# THE EISENBUD-GREEN-HARRIS CONJECTURE FOR DEFECT TWO QUADRATIC IDEALS

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Abstract. The Eisenbud-Green-Harris (EGH) conjecture states that a homogeneous ideal in a polynomial ring $K[x_1, \ldots, x_n]$ over a field $K$ that contains a regular sequence $f_1, \ldots, f_n$ with degrees $a_i$, $i = 1, \ldots, n$ has the same Hilbert function as a lex-plus-powers ideal containing the powers $x_i^{a_i}$, $i = 1, \ldots, n$. In this paper, we discuss a case of the EGH conjecture for homogeneous ideals generated by $n+2$ quadrics containing a regular sequence $f_1, \ldots, f_n$ and give a complete proof for EGH when $n = 5$ and $a_1 = \cdots = a_5 = 2$.

## 1. Introduction

Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over a field $K$ with the homogeneous lexicographic order in which $x_1 > \cdots > x_n$ and with the standard grading $R = \bigoplus_{i \geq 0} R_i$. We denote the Hilbert function of a $\mathbb{Z}$-graded $R$-module $M$ by $\mathrm{Hilb}_M(i) := \dim_K M_i$, where $M_i$ is the homogeneous component of $M$ in degree $i$. When $I$ is a homogeneous ideal of $R$ and $M$ is $R$, or $I$, or $R/I$, the Hilbert function has value 0 when $i < 0$. When the Hilbert function of $M$ is 0 in negative degree, we may discuss the Hilbert function of $M$ by giving the sequence of its values, and we refer to this sequence of integers as the *O-sequence* of $M$.

In 1927, Macaulay [13] showed that the Hilbert function of any homogeneous ideal of $R$ is attained by a lexicographic ideal in $R$. Later, in Kruskal-Katona's theorem [11, 12], it is shown that the polynomial ring $R$ in Macaulay's result can be replaced with the quotient $R/(x_1^2, \ldots, x_n^2)$. After this result, Clement and Lindström, in [5], generalized the result to $R/(x_1^{a_1}, \ldots, x_n^{a_n})$ if $a_1 \leq \cdots \leq a_n < \infty$.

In [7] Eisenbud, Green and Harris conjectured a generalization of the Clement-Lindström result. Let $\underline{\mathbf{a}} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, where $2 \leq a_1 \leq \ldots \leq a_n$.

**Conjecture 1.1** (Eisenbud-Green-Harris ($\mathrm{EGH}_{\underline{\mathbf{a}},n}$) Conjecture [7]). *If $I$ is a homogeneous ideal in $R = K[x_1, \ldots, x_n]$ containing a regular sequence $f_1, f_2, \ldots, f_n$ with degrees $\deg f_i = a_i$, then there is a monomial ideal $\mathcal{L} = (x_1^{a_1}, \ldots, x_n^{a_n}) + J$, where $J$ is a lexicographic ideal in $R$, such that $R/\mathcal{L}$ and $R/I$ have the same Hilbert function.*

Although there has been some progress on the conjecture, it remains open. The conjecture is shown to be true for $n = 2$ by Richert in [14]. Francisco [8] shows the conjecture for almost complete intersections. Caviglia and Maclagan in [2] prove the result if $a_i > \sum_{j=1}^{i-1}(a_j - 1)$ for $2 \leq i \leq n$. The rapid growth required for the degrees does not yield much insight into cases like the one in which the regular sequence consists of quadratic forms. When $n = 3$, Cooper in [6] proves the EGH conjecture for the cases where $(a_1, a_2, a_3) = (2, a_2, a_3)$ and $(a_1, a_2, a_3) = (3, a_2, a_3)$ with $a_2 \leq a_3 \leq a_2 + 1$.

One of the most intriguing cases is when $a_1 = \cdots = a_n = 2$ for any $n \geq 2$, which is the case for which Eisenbud, Green and Harris originally stated their conjecture. It is known that the conjecture holds for homogeneous ideals minimally generated by generic quadrics: the case where $\mathrm{char}\, K = 0$ was proved by Herzog and Popescu [10] and the case of arbitrary characteristic

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was proved by Gasharov [9] around the same time. There have been several other results on the EGH conjecture. More recently, the case when every $f_i$, $i = 1, \ldots, n$, in the regular sequence is a product of linear forms is settled by Abedelfatah in [1], and results on the EGH conjecture using linkage theory are given by Chong [4].

In this paper we focus on the case when the degrees of the elements of the regular sequence are $a_1 = \cdots = a_n = 2$. In [14], Richert claimed that the conjecture for quadratic regular sequences is true for $2 \le n \le 5$, but this work has not been published, and other researchers have been unable to verify this for $n = 5$ thus far. Chen, in [3], has given a proof for the case where $n \le 4$ when $a_1 = \cdots = a_n = 2$.

In §2 we recall some definitions and results from the papers of Francisco [8], Caviglia-Maclagan [2] and Chen [3]. In §3 we study homogeneous ideals $I$ generated by $n+2$ quadratic forms in $n$ variables containing a regular sequence of length $n$, and Theorem 3.17 shows that there is a monomial ideal $\mathcal{L} = (x_1^2, \ldots, x_n^2) + J$, where $J$ is a lexicographic ideal in $R$, such that $R/I$ and $R/\mathcal{L}$ have the same Hilbert function in degree 2 and 3 (i.e., $\mathrm{EGH}_{(2,\ldots,2),n}(2)$ holds: see Definition 2.5). In §4 we give a proof to the claim of Richert for the quadratic regular sequence case when $n = 5$.

