \cdot \vec{p}_1 \\
x_2g_\delta &= \vec{g} \cdot \vec{\ell}_2 + \tilde{f} \cdot \vec{p}_2 \\
x_3g_\delta &= \vec{g} \cdot \vec{\ell}_3 + \tilde{f} \cdot \vec{p}_3 \\
x_4g_\delta &= \vec{g} \cdot \vec{\ell}_4 + \tilde{f} \cdot \vec{p}_4
\end{align*}
\]
where $\vec{\ell}_i$ and $\vec{p}_i$ are column vectors of linear forms of lengths $\delta - 1$ and $n$, respectively.

We observe that each $x_iLg_\delta$ is in $J$, and write $x_iLg_\delta = \tilde{f} \cdot \vec{Q}_i$ where $\vec{Q}_i$ is a column of quadratic forms for $i = 1, 2, 3, 4$. Therefore:
\[
Lg_\delta = \tilde{f} \cdot (q_{i,j}).
\]

We proceed to the proof by induction on the number of generators of quadratic forms. In the base case, we assume that $Lg_\delta \neq 0$ in $A$. Then we get an $n \times (\delta - 1)$ matrix $(q_{i,j}) = (q_1, q_2, \ldots, q_{\delta-1})$ of linear forms such that
\[
\tilde{f} \cdot \vec{Q}_i = \vec{g} \cdot (\vec{Q}_1 \vec{Q}_2, \vec{Q}_3, \vec{Q}_4).
\]

Let $M_1 = \begin{pmatrix} x_2 & x_3 & x_4 & 0 & 0 \\
-x_1 & 0 & x_3 & x_4 & 0 \\
0 & -x_1 & 0 & -x_2 & 0 \\
0 & 0 & -x_1 & 0 & -x_2 - x_3 \end{pmatrix}$. Note that $(x_1, x_2, \ldots, x_4) \cdot M_1 = 0$.

Multiplying the equation (1) by $M_1$ from right gives that $\tilde{f} \cdot (\vec{Q}_1 \vec{Q}_2, \vec{Q}_3, \vec{Q}_4) = 0$, and so all entries are 0 in
\[
\begin{align*}
\tilde{f} (x_2\vec{Q}_1 - x_1\vec{Q}_2 & \quad x_3\vec{Q}_1 - x_1\vec{Q}_3 & \quad x_4\vec{Q}_1 - x_1\vec{Q}_4 & \quad x_3\vec{Q}_2 - x_2\vec{Q}_3 & \quad x_4\vec{Q}_2 - x_2\vec{Q}_4 & \quad x_4\vec{Q}_3 - x_3\vec{Q}_4).
\end{align*}
\]

By Lemma 3.13, there are alternating $n \times n$ matrices $B_{12}, B_{13}, B_{14}, B_{23}, B_{24}, B_{34}$ of linear forms such that
\[
\begin{pmatrix} x_2\vec{Q}_1 - x_1\vec{Q}_2 & \cdots & x_4\vec{Q}_3 - x_3\vec{Q}_4 \end{pmatrix} = \begin{pmatrix} B_{12} & \cdots & B_{34} \end{pmatrix} \tilde{f} \vec{Q}.
\]

Similarly, consider the matrix $M_2 = \begin{pmatrix} x_3 & x_4 & 0 & 0 \\
-x_2 & 0 & x_4 & 0 \\
0 & -x_2 & -x_3 & 0 \\
x_1 & 0 & 0 & x_4 \\
0 & x_1 & 0 & -x_3 \\
0 & 0 & x_3 & x_2 \end{pmatrix}$ such that $M_1 \cdot M_2 = 0$ and multiply equation (2) by $M_2$ from right to obtain:
\[
\begin{pmatrix} x_3B_{12} - x_2B_{13} + x_1B_{23} \cdots & x_4B_{23} - x_3B_{24} + x_2B_{34} \end{pmatrix} \tilde{f} \vec{Q} = 0.
\]

Then again by Lemma 3.13, there are alternating $n \times n$ matrices
\[
C_1^{123}, \ldots, C_n^{123}, C_1^{124}, \ldots, C_n^{124}, \ldots, C_1^{234}, \ldots, C_n^{234}
\]
of scalars such that
\[ x_3B_{12} - x_2B_{13} + x_1B_{23} = \begin{pmatrix} \bar{f}C_1^{123} \\ \vdots \\ \bar{f}C_n^{123} \end{pmatrix} \]

\[ x_4B_{12} - x_2B_{14} + x_1B_{24} = \begin{pmatrix} \bar{f}C_1^{124} \\ \vdots \\ \bar{f}C_n^{124} \end{pmatrix} \]

(3)

\[ x_4B_{13} - x_3B_{14} + x_1B_{34} = \begin{pmatrix} \bar{f}C_1^{134} \\ \vdots \\ \bar{f}C_n^{134} \end{pmatrix} \]

\[ x_4B_{23} - x_3B_{24} + x_2B_{34} = \begin{pmatrix} \bar{f}C_1^{234} \\ \vdots \\ \bar{f}C_n^{234} \end{pmatrix} \]

Repeating the previous steps with \( M_3 = \begin{pmatrix} x_4 \\ -x_3 \\ x_2 \\ -x_1 \end{pmatrix} \), so that \( M_2 \cdot M_3 = 0 \), we get

\[ \bar{f}(x_4C_1^{123} - x_3C_i^{124} + x_2C_i^{134} - x_1C_i^{234}) = 0. \]

Then, finally, \( x_4C_1^{123} - x_3C_i^{124} + x_2C_i^{134} - x_1C_i^{234} = 0 \) for all \( i = 1, 2, \ldots, n \). Hence, \( C_i^{123} = C_i^{124} = C_i^{134} = C_i^{234} = 0 \) for all \( i = 1, 2, \ldots, n \).

Thus, in (3) we get \( x_3B_{12} - x_2B_{13} + x_1B_{23} = 0 \). This shows that \( x_3 \) divides every entry in \( x_2B_{13} - x_1B_{23} \). Therefore we may rewrite \( B_{13} = x_3B_{13} + D_{13} \) and \( B_{23} = x_3B_{23} + D_{23} \), where \( B_{13} \) and \( B_{23} \) are alternating matrices of scalars, \( D_{13} \) and \( D_{23} \) are alternating matrices of linear forms that do not contain \( x_3 \), and \( x_2D_{13} - x_1D_{23} = 0 \). We obtain the following

\[ B_{12} = \frac{1}{x_3}(x_2B_{13} - x_1B_{23}) = x_2\bar{B}_{13} - x_1\bar{B}_{23} \]

Returning to equation (2), we obtain \( x_2\bar{Q}_1 - x_1\bar{Q}_2 = B_{12}\bar{f}^T = (x_2\bar{B}_{13} - x_1\bar{B}_{23})\bar{f}^T \). Consequently,

\[ x_1(\bar{Q}_2 - \bar{B}_{23}\bar{f}^T) = x_2(\bar{Q}_1 - \bar{B}_{13}\bar{f}^T) \]

which tells us that \( x_1 \) divides every entry of \( \bar{Q}_1 - \bar{B}_{13}\bar{f}^T \). It follows that

\[ \bar{f}(\bar{Q}_1 - \bar{B}_{13}\bar{f}^T) = \bar{f}\bar{Q}_1 \quad \text{as} \quad \bar{B}_{13} \quad \text{is alternating and} \quad \bar{f}\bar{B}_{13}\bar{f}^T = 0 \]

\[ = x_1Lg_6 \quad \text{by equation (1)}. \]

This shows that \( Lg_6 = \frac{1}{x_1}(\bar{Q}_1 - \bar{B}_{13}\bar{f}^T) \in (f_1, \ldots, f_n)_3 \), which contradicts our assumption \( L \notin \text{Ann}_A(g_6) \). \( \square \)
Corollary 3.15. Let $I = \mathfrak{f} + \mathfrak{g} \subseteq R$ be a defect $\delta$ quadratic ideal with $2 \leq \delta \leq n - 1$. Suppose that

\[ \dim_{K} I_{3} \geq \dim_{K} L_{3} \]

where $L = (x_{1}^{2}, \ldots, x_{n}^{2}) + (x_{1}x_{2}, x_{1}x_{3}, \ldots, x_{1}x_{\delta+1})$ is the defect $\delta$ lex-plus-powers ideal of $R$. That is, $\text{EGH}_{2,n}(2)$ holds for any defect $\delta$ quadratic ideal with property $(\dagger)$.

Proof. Notice that $\dim_{K} L_{3} = n^{2} + n\delta - \frac{\delta(\delta+3)}{2}$. We use induction on $\delta$. Let $J = \mathfrak{f} + (g_{1}, \ldots, g_{\delta-1})$ be the defect $\delta - 1$ quadratic ideal.

\[
\begin{aligned}
\dim_{K} I_{3} &= \dim_{K} J_{3} + n - \dim_{K} (J_{3} \cap g_{\delta}R_{1}) \\
&\geq \left(n^{2} + (\delta - 1)n - \frac{(\delta - 1)(\delta + 2)}{2}\right) + n - 3 = n^{2} + n\delta - \frac{(\delta + 3)}{2} + \delta - 3 \\
&\geq n^{2} + n\delta - \frac{(\delta)(\delta + 3)}{2}.
\end{aligned}
\]

We notice that a special case of Corollary 3.15 when $\delta = 2$ shows that the inequality is strict.