## 2. Background and Preliminaries

In this section we recall some definitions and state some known results that are used throughout the paper.

**Definition 2.1.** Let $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$ be monomials in $R$ of the same degree. We say that $u$ is greater than $v$ with respect to the *lexicographic* (or *lex*) order if there exists an $i$ such that $a_i > b_i$ and $a_j = b_j$ for all $j < i$.

A monomial ideal $J \subseteq R$ is called a *lexicographic ideal* (or *lex ideal*) if, for all degrees $d$, the $d$-th degree component of $J$, denoted by $J_d$, is spanned over the base field $K$ by an initial segment of the degree $d$ monomials in the lexicographic order.

**Definition 2.2.** Given $2 \le a_1 \le \cdots \le a_n$, a *lex-plus-powers ideal* (LPP ideal) $\mathcal{L}$ is a monomial ideal in $R$ that can be written as $\mathcal{L} = (x_1^{a_1}, \ldots, x_n^{a_n}) + J$ where $J$ is a lex ideal in $R$.

This definition agrees with the one in [2]. Some authors require that the $x_i^{a_i}$ be minimal generators of $\mathcal{L}$, which we do not. However, since we consider only nondegenerate homogeneous ideals in this paper, i.e., ideals contained in $(x_1, \ldots, x_n)^2$, in the case where $a_1 = \cdots = a_n = 2$ it is automatic that the $x_i^2$ are minimal generators of the ideal under consideration.

In [8] Francisco showed the following for almost complete intersections.

**Theorem 2.3** (Francisco [8])**.** *Let integers $2 \le a_1 \le a_2 \le \cdots \le a_n$ and $d \ge a_1$ be given. Let the ideal $I$ have minimal generators $f_1, \ldots, f_n, g$ where $f_1, \ldots, f_n$ form a regular sequence with $\deg f_i = a_i$ and $g$ has degree $d$. Let $\mathcal{L} = (x_1^{a_1}, \ldots, x_n^{a_n}, m)$ be the lex-plus-powers ideal where $m$ is the greatest monomial in lex order in degree $d$ that is not in $(x_1^{a_1}, \ldots, x_n^{a_n})$. Then $\mathrm{Hilb}_{R/I}(d+1) \le \mathrm{Hilb}_{R/\mathcal{L}}(d+1)$.*

Note that, necessarily, $d \le \sum_{i=1}^n (a_i - 1)$, since $(f_1, \ldots, f_n)$ contains all forms of degree larger than that. If $a_1 = \cdots = a_n = 2$, then $d \le n$.

The following corollary is an immediate consequence of Theorem 2.3 above. If $g \in R$ is a nonzero form of degree $i$ we write $gR_j$ for the vector space $\{gh : h \in R_j\} \subseteq R_{i+j}$.

**Corollary 2.4.** *Let $I = (f_1, \ldots, f_n, g)$ be an almost complete intersection as in Theorem 2.3 above such that $a_1 = \cdots = a_n = 2$. Then*

$$\dim_K \big((f_1, \ldots, f_n)_{d+1} \cap gR_1\big) \le d.$$

*Proof.* We can write

$$\dim_K I_{d+1} = \dim_K (f_1, \ldots, f_n)_{d+1} + \dim_K gR_1 - \dim_K \big((f_1, \ldots, f_n)_{d+1} \cap gR_1\big),$$

where $\dim_K gR_1 = n$. Then by Theorem 2.3, we have

$$\dim_K I_{d+1} \geq \dim_K(x_1^2, \ldots, x_n^2, x_1 \cdots x_d)_{d+1} = \dim_K(x_1^2, \ldots, x_n^2)_{d+1} + n - d$$

Since $\mathrm{Hilb}_{R/(f_1, \ldots, f_n)}(i) = \mathrm{Hilb}_{R/(x_1^2, \ldots, x_n^2)}(i)$ for all $i \geq 0$, we can conclude that

$$\dim_K \big((f_1, \ldots, f_n)_{d+1} \cap gR_1\big) \leq d.$$

□

The next statement is a weaker version of the $\mathrm{EGH}_{\underline{\mathbf{a}},n}$ conjecture. It focuses on the Hilbert function of the given homogeneous ideal only at the two consecutive degrees $d$ and $d+1$ for some non-negative integer $d$.

**Definition 2.5** ($\mathrm{EGH}_{\underline{\mathbf{a}},\mathbf{n}}(\mathbf{d})$)**.** Following Caviglia-Maclagan [2], we say that "$\mathrm{EGH}_{\underline{\mathbf{a}},n}(d)$ holds" if for any homogeneous ideal $I \in K[x_1, \ldots, x_n]$ containing a regular sequence of degrees $\underline{\mathbf{a}} = (a_1, \ldots, a_n)$, where $2 \leq a_1 \leq \cdots \leq a_n$, there exists a lex-plus-powers ideal $\mathcal{L}$ containing $\{x_i^{a_i} : 1 \leq i \leq n\}$ such that

$$\dim_K I_d = \dim_K \mathcal{L}_d \quad \text{and} \quad \dim_K I_{d+1} = \dim_K \mathcal{L}_{d+1}\,.$$