Corollary 3.16. Let $I = \mathfrak{f} + (g, h)$ be a defect two ideal generated by quadrics in $R$. If $\text{Ann}_{A_{1}}(g) = \text{Span}\{L\}$ for some $L \subseteq R_{1}$ and $L$ does not annihilate $h$ in $A = R/\mathfrak{f}$, then

\[
\dim_{K} I_{3} \geq n^{2} + 2n - 4 > \dim_{K}(x_{1}^{2}, \ldots, x_{n}^{2}, x_{1}x_{2}, x_{1}x_{3})_{3}
\]

Proof. The result follows from Proposition 3.14 as

\[
\dim_{K} I_{3} = n^{2} + 2n - \dim_{K}(f_{3} \cap gR_{1}) - \dim_{K}(J_{3} \cap hR_{1}) = \dim_{K}(\text{Ann}_{A_{1}}(g)) = 1 \leq 3
\]

which is $\geq n^{2} + 2n - 4$.

Finally, we give an affirmative answer to the Main Question 3.4.

Theorem 3.17. Let $I = \mathfrak{f} + (g, h) \subseteq R = K[x_{1}, \ldots, x_{n}]$ for $n \geq 5$ be a defect two ideal quadratic ideal. Then

\[
\dim_{K} I_{3} \geq n^{2} + 2n - 5.
\]

More precisely, $\text{EGH}_{2,n}(2)$ holds for homogeneous defect two quadratic ideals in $R$ for any $n \geq 5$.

Proof. If the given defect two ideal satisfies Proposition 3.6, then, by Chen’s result, the theorem is proved.

Assume that $\dim_{K}(f_{3} \cap gR_{1}) \neq 2$ for any $g' \in Kg + Kh \setminus \{0\}$. If $\dim_{K}(f_{3} \cap gR_{1}) = \dim_{K}(f_{3} \cap hR_{1}) = 0$, by Proposition 3.12, we can always find another quadratic form $q' \in Kg + Kh \setminus \{0\}$ so that $q'$ has a linear annihilator in $A$. Then we can apply Corollary 3.16. If $\dim_{K}(f_{3} \cap gR_{1}) = \dim_{K}(f_{3} \cap hR_{1}) = 1$ and the same linear form annihilates both $g$ and $h$ in $A$, by Proposition 3.11, we have a situation contradicts our assumption.

Corollary 3.18. $\text{EGH}_{2,n}(2)$ holds for every defect two ideal containing a regular sequence of quadratic forms.

Proof. This result follows from Lemma 2.6 and Theorem 3.17.
4. THE EGH CONJECTURE WHEN \( n = 5 \) AND \( a_1 = \cdots = a_5 = 2 \)

In this section \( R = K[x_1, \ldots, x_5] \) and \( J = (f_1, \ldots, f_5) + (g_1, \ldots, g_5) = f + g \) is a homogeneous defect \( \delta \) ideal in \( R \), where \( f_1, \ldots, f_5 \) is a regular sequence of quadrics and \( \deg g_j \geq 2 \) for \( j = 1, \ldots, \delta \). Throughout, we shall write \( A := R/f \), which is a graded Gorenstein local Artin ring. We will show the existence of a lex-plus-powers ideal \( L \subseteq R/\ell A \) containing \( x_i^2 \) for \( i = 1, \ldots, 5 \) with the same Hilbert function as \( I \) by proving the following main theorem.

**Theorem 4.1.** The EGH conjecture holds for all homogeneous ideals containing a regular sequence of quadrics in \( K[x_1, \ldots, x_5] \).

Lemma 2.8 of Caviglia-Macalagan tells us that \( EGH_{2,5}(d) \) holds if and only if \( EGH_{2,5}(5-d-1) \) holds. Thus it will be enough to show \( EGH_{2,5}(d) \) when \( d = 0, 1, 2 \). By Remark 2.9 we know that \( EGH_{2,5}(d) \) is true when \( d = 0, 1 \), therefore \( EGH_{2,5}(3) \) and \( EGH_{2,5}(4) \) both hold as well.

Our goal in this section is to prove \( EGH_{2,5}(2) \) for any homogeneous ideal containing a regular sequence of quadrics: this will complete the proof of \( EGH_{2,5} \). To achieve this, it suffices to understand \( EGH_{2,5}(2) \) for quadratic ideals with arbitrary defect \( \delta \) (but, of course, \( \delta \leq 10 \), since \( \dim_K R/2 = 15 \)), by Lemma 2.6.

**Remark 4.2.** As a result of Corollary 3.18, we see that \( EGH_{2,5} \) holds for any defect \( \delta = 2 \) quadratic ideal in \( K[x_1, \ldots, x_5] \) for \( n = 5 \).

To accomplish our goal we will prove \( EGH_{2,5}(2) \) for defect \( \delta \geq 3 \) quadratic ideals. In the next subsection, we prove that if one knows the case where \( \delta = 3 \), on obtains all the cases for \( \delta \geq 4 \).

In the final subsection we finish the proof by establishing \( EGH_{2,5}(2) \) for \( \delta = 3 \).

**Quadratic ideals with defect \( \delta \geq 4 \).**

**Lemma 4.3.** If \( EGH_{2,5}(2) \) holds for all defect three quadratic ideals, then it holds for all quadradic ideals with defect \( \delta \geq 4 \).

**Proof.** Let \( I = (f_1, \ldots, f_5, g_1, g_2, g_3, g_4) = f + g \subseteq R \) be a defect 4 homogeneous ideal generated by quadrics, where \( f_1, \ldots, f_5 \) form a regular sequence. By assumption the defect three quadratic ideal \( J = (f + (g_1, g_2, g_3)) \subseteq I \) satisfies \( EGH_{2,5}(2) \), that is, \( \dim_K J_3 \geq 31 \).

Let \( L = (x_2^2, x_1^2, x_1 x_2, x_1 x_3, x_1 x_4, x_1 x_5) \) be the LPP ideal with \( \dim_K L_2 = \dim_K I_2 = 9 \). Then we get \( \dim_K I_3 \geq \dim_K J_3 \geq 31 = \dim_K L_3 \), as we need for the case of defect \( \delta = 4 \).

Now assume \( 5 \leq \delta \leq 10 \). Let \( \delta I \) denote an arbitrary defect \( \delta \) quadratic ideal, and let \( \delta L \) denote the lex-plus-power ideal with defect \( \delta \geq 5 \). More precisely, \( \delta L := (x_2^2, \ldots, x_5^2) + (m_1, \ldots, m_8) \) where \( m_i \) are the next greatest quadratic square-free monomials with respect to lexicographic order. We need to show that \( \text{Hilb}_{R/\delta I}(3) \leq \text{Hilb}_{R/\delta L}(3) \).

We assume that \( \text{Hilb}_{R/\delta I}(3) \geq \text{Hilb}_{R/\delta L}(3) + 1 \), and we shall obtain a contradiction.

Using duality for Gorenstein rings, we know that for \( 0 \leq d \leq 5 \) we have that

\[
\text{Hilb}_{R/\delta I}(d) = \text{Hilb}_{R/f}(d) - \text{Hilb}_{R/(f_2, \ldots, f_5)}(5-d).
\]

Then, for \( d = 3 \), using the assumption we get

\[
\text{Hilb}_{R/(f_2, \ldots, f_5)}(3) = \text{Hilb}_{R/f}(3) - \text{Hilb}_{R/\delta I}(3)
\]

\[
\leq 9 - \text{Hilb}_{R/\delta L}(3) = \begin{cases} 7 & \text{if } \delta = 5, \\ 8 & \text{if } \delta = 6, 7, \\ 9 & \text{if } \delta = 8, 9, 10. \end{cases}
\]

We next show that \( \dim_K (f : \delta I)_1 = 0 \). If there is a nonzero linear form \( \ell \in f : \delta I \) then \( \dim_K \text{Ann}_{A_2} \ell A \geq \delta \geq 5 \), so we get that \( \dim_K A_3/\ell A_2 \geq 5 \). On the other hand, we see that \( A_3/\ell A_2 \cong [R/(f_1, \ldots, f_4, f_5, l)]_3 \) where the \( \bar{f}_i \) are the images of the \( f_i \), and the dimension of \( [R/(f_1, \ldots, f_4, f_5, l)]_3 \) as a \( K \)-vector space is at most 4.
Then we can find a defect $\gamma$ quadratic ideal $\gamma J \subseteq I$ for $\gamma = 3, 2, 1$ if the defect of $\delta I$ is $\delta = 5$ or $\delta = 6, 7$ or $\delta = 8, 9, 10$, respectively. We then have the inequalities shown below, where the first is obvious and the second follows by comparison with Hilbert functions of quotients by LPP ideals in degree 3 and the fact that, by assumption, $\text{EGH}_{2,5}(2)$ holds for quadratic ideals with defect less than or equal to three.

$$
\text{Hilb}_{R/(\delta I)}(3) \leq \text{Hilb}_{R/\gamma J}(3) \leq \begin{cases} 4 & \text{if } \gamma J \text{ is a defect } \gamma = 3 \text{ quadratic ideal when } \delta = 5, \\ 5 & \text{if } \gamma J \text{ is a defect } \gamma = 2 \text{ quadratic ideal when } 6 \leq \delta \leq 7, \\ 7 & \text{if } \gamma J \text{ is a defect } \gamma = 1 \text{ quadratic ideal when } 8 \leq \delta \leq 10. 
\end{cases}
$$

However, each of the cases above contradicts the following equality:

$$
\text{Hilb}_{R/(\delta I)}(3) = \text{Hilb}_{R/\delta I}(2) - \text{Hilb}_{R/\delta I}(2) = \delta.
$$