**Lemma 2.6.** *The condition $\mathrm{EGH}_{(d,\ldots,d),n}(d)$ on a polynomial ring $R = K[x_1, \ldots, x_n]$ is equivalent to the statement that for the ideal $I$ generated by $n + \delta$ $K$-linearly independent forms of degree $d$ containing a regular sequence, one has that $\dim_K I_{d+1} \geq \dim_K \mathcal{L}_{d+1}$, where $\mathcal{L} = (x_1^d, \ldots, x_n^d) + J'$ and $J'$ is minimally generated by the greatest in lex order $\delta$ forms of degree $d$ not already in $(x_1^d, \ldots, x_n^d)$.*

*Proof.* If there is an LPP ideal $(x_1^d, \ldots, x_n^d) + J$, where $J$ is a lex ideal, with the same Hilbert function as $I$ in degrees $d$ and $d+1$, it is clear that $J_d$ must be spanned over $K$ by the specified generators of $J'$, so that $(x_1^d, \ldots, x_n^d) + J' \subseteq (x_1^d, \ldots, x_n^d) + J$, which implies the specified inequality on the Hilbert functions. Moreover, when that inequality holds we may increase $\mathcal{L} := (x_1^d, \ldots, x_n^d) + J'$ to an LPP ideal with the same Hilbert function as $I$ in degrees $d$ and $d+1$: if $\Delta = \mathrm{Hilb}_I(d+1) - \mathrm{Hilb}_{\mathcal{L}}(d+1)$, we may simply include the greatest (in lex order) $\Delta$ forms of degree $d+1$ not already in $\mathcal{L}$. □

**Remark 2.7.** We shall eventually be focused on $\mathrm{EGH}_{\underline{\mathbf{a}},n}(d)$ in the case where $a_1 = \cdots = a_n = d = 2$, simply referred as $\mathrm{EGH}_{(2,\ldots,2),n}(2)$ or $\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$. We shall routinely make use of this lemma in this case of quadratic regular sequence and $d = 2$.

**Lemma 2.8** (Caviglia-Maclagan [2])**.** *Fix $\underline{\mathbf{a}} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ where $2 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and set $s = \sum_{i=1}^{n}(a_i - 1)$. Then for any $0 \leq d \leq s-1$, $\mathrm{EGH}_{\underline{\mathbf{a}},n}(d)$ holds if and only if $\mathrm{EGH}_{\underline{\mathbf{a}},n}$(s-1-d) holds.*
*Furthermore, the $\mathrm{EGH}_{\underline{\mathbf{a}},n}$ conjecture holds if and only if $\mathrm{EGH}_{\underline{\mathbf{a}},n}(d)$ holds for all degrees $d \geq 0$.*

From now on, we always assume $\underline{\mathbf{a}} = \underline{\mathbf{2}} = (2, \ldots, 2)$ for $n \geq 2$, unless it is stated otherwise.

**Remark 2.9.** For any $n \geq 2$, $\mathrm{EGH}_{\underline{\mathbf{2}},n}(0)$ holds trivially. In [3, Proposition 2.1], Chen showed that $\mathrm{EGH}_{\underline{\mathbf{2}},n}(1)$ is true for any $n \geq 2$.

Chen proved the following.

**Theorem 2.10** (Chen [3])**.** *The $\mathrm{EGH}_{\underline{\mathbf{2}},n}$ conjecture holds when $2 \leq n \leq 4$.*

Chen's proof of this uses Lemma 2.8 above, and the observation that, when $n = 4$, to demonstrate that the $\mathrm{EGH}_{\underline{\mathbf{2}},4}$ conjecture is true, it suffices to show that $\mathrm{EGH}_{\underline{\mathbf{2}},4}(0)$ and $\mathrm{EGH}_{\underline{\mathbf{2}},4}(1)$ are true.

## 3. $\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$ FOR DEFECT TWO IDEALS

In this section, we focus on the homogeneous ideals in $K[x_1, \ldots, x_n]$ for $n \geq 5$ that are generated by $n+2$ quadratic forms containing a regular sequence. In particular, we study their Hilbert functions in degree 3.

**Definition 3.1.** If $I$ is a homogeneous ideal minimally generated by $n+\delta$ forms that contain a regular sequence of length $n$, then $I$ is said to be a *defect* $\delta$ ideal.

Clearly, when $\delta = 0$ then $I$ is generated by a regular sequence, it is a complete intersection, and we understand the Hilbert function completely. If $\delta = 1$, then $I$ is an almost complete intersection.

**Definition 3.2.** We call a homogeneous ideal a *quadratic ideal* if it is generated by quadratic forms.

Let $I = (f_1, \ldots, f_n, g, h)$ be a homogeneous ideal minimally generated by $n+2$ quadrics where $f_1, \ldots, f_n$ form a regular sequence. We call such an ideal a *defect two ideal generated by quadrics* or simply a *defect two quadratic ideal.* More generally, if a quadratic ideal is a defect $\delta$ ideal, then we call it *defect* $\delta$ quadratic ideal.

**Example 3.3.** The lex-plus-powers ideal $\mathcal{L} = (x_1^2, \ldots, x_n^2, x_1x_2, x_1x_3)$ in $R$ is also a defect two quadratic ideal.

Further, for any homogeneous defect two quadratic ideal $I$, we have the equality

$$\dim_K I_2 = n+2 = \dim_K \mathcal{L}_2\,.$$

**Main Question 3.4** ($\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$ for defect two quadratic ideals)**.** For any $n \geq 5$, is it true that

$$\dim_K I_3 \geq n^2 + 2n - 5 = \dim_K \mathcal{L}_3?$$

An affirmative answer for this question is proved completely in Theorem 3.17 below.