Thus, we get $\text{Hilb}_{R/\delta I}(3) \leq \text{Hilb}_{R/\delta I}(3)$ for any defect $\delta \geq 5$ quadratic ideal $\delta I$ in $R$. □

**Defect three quadratic ideals.**

**Lemma 4.4.** Let $I = \bar{f} + (g_1, g_2, g_3)$ be a defect three quadratic ideal in the polynomial ring $R$. Then, for any $1 \leq i_1 < i_2 \leq 3$,

$$\dim_K(\bar{f} : (g_{i_1}, g_{i_2}))_1 \leq 1,$$

and, furthermore, $\dim_K(\bar{f} : (g_1, g_2, g_3))_1 \leq 1$.

**Proof.** Suppose that $\dim_K(\bar{f} : (g_1, g_2))_1 \geq 2$, and assume there are $\ell_1, \ell_2 \in R_1$ such that $\ell_i g_1, \ell_i g_2 \in \bar{f}$ for both $i = 1, 2$. Without loss of generality we assume that $\ell_1 = x_1$ and $\ell_2 = x_2$.

Therefore, we can write $(x_1, x_2, f_1, \ldots, f_5) \subseteq \bar{f} : I$. Then

$$2 = \text{Hilb}(f_1, \ldots, f_5, g_1, g_2)/\bar{f}(2) = \text{Hilb}_{R/(\bar{f} : (f_1, \ldots, f_5, g_1, g_2))}(5 - 2), \quad (\text{by duality})$$

$$\leq \text{Hilb}_{R/(x_1, x_2, f_1, \ldots, f_5)}(3)$$

$$= \text{Hilb}_{K[x_3, x_4, x_5]/(f_1, \ldots, f_5)}(3), \quad (\text{where } \bar{f}_i \text{ is the image of } f_i \text{ in } K[x_3, x_4, x_5],)$$

$$\leq \binom{5 - 2}{3} = 1,$$

which is a contradiction. □

Hence, working in the graded Gorenstein Artin $K$-algebra $A = R/\bar{f}$, we have from the lemma just above that $\text{Ann}_{A_1}(g_1, g_2)$ is a $K$-vector space of dimension at most one, and, therefore

$$\dim_K \text{Ann}_{A_1}(g_1, g_2, g_3) \leq 1$$

since $\text{Ann}_{A_1}(g_1, g_2, g_3) \subseteq \text{Ann}_{A_1}(g_1, g_2)$.

**Remark 4.5.** By Remark 4.2 we know that for any defect two quadratic ideal $J$ in $R$, $\dim_K J_3$ is at least 30. Then $\text{EGH}_{2,5}(2)$ holds for the defect three quadratic ideals $I$ containing a defect two quadratic ideal $J$ with $\dim_K J_3 \geq 31$, as $\text{Hilb}_{R/\bar{f}}(3) \leq \text{Hilb}_{R/I} \leq 4$.

We henceforth focus on defect three quadratic ideals $I = \bar{f} + (g_1, g_2, g_3)$ in $R$ such that every defect two quadratic ideal $J \subseteq I$ containing $\bar{f}$ has $\dim_K J_3 = 30$.

For such defect three quadratic ideals, we observe the following.

**Remark 4.6.** Consider the ideal $\mathcal{I} = (g_1, g_2, g_3)A$ in the Gorenstein ring $A$ such that any ideal $(g_1, g_2)A$ contained in $\mathcal{I}$ has degree three component of dimension $\dim_K (g_1, g_2)A_1 = 5$. Assuming that $\dim_K \text{Ann}_{A_1}(g_1) = 1$, we have that $\text{Ann}_{A_1}(g_1, g_2, g_3) = \text{Ann}_{A_1}(g_1)$.

Furthermore, if $g_1A_1$ is 5-dimensional, that is, there is no linear form that annihilates $g_1$ in $A$, then for any quadric $g$ in $K g_1 + K g_2 + K g_3$ the vector space $gA_1 \subseteq A_3$ is either 3 or 5 dimensional.
Proof. Let $\dim_K \Ann_A_1(g_1) = 1$, and let the linear form $L$ annihilate $g_1$ but not some form $g' \in Kg_2 + Kg_3$ in $A$. We define a defect two quadratic ideal

$$J = (f_1, \ldots, f_5, g_1, g') \subseteq f + (g_1, g_2, g_3)$$

in $R$. Hence, by Corollary 3.16, we know already that $\dim_K J_3 \geq 31$, which means that $\dim_K (g_1, g')A_1 = 6$. This contradicts our assumption. Thus, $L$ must be in $\Ann_A_1(g_1, g_2, g_3)$. \hfill $\square$

Recall that the following holds, by Proposition 3.14, when $\delta = 3$.