**Notation 3.5.** Throughout the rest of this paper we write $\mathfrak{f}$ for the ideal $(f_1, \ldots, f_n)R$ when $f_1, \ldots, f_n$ is a regular sequence of quadratic forms, and in the defect $\delta$ quadratic ideal case we write $\mathfrak{g}$ for the additional generators $g_1, \ldots, g_\delta$ of the quadratic ideal. Here, $f_1, \ldots, f_n, g_1, \ldots, g_\delta$ are assumed to be linearly independent over $K$. Moreover, henceforth, we write $J$ for the ideal $\mathfrak{f} + (g_1, \ldots, g_{\delta-1})$. However, when $\delta = 1$ or 2 we may write $g, h$ for $g_1, g_2$, so that whenever $\delta = 2$ we henceforth write $J$ for the ideal $\mathfrak{f} + (g_1) = \mathfrak{f} + (g)$. We denote the graded Gorenstein Artin $K$-algebra $R/\mathfrak{f}$ by $A$.

We know that, if $a_1 = \cdots = a_n = \deg g = 2$, Theorem 2.3 shows that

$$\dim_K J_3 \geq n^2 + n - 2$$

and then Corollary 2.4 gives $\dim_K \big(\mathfrak{f}_3 \cap gR_1\big) \leq 2$.

**Remark 3.6.** In [3, Proposition 3.7] Chen gave a positive answer to the Question 3.4 for defect two quadratic ideals $I = \mathfrak{f} + (g, h)$ if $\dim_K \big(\mathfrak{f}_3 \cap gR_1\big) = 2$. We shall make repeated use of this fact in the sequel.

In this section we show $\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$ for a defect two quadratic ideal $I = \mathfrak{f} + (g, h)$ under the condition that $\dim_K \big(\mathfrak{f}_3 \cap g'R_1\big) \leq 1$ for all $g' \in Kg + Kh - \{0\}$: this covers all the cases for which Chen's result in Proposition 3.6 is not applicable.

**Lemma 3.7.** *As in Notation 3.5, $J$ is the defect 1 quadratic ideal $\mathfrak{f} + gR$. Then:*

$$\dim_K I_3 = n^2 + 2n - \dim_K \big(\mathfrak{f}_3 \cap gR_1\big) - \dim_K \big(J_3 \cap hR_1\big).$$

*Consequently, for the cases that are not covered by the Proposition 3.6 we have:*

**(i)** *If* $\dim_K(\mathfrak{f}_3 \cap gR_1) = 1$ *then* $\dim_K I_3 = n^2 + 2n - 1 - \dim_K(J_3 \cap hR_1)$*, and* $\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$ *holds for a defect two quadratic ideal* $I$ *if and only if* $\dim_K(J_3 \cap hR_1) \leq 4$*.*

**(ii)** *If* $\dim_K(\mathfrak{f}_3 \cap gR_1) = 0$ *then* $\dim_K I_3 = n^2 + 2n - \dim_K(J_3 \cap hR_1)$*, and* $\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$ *holds for* $I$ *if and only if* $\dim_K(J_3 \cap hR_1) \leq 5$*.*

*Proof.* We have:

$$\begin{aligned}\dim_K I_3 &= \dim_K J_3 + \dim_K(hR_1) - \dim_K(J_3 \cap hR_1)\\ &= \dim_K \mathfrak{f}_3 + \dim_K(gR_1) - \dim_K(\mathfrak{f}_3 \cap gR_1) + \dim_K(hR_1) - \dim_K(J_3 \cap hR_1)\\ &= n^2 + 2n - \dim_K(\mathfrak{f}_3 \cap gR_1) - \dim_K(J_3 \cap hR_1),\end{aligned}$$

and then (i) and (ii) are immediate. □

**Remark 3.8.** Let $n = 5$, so that $\mathfrak{f} = (f_1, \ldots, f_5)$. For a defect two quadratic ideal $I = (\mathfrak{f}, g, h) \subseteq K[x_1, \ldots, x_5]$, if $\dim_K(\mathfrak{f}_3 \cap gR_1) = 0$ then clearly $\dim_K((\mathfrak{f}, g)_3 \cap hR_1) \leq \dim_K(hR_1) \leq 5$, therefore $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ holds for such an ideal $I$. However, we must give an argument to cover all possible cases, that is, when $\dim_K(\mathfrak{f}_3 \cap gR_1) = 1$, to be able to confirm $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ for every defect two quadratic ideal. In the last section, we discuss the EGH conjecture for $n = 5$ and $a_1 = \cdots = a_5 = 2$ in detail.

Next, we proceed with two useful lemmas.

**Lemma 3.9.** *Let* $A$ *be the graded Gorenstein Artin* $K$*-algebra* $R/\mathfrak{f}$ *with* $\dim_K A_1 = n$*. Let* $g, h$ *be two quadratic forms such that* $gA_1 = hA_1$*. Then* $\mathrm{Ann}_{A_1} g = \mathrm{Ann}_{A_1} h$*.*

*Moreover,* $\mathrm{Ann}_{A_i}(g) = \mathrm{Ann}_{A_i}(h)$ *if* $i \neq n - 2$*.*

*Proof.* Suppose that the linear annihilator space of $g$, $\mathrm{Ann}_{A_1} g$, has dimension $a$ and $gA_1 = hA_1$. Thus $gA_1$ has dimension $n - a$ and clearly $hA_1$ and $\mathrm{Ann}_{A_1} h$ have dimensions $n - a$ and $a$, respectively.