**Proposition 4.7.** Let $I = f + (g_1, g_2, g_3) \subseteq K[x_1, \ldots, x_5]$ be a defect 3 quadratic ideal.

As usual, let $A = R/\mathfrak{f}$. If there is a linear form $L \in \Ann_A(g_1, g_2)$ such that $L \notin \Ann_A(g_3)$, then

$$\dim_K ((f + (g_1, g_2))_3 \cap g_3R_1) \leq 3.$$

\hfill $\square$

When a defect three quadratic ideal $I$ satisfies the condition of the above proposition, we notice a sharp bound for $\Hilb_{R/I}(3)$.

**Corollary 4.8.** Given a defect three quadratic ideal $I = f + (g_1, g_2, g_3)$ in $R = K[x_1, \ldots, x_5]$, and, as usual, let $A = R/\mathfrak{f}$, which is a graded Gorenstein Artin ring. If $\dim_K \Ann_A(g_1, g_2) = 1$ and $\Ann_A(g_1, g_2, g_3) = 0$ then

$$\dim_K I_3 \geq 32 > \dim_K \mathcal{L}_3,$$

where $\mathcal{L} = (x_1^2, \ldots, x_5^2, x_1x_2, x_1x_3, x_1x_4)$.

**Proof.** By assumption there is a linear form in $\Ann_A(g_1, g_2)$, say $L$, such that $L$ does not annihilate $g_3$. Hence, Proposition 4.7 gives us $\dim_K ((f + (g_1, g_2))_3 \cap g_3R_1) \leq 3$. Then we get

$$\dim_K (f + (g_1, g_2))_3 \cap g_3R_1 = \dim_K (f + (g_1, g_2))_3 + \dim_K g_3R_1$$

$$- \dim_K ((f + (g_1, g_2))_3 \cap g_3R_1)$$

$$\geq 30 + 5 - 3 = 23 > 31 = \dim_K \mathcal{L}_3.$$
Proof. First, by Remark 4.5 we note that it suffices to consider any defect two quadratic ideal $J \subseteq I$ with Hilb$_{R/I}(3) = 5$.

Suppose that $\dim_K \text{Ann}_{A_1}(g_1, g_2, g_3) = 0$. Then, clearly, no $g_i$, for $i = 1, 2, 3$ has a 1-dimensional linear annihilator space in $A_1$, since, otherwise, by Remark 4.6, we obtain that $\dim_K \text{Ann}_{A_1}(g_1, g_2, g_3) = 1$, which contradicts our assumption. Thus, for the rest of the proof we may assume that each $g_i A_1$, $i = 1, 2, 3$, is either 3 or 5 dimensional.

If all forms $g$ in $Kg_1 + Kg_2 + Kg_3$ are such that $\dim_K g A_1 = 3$ then we can find two independent quadratic forms whose linear annihilator spaces intersect in 1-dimensional space, and the result follows from Corollary 4.8.

Let $g_1 A_1$ be a 5-dimensional subspace of $A_3$ and for every $g \in Kg_2 + Kg_3$, $g A_1$ has dimension either 3 or 5.

We complete the proof by obtaining a contradiction. We assume that Hilb$_{R/I}(3) = 5$. In other words, the space $W = (Kg_1 + Kg_2 + Kg_3)A_1 \subseteq A_3$ is 5-dimensional. Then we get $W = g_1 A_1 = (Kg_2 + Kg_3)A_1$.

Consider the multiplication maps by $g_1, g_2$ and $g_3$ from $A_1$ to the subspace $W$ of $A_3$. By adjusting the bases of $A_1$ and $W$ we can assume the matrix of $g_1$ is the identity matrix $I_5$ of size 5. Denote the matrices of $g_2$ and $g_3$ by $\alpha$ and $\beta$, respectively. We can assume that $\alpha$ and $\beta$ are both singular, and so have rank 3, by subtracting the suitable multiples of $I_5$ from them if they are not singular.

We see that all matrices $z I_5 + x \alpha + y \beta$ must have at most two eigenvalues, otherwise we can form a linear combination whose kernel is 1-dimensional, which corresponds to a quadratic form with 1-dimensional linear annihilator space. Then there are two main cases: one is that every matrix in the space spanned by $I_5, \alpha$ and $\beta$ has one eigenvalue. The other is that almost all matrices in the form $z I_5 + x \alpha + y \beta$ have two eigenvalues, since the subset with at most one eigenvalue is Zariski closed.

Define $D(x, y, z) = \det(z I_5 - x \alpha - y \beta)$, a homogeneous polynomial in $x, y, z$ of degree 5 that is monic in $z$. Note that $D$ is also the characteristic polynomial, in $z$, of $x \alpha + y \beta$. Notice that the singular matrices in the subspace of $5 \times 5$ matrices spanned by $I_5, \alpha$ and $\beta$ are defined by the vanishing of $D$.