Notice that $gA(-2) \cong A/\mathrm{Ann}_A(g)$, hence it is Gorenstein and it has a symmetric O-sequence

$$(0, 0, 1, n - a, e_4, e_5, \ldots, e_5, e_4, n - a, 1),$$

where $e_i$ denotes the dimension of $[gA]_i$ and $e_i = e_{n-i+2}$ for $2 \leq i \leq n$. Then the Hilbert function of $A/gA$ is

$$(1, n, \binom{n}{2} - 1, \binom{n}{3} - n + a, \binom{n}{4} - e_4, \ldots, \binom{n}{3} - e_5, \binom{n}{2} - e_4, a, 0).$$

Since $\mathrm{Ann}_A(g) \cong \mathrm{Hom}_K(A/gA, A) \cong (A/gA)^\vee$, the Hilbert function of $\mathrm{Ann}_A(g)$ is

$$(0, a, \binom{n}{2} - e_4, \ldots, \binom{n}{4} - e_4, \binom{n}{3} - n + a, \binom{n}{2} - 1, n, 1).$$

Recall that $gA_1 = hA_1$, $gA_i = hA_i$ for all $i \geq 2$, so $(g, h)A$ has the Hilbert function

$$(0, 0, 2, \underbrace{n - a, e_4, \ldots, e_4, n - a, 1}_{\text{the same as for } gA}).$$

Then the O-sequence of $A/(g, h)$ becomes

$$(1, n, \binom{n}{2} - 2, \binom{n}{3} - n + a, \binom{n}{4} - e_4, \ldots, \binom{n}{3} - e_5, \binom{n}{2} - e_4, a, 0),$$

and it follows that $\mathrm{Ann}_A(g, h)$ has the Hilbert function

$$(0, a, \binom{n}{2} - e_4, \ldots, \binom{n}{4} - e_4, \binom{n}{3} - n + a, \binom{n}{2} - 2, n, 1).$$

We know that $\mathrm{Ann}_A(g, h) = \mathrm{Ann}_A(g) \cap \mathrm{Ann}_A(h)$, and in degree 1, $\mathrm{Ann}_A(g, h)$ has dimension $a$, so $\mathrm{Ann}_A(g, h) = \mathrm{Ann}_{A_1}(g) = \mathrm{Ann}_{A_1}(h)$. Further, $\mathrm{Ann}_A(g)$ and $\mathrm{Ann}_A(h)$ are the same in every degrees except in degree $n - 2$. □

**Lemma 3.10.** *Let $g, h$ be two quadratic forms in a graded Gorenstein Artin $K$-algebra $A$ such that $gA_i = hA_i$ and $g, h$ have the same annihilator space $V$ in $A_i$ for some $i \geq 1$. Then there exists $g' \in Kg + Kh - \{0\}$ such that*

$$\dim_K \operatorname{Ann}_{A_i}(g') \geq \dim_K V + 1.$$

*Proof.* Consider the multiplication maps by $g$ and $h$,

$$\phi_g : A_i/V \to gA_i \quad \text{and} \quad \phi_h : A_i/V \to hA_i$$

whose images $gA_i$, $hA_i$ are subspaces in $A_{i+2}$ and $gA_i = hA_i$ by assumption. Then there is a automorphism

$$T : A_i/V \to A_i/V$$

such that $g\ell = hT(\ell)$ for any $\ell \in A_i/V$. However, $T$ has at least one nonzero eigenvector $u$ with $T(u) = cu$ for some $c \in K$. Say $\ell_u$ be a form in degree $i$ represented by this eigenvector $u$ in $A_i$ and not in the annihilator space $V$, thus $g\ell_u = hc\ell_u$. Then there is a quadratic form $g' := g - ch \in Kg + Kh - \{0\}$ such that $g'$ annihilated by the space $V$ and also by $\ell_u \in A_i \setminus V$. Hence $\dim_K \operatorname{Ann}_{A_i}(g') \geq \dim_K V + 1$. $\square$

From now on, $I = (f_1, \ldots, f_n, g, h) = \mathfrak{f} + (g, h)$ is a homogeneous ideal where $\dim_K \big(\mathfrak{f}_3 \cap g'R_1\big) \neq 2$ for a quadratic form $g' \in Kg + Kh - \{0\}$, which means that $\dim_K g'A_1 \neq n - 2$. Therefore $\dim_K g'A_1$ is either $n$ or $n - 1$.

**Proposition 3.11.** *For the graded Gorenstein Artin $K$-algebra $A$, if $gA_1 = hA_1$ with $\dim_K gA_1 = n - 1 = \dim_K hA_1$, that is $\dim_K(\mathfrak{f}_3 \cap gR_1) = \dim_K(\mathfrak{f}_3 \cap hR_1) = 1$, then $\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$ holds for the homogeneous defect two quadratic ideal $I = \mathfrak{f} + (g, h)$.*

*Proof.* Since $\dim_K \operatorname{Ann}_{A_1}(g) = \dim_K \operatorname{Ann}_{A_1}(h) = 1$ there is some $g' \in Kg + Kh - \{0\}$ with $\dim_K \operatorname{Ann}_{A_i}(g') = 2$ by Lemma 3.10. In consequence, $\dim_K \big(\mathfrak{f}_3 \cap g'R_1\big) = 2$, and so we are done by Proposition 3.6. $\square$

**Proposition 3.12.** *For the graded Gorenstein Artin $K$-algebra $A$, if $\dim_K gA_1 = \dim_K hA_1 = n$, then there exists a quadratic form $g'$ in $Kg + Kh$ with a nonzero linear annihilator in $A$.*

*Proof.* By assumption $\dim_K A_1 = \dim_K gA_1 = \dim_K hA_1 = n$, and so we may consider again the multiplication maps $\phi_g : A_1 \to gA_1$ and $\phi_h : A_1 \to hA_1$. Then we obtain a automorphism $T : A_1 \to A_1$ and there exists an nonzero linear form $\ell \in A_1$ such that $T(\ell) = c\ell$ for some $c \in K$, that is $g\ell = ch\ell$. Consider $g' = g - ch \in Kg + Kh$. Clearly, $\ell \in \operatorname{Ann}_{A_1}(g')$. $\square$

Next we assume that there is a linear annihilator $L \in A_1$ of $g$ where $Lh \neq 0$ over the Gorenstein ring $A = R/\mathfrak{f}$. This case may come up either when $\dim_K gA_1 = \dim_K hA_1 = n - 1$ and the linear annihilator spaces $\operatorname{Ann}_{A_1}(g)$ and $\operatorname{Ann}_{A_1}(h)$ are distinct, or when $\dim_K gA_1 = n - 1$ and $\dim_K hA_1 = n$.

We shall make repeated use of the following result, which is Lemma 3.3 of Chen's paper [3].

**Lemma 3.13** (Chen [3])**.** *If $f_1, \ldots, f_n$ is a regular sequence of 2-forms in $R$ and we have a relation $u_1 f_1 + u_2 f_2 + \cdots + u_n f_n = 0$ for some $t$-forms $u_1, \ldots, u_n$, then $u_1, \ldots, u_n \in (f_1, \ldots, f_n)_t$. More precisely, we have that $t \geq 2$ and there exists a skew-symmetric $n \times n$ matrix $B$ of $(t-2)$-forms such that $(u_1\, u_2 \cdots u_n) = (f_1\, f_2 \cdots f_n)B$.* $\square$

**Proposition 3.14.** *Let $I = \mathfrak{f} + \mathfrak{g}$ be a defect $\delta$, where $2 \leq \delta \leq n - 1$, quadratic ideal of $R$ as in Notation 3.5. If there is a linear form $L$ in $\operatorname{Ann}_A(g_1, \ldots, g_{\delta-1})$ such that $Lg_\delta \neq 0$ in $A$, then*

$$\dim_K \big((f_1, \ldots, f_n, g_1, \ldots, g_{\delta-1})_3 \cap g_\delta R_1\big) \leq 3$$

Chen [3] used an argument involving the Koszul relations on $(x_1, \ldots, x_r)$ for $r \leq n$ while introducing another proof for Theorem 2.3. In the proof of this proposition we use a very similar argument.

*Proof.* As in Notation 3.5, let $J = \mathfrak{f} + (g_1, \ldots, g_{\delta-1})$, and denote the row vector of the regular sequence $f_1, \ldots, f_n$ by $\vec{\mathbf{f}}$ and the row vector of quadratic forms $g_1, \ldots, g_{\delta-1}$ by $\vec{\mathbf{g}}$.

Suppose $\dim_K(J_3 \cap g_\delta R_1) \geq 4$, and without loss of generality we may assume that

$$\begin{aligned} x_1 g_\delta &= \vec{\mathbf{g}} \cdot \vec{\ell_1} + \vec{\mathbf{f}} \cdot \vec{p_1} \\ x_2 g_\delta &= \vec{\mathbf{g}} \cdot \vec{\ell_2} + \vec{\mathbf{f}} \cdot \vec{p_2} \\ x_3 g_\delta &= \vec{\mathbf{g}} \cdot \vec{\ell_3} + \vec{\mathbf{f}} \cdot \vec{p_3} \\ x_4 g_\delta &= \vec{\mathbf{g}} \cdot \vec{\ell_4} + \vec{\mathbf{f}} \cdot \vec{p_4} \end{aligned}$$

where $\vec{\ell_i}$ and $\vec{p_i}$ are column vectors of linear forms of lengths $\delta - 1$ and $n$, respectively.

We assume that there is a linear form $L$ such that $Lg_i = 0$ for each $i = 1, \ldots, \delta - 1$ but $Lg_\delta \neq 0$ in $A$. Then we get am $n \times (\delta - 1)$ matrix $(q_{i,j}) = \begin{pmatrix} \vec{q_1} & \vec{q_2} & \cdots & \vec{q}_{\delta-1} \end{pmatrix}$ of linear forms such that

$$L\vec{\mathbf{g}} = \vec{\mathbf{f}} \cdot (q_{i,j}).$$

We observe that each $x_i L g_\delta$ is in $\mathfrak{f}$, and write $x_i L g_\delta = \vec{\mathbf{f}} \cdot \vec{Q_i}$ where $\vec{Q_i}$ is a column of quadratic forms for $i = 1, 2, 3, 4$. Therefore:

$$Lg_\delta \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} = \vec{\mathbf{f}} \cdot \begin{pmatrix} \vec{Q_1} & \vec{Q_2} & \vec{Q_2} & \vec{Q_4} \end{pmatrix}. \tag{1}$$