If the determinant $D$ is square-free (as the characteristic polynomial in $z$), then the ideal $(D)$ is a radical ideal and it cannot contain a nonzero polynomial of degree less than 5, which contradicts the fact that all size 4 minors of a singular matrix must vanish, since in our situation these singular matrices have rank 3. Therefore the size 4 minors, whose degrees are at most 4, are in the radical $(D)$.

If the determinant $D$ is not square-free, then its squared factor must be linear or quadratic: in the latter case the other factor is linear, so that in either case $D$ has a linear factor, say $z - ax - by$.

Consider the independent matrices $\alpha' = a I_5 - \alpha$, $\beta' = b I_5 - \beta$. Then we think of any linear combination of them, say $r \alpha' + s \beta' = r(a I_5 - \alpha) + s(b I_5 - \beta) = (ar + bs)I_5 - r \alpha - s \beta$. As $z - ax - by$ is a factor of $D(x, y, z)$, and hence, $D$ vanishes for $x = r, y = s, z = ar + bs$. This means that every linear combination of $\alpha'$ and $\beta'$ is singular. Therefore, we can replace $\alpha, \beta$ by $\alpha'$ and $\beta'$ and so we can assume that we are in the case where every linear combination of the two non-identity matrices is singular, and, if not 0, of rank 3. By Lemma 4.10, this implies that the kernels of $\alpha'$ and $\beta'$ cannot be disjoint, so we are done by Proposition 4.9 and Corollary 4.8.

In order to prove EGH$_{2,5}(2)$ for every defect three quadratic ideal $I = f + (g_1, g_2, g_3)$ in $R = K[x_1, \ldots, x_5]$ we must also discuss the cases when there is a nonzero linear form $L \in \text{Ann}_{A_1}(g_1, g_2, g_3)$.

Proposition 4.12. Let $I = f + (g_1, g_2, g_3)$ be a defect three quadratic ideal in $R$. If $\text{Ann}_{A_1}(g_1, g_2, g_3)$ is a 1-dimensional $K$-subspace of $A_1$, say $KL$, then

Hilb$_{R/I}(3) = 4$. 

\[\]
Claim 4.13. One of the quadratic forms $f_i$ in the regular sequence has the linear factor $L$.

Proof of claim. As $g_1, g_2, g_3 \in \Ann_A(L) \subseteq A_2$ for $L \in \Ann_A(g_1, g_2, g_3)$ we know that
\[
\dim_K \Ann_A(L) \geq 3.
\]
This tells us that $\dim_K LA_2 \leq 7$, which implies
\[
\dim_K (A_3/LA_2) = \dim_K [A/\LA]^3 \geq 3
\]
as $\dim_K A_3 = 10$.

Case 2. Let $f_i$ be the image of $f_i$ modulo $x_5$.
Suppose that $\bar{f} = (\bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4, \bar{f}_5)$ is an almost complete intersection in $K[x_1, x_2, x_3, x_4]$.

Thus,
\[
\dim_K \left[ \frac{K[x_1, x_2, x_3, x_4]}{\bar{f}} \right]_{3} \leq 2 = \dim_K \left[ \frac{K[x_1, x_2, x_3, x_4]}{(x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2)} \right]_{3}.
\]
This contradicts (4).

Hence the images of $f_i$ modulo $L$ form a regular sequence in $K[x_1, \ldots, x_4]$, that is, one of them has a linear factor $x_5$. □

As a result of the claim, after a suitable change of variables, we may assume that the linear annihilator is $L = x_5$ and may consider $I$ in two possible forms: either $I$ is in the form of (5) in Case 1 below, where $f_1, f_2, f_3, f_4, x_1x_5$ is the regular sequence, or $I$ is as in (6) in Case 2 below, where $f_1, f_2, f_3, f_4, x_1^3$ form a quadratic regular sequence in $I$.

Case 1. Suppose that $f_5 = x_1x_5$. Then we can assume that $g_1 = x_1x_2, g_2 = x_1x_3, g_3 = x_1x_4$. Furthermore, after we alter the $f_i$ by getting rid of all the terms containing $x_1$ except $x_1^2$, we may assume that the defect three quadratic ideal $I$ looks like
\[
I = (f_1, f_2, f_3, f_4 + cx_1^2, x_1x_5, x_1x_2, x_1x_3, x_1x_4),
\]
where $f_1, f_2, f_3, f_4$ form a regular sequence in $K[x_2, x_3, x_4, x_5]$ and $c \in K$.

Proposition 4.14. Let $I = (f_1, f_2, f_3, f_4 + cx_1^2, x_1x_5, x_1x_2, x_1x_3, x_1x_4)$ be a defect three quadratic ideal in $R$ where $f_1, f_2, f_3, f_4$ is an $K[x_2, x_3, x_4, x_5]$-sequence. Then
\[
\Hilb_{R/1}(3) = 4 = \Hilb_{R/\mathcal{L}}(3)
\]
where $\mathcal{L} = (x_1^2, \ldots, x_5^2, x_1x_2, x_1x_3, x_1x_4)$.