Let $M_1 = \begin{pmatrix} x_2 & x_3 & x_4 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & x_3 & x_4 & 0 \\ 0 & -x_1 & 0 & -x_2 & 0 & x_4 \\ 0 & 0 & -x_1 & 0 & -x_2 & -x_3 \end{pmatrix}$. Note that $\begin{pmatrix} x_1 & x_2 & \cdots & x_4 \end{pmatrix} \cdot M_1 = 0$. Multiplying the equation (1) by $M_1$ from right gives that $\vec{\mathbf{f}} \cdot (\vec{Q_1}\ \vec{Q_2}\ vecQ_3\ \vec{Q_4}) = 0$, and so all entries are 0 in

$$\vec{\mathbf{f}} \begin{pmatrix} x_2\vec{Q_1} - x_1\vec{Q_2} & x_3\vec{Q_1} - x_1\vec{Q_3} & x_4\vec{Q_1} - x_1\vec{Q_4} & x_3\vec{Q_2} - x_2\vec{Q_3} & x_4\vec{Q_2} - x_2\vec{Q_4} & x_4\vec{Q_3} - x_3\vec{Q_4} \end{pmatrix}$$

By Lemma 3.13, there are alternating $n \times n$ matrices $B_{12}, B_{13}, B_{14}, B_{23}, B_{24}, B_{34}$ of <u>linear</u> forms such that

$$\begin{pmatrix} \underbrace{x_2\vec{Q_1} - x_1\vec{Q_2}}_{\substack{\text{a column vector} \\ \text{of cubic forms}}} & \cdots & x_4\vec{Q_3} - x_3\vec{Q_4} \end{pmatrix} = \begin{pmatrix} B_{12}\vec{\mathbf{f}}^T & \cdots & B_{34}\vec{\mathbf{f}}^T \end{pmatrix} \tag{2}$$

Similarly, consider the matrix $M_2 = \begin{pmatrix} x_3 & x_4 & 0 & 0 \\ -x_2 & 0 & x_4 & 0 \\ 0 & -x_2 & -x_3 & 0 \\ x_1 & 0 & 0 & x_4 \\ 0 & x_1 & 0 & -x_3 \\ 0 & 0 & x_1 & x_2 \end{pmatrix}$ such that $M_1 \cdot M_2 = \mathbf{0}$ and multiply equation (2) by $M_2$ from right to obtain:

$$\begin{pmatrix} \underbrace{(x_3B_{12} - x_2B_{13} + x_1B_{23})}_{\substack{n \times n\,\text{matrix of} \\ \text{quadratic forms}}}\vec{\mathbf{f}}^T & \cdots & (x_4B_{23} - x_3B_{24} + x_2B_{34})\vec{\mathbf{f}}^T \end{pmatrix} = \mathbf{0}.$$

Then again by Lemma 3.13, there are alternating $n \times n$ matrices

$$C_1^{123}, \ldots, C_n^{123}, C_1^{124}, \ldots, C_n^{124}, \ldots, C_1^{234}, \ldots, C_n^{234}$$

of <u>scalars</u> such that

$$
(3) \qquad
\begin{aligned}
x_3B_{12} - x_2B_{13} + x_1B_{23} &= \begin{pmatrix} \vec{\mathbf{f}}C_1^{123} \\ \vdots \\ \vec{\mathbf{f}}C_n^{123} \end{pmatrix} \\
x_4B_{12} - x_2B_{14} + x_1B_{24} &= \begin{pmatrix} \vec{\mathbf{f}}C_1^{124} \\ \vdots \\ \vec{\mathbf{f}}C_n^{124} \end{pmatrix} \\
x_4B_{13} - x_3B_{14} + x_1B_{34} &= \begin{pmatrix} \vec{\mathbf{f}}C_1^{134} \\ \vdots \\ \vec{\mathbf{f}}C_n^{134} \end{pmatrix} \\
x_4B_{23} - x_3B_{24} + x_2B_{34} &= \begin{pmatrix} \vec{\mathbf{f}}C_1^{234} \\ \vdots \\ \vec{\mathbf{f}}C_n^{234} \end{pmatrix}
\end{aligned}
$$

Repeating the previous steps with $M_3 = \begin{pmatrix} x_4 \\ -x_3 \\ x_2 \\ -x_1 \end{pmatrix}$, so that $M_2 \cdot M_3 = \mathbf{0}$, we get

$$
\mathbf{0} = \begin{pmatrix} B_{12} & B_{13} & B_{14} & B_{23} & B_{24} & B_{34} \end{pmatrix} M_2 M_3 = \begin{pmatrix} \vec{\mathbf{f}}C_1^{123} & \vec{\mathbf{f}}C_1^{124} & \vec{\mathbf{f}}C_1^{134} & \vec{\mathbf{f}}C_1^{234} \\ \vdots & \vdots & \vdots & \vdots \\ \vec{\mathbf{f}}C_n^{123} & \vec{\mathbf{f}}C_n^{124} & \vec{\mathbf{f}}C_n^{134} & \vec{\mathbf{f}}C_n^{234} \end{pmatrix} M_3
$$

and then for all $i = 1, 2, \ldots, n$ we obtain

$$
\vec{\mathbf{f}}(x_4C_i^{123} - x_3C_i^{124} + x_2C_i^{134} - x_1C_i^{234}) = 0.
$$

Then, finally, $x_4C_i^{123} - x_3C_i^{124} + x_2C_i^{134} - x_1C_i^{234} = 0$ for all $i = 1, 2, ..., n$. Hence,

$$
C_i^{123} = C_i^{124} = C_i^{134} = C_i^{234} = 0 \quad \text{for all} \quad i = 1, 2, ..., n.
$$