Proof. One can easily see that $I$ contains all cubic monomials divisible by $x_1$ since $x_1x_i \in I$ for all $i = 2, 3, 4, 5$ and $f_4$ is a quadratic form in $K[x_2, x_3, x_4, x_5]$, therefore $x_1f_4 \in I$ and so is $x_1^3$. Thus, the Hilbert functions of $R/I$ and $k[x_2, x_3, x_4, x_5]/I \cap K[x_2, x_3, x_4, x_5]$ agree in degree 3. So $\Hilb_{R/1}(3) = \Hilb_{K[x_2, x_3, x_4, x_5]/I \cap K[x_2, x_3, x_4, x_5]}(3) = \Hilb_{K[x_2, x_3, x_4, x_5]/(f_1, f_2, f_3, f_4)}(3) = 4$. □

Case 2. Suppose that $f_5 = x_1^2$ by altering the variables and generators, and then we can assume that $g_1 = x_1x_5, g_2 = x_2x_5, g_3 = x_3x_5$. As we did in the case above, we get rid of all the terms containing $x_5$ except $x_4x_5$ in the $f_i$, and so the defect three quadratic ideal can be written as follows:
\[
I = (f_1, f_2, f_3, f_4 + cx_4x_5, x_1^2, x_1x_5, x_2x_5, x_3x_5),
\]
where $f_1, f_2, f_3, f_4$ form a regular sequence in $K[x_1, x_2, x_3, x_4]$ and $c \in K$.

Lemma 4.15. Let $a = (f_1, f_2, f_3, f_4 + x_4x_5, x_1^2) : (x_1x_5, x_2x_5, x_3x_5)$ be the colon ideal in $R$. Then we have $\Hilb_{R/a}(2) = 6$.  

Proof. It suffices to show \( \dim_K a_2 = 9 \).

We know that \( x_1x_5, x_2x_5, x_3x_5, x_4x_5, x_5^2 \) are all in \( a_2 \), and \( f_1, f_2, f_3, f_4 \in a_2 \) as well. Thus we see that \( \dim_K a_2 \geq 9 \).

If there is another independent quadratic form in \( a \), it must be in \( R[x_5] \), as we have all quadratic monomials containing \( x_5 \), so call it \( Q \) in \( R[x_5] \). Then we consider the cubic form \( H = x_5Q \). Clearly \( H \) is not in the \( R_1 \)-span of \( f_1, f_2, f_3, f_4, x_5^2 \), therefore we can define the ideal \( J = (f_1, f_2, f_3, f_4, x_5^2, H) \), which is an almost complete intersection in \( R \). Then we get \( \dim_K ((f_1, f_2, f_3, f_4, x_5^2) \cap HR_1) \geq 4 \) as \( x_1H, x_2H, x_3H \) and \( x_5H \) are in \( (f_1, f_2, f_3, f_4, x_5^2) \), but by Corollary 2.4 this dimension must be at most 3. This proves that there cannot be such a quadratic form \( Q \) in \( a \).

\[ \square \]

Proposition 4.16. Let \( I = (f_1, f_2, f_3, f_4 + x_4x_5, x_5^2, x_1x_5, x_2x_5, x_3x_5) \) be a defect three quadratic ideal in \( R \) where \( f_1, f_2, f_3, f_4 \) is an \( R[x_5] \)-sequence. Then

\[ \text{Hilb}_{R/I}(3) = 4 = \text{Hilb}_{R/L}(3) \]

where \( L = (x_1^2, \ldots, x_5^2, x_1x_2, x_1x_3, x_1x_4) \).

Proof. Using the duality of Gorenstein algebras, again we can obtain

\[ \text{Hilb}_{R/I}(3) = \text{Hilb}_{R/(f_1, f_2, f_3, f_4 + x_4x_5, x_5^2)}(3) - \text{Hilb}_{R/\mathfrak{a}}(5 - 3), \]

where \( \mathfrak{a} \) is the colon ideal \((f_1, f_2, f_3, f_4 + x_4x_5, x_5^2) : I\).

Then proof is done, since \( \text{Hilb}_{R/(f_1, f_2, f_3, f_4 + x_4x_5, x_5^2)}(3) = 10 \) and \( \text{Hilb}_{R/\mathfrak{a}}(2) = 6 \) by the above lemma. \( \square \)

References

Department of Mathematics, University of Michigan, 530 Church Street East Hall, MI 48109
Current address: Department of Mathematics, University of Connecticut, 341 Mansfield Road, CT, 06269
E-mail address: gunturku@umich.edu

Department of Mathematics, University of Michigan, 530 Church Street East Hall, MI 48109
E-mail address: hochster@umich.edu