Thus, in (3) we get $x_3B_{12} - x_2B_{13} + x_1B_{23} = 0$. This shows that $x_3$ divides every entry in $x_2B_{13} - x_1B_{23}$. Therefore we may rewrite $B_{13} = x_3\widetilde{B_{13}} + D_{13}$ and $B_{23} = x_3\widetilde{B_{23}} + D_{23}$, where $\widetilde{B_{13}}$ and $\widetilde{B_{23}}$ are alternating matrices of scalars, $D_{13}$ and $D_{23}$ are alternating matrices of linear forms that do not contain $x_3$, and $x_2D_{13} - x_1D_{23} = 0$. We obtain the following

$$
B_{12} = \frac{1}{x_3}(x_2B_{13} - x_1B_{23}) = x_2\widetilde{B_{13}} - x_1\widetilde{B_{23}}
$$

Returning to equation (2), we obtain $x_2\vec{Q_1} - x_1\vec{Q_2} = B_{12}\vec{\mathbf{f}}^T = (x_2\widetilde{B_{13}} - x_1\widetilde{B_{23}})\vec{\mathbf{f}}^T$. Consequently,

$$
x_1\big(\vec{Q_2} - \widetilde{B_{23}}\vec{\mathbf{f}}^T\big) = x_2\big(\vec{Q_1} - \widetilde{B_{13}}\vec{\mathbf{f}}^T\big)
$$

which tells us that $x_1$ divides every entry of $\vec{Q_1} - \widetilde{B_{13}}\vec{\mathbf{f}}^T$. It follows that

$$
\begin{aligned}
\vec{\mathbf{f}}\big(\vec{Q_1} - \widetilde{B_{13}}\vec{\mathbf{f}}^T\big) &= \vec{\mathbf{f}}\vec{Q_1} && \text{as } \widetilde{B_{13}} \text{ is alternating and } \vec{\mathbf{f}}\widetilde{B_{13}}\vec{\mathbf{f}}^T = 0 \\
&= x_1Lg_\delta && \text{by equation (1).}
\end{aligned}
$$

This shows that $Lg_\delta = \vec{\mathbf{f}}\frac{1}{x_1}\Big(\vec{Q_1} - \widetilde{B_{13}}\vec{\mathbf{f}}^T\Big) \in (f_1, \ldots, f_n)_3$, which contradicts our assumption $L \notin \mathrm{Ann}_A(g_\delta)$. □

**Corollary 3.15.** *Let $I = \mathfrak{f} + \mathfrak{g} \subseteq R$ be a defect $\delta$ quadratic ideal with $2 \le \delta \le n-1$. Suppose that*

$$(\dagger) \qquad \operatorname{Ann}_{A_1}(g_1, \ldots, g_{\delta-1}, g_\delta) \setminus \operatorname{Ann}_{A_1}(g_\delta) \neq \emptyset.$$

*Then*

$$\dim_K I_3 \ge \dim_K \mathcal{L}_3$$

*where $\mathcal{L} = (x_1^2, \ldots, x_n^2) + (x_1x_2, x_1x_3, \ldots, x_1x_{\delta+1})$ is the defect $\delta$ lex-plus-powers ideal of $R$. That is, $\mathrm{EGH}_{\mathbf{2},n}(2)$ holds for any defect $\delta$ quadratic ideal with property $(\dagger)$.*

*Proof.* Notice that $\dim_K \mathcal{L}_3 = n^2 + n\delta - \frac{\delta(\delta+3)}{2}$. We use induction on $\delta$. Let $J = \mathfrak{f} + (g_1, \ldots, g_{\delta-1})$ be the defect $\delta - 1$ quadratic ideal.

$$\begin{aligned}
\dim_K I_3 &= \dim_K J_3 + n - \dim_K \left(J_3 \cap g_\delta R_1\right) \\
&\ge \left(n^2 + (\delta-1)n - \frac{(\delta-1)(\delta+2)}{2}\right) + n - 3 = n^2 + n\delta - \frac{\delta(\delta+3)}{2} + \delta - 3 \\
&\ge n^2 + n\delta - \frac{(\delta)(\delta+3)}{2}.
\end{aligned}$$

∎

We notice that a special case of Corollary 3.15 when $\delta = 2$ shows that the inequality is strict.

**Corollary 3.16.** *Let $I = \mathfrak{f} + (g, h)$ be a defect two ideal generated by quadrics in $R$. If $\operatorname{Ann}_{A_1}(g) = \operatorname{Span}\{L\}$ for some $L \in R_1$ and $L$ does not annihilate $h$ in $A = R/\mathfrak{f}$, then*

$$\dim_K I_3 \ge n^2 + 2n - 4 > \dim_K (x_1^2, \ldots, x_n^2, x_1x_2, x_1x_3)_3$$

*Proof.* The result follows from Proposition 3.14 as

$$\dim_K I_3 = n^2 + 2n - \underbrace{\dim_K(\mathfrak{f}_3 \cap gR_1)}_{=\dim_K \operatorname{Ann}_{A_1}(g)=1} - \underbrace{\dim_K(J_3 \cap hR_1)}_{\le 3}$$

which is $\ge n^2 + 2n - 4$.

∎

Finally, we give an affirmative answer to the Main Question 3.4.

**Theorem 3.17.** *Let $I = \mathfrak{f} + (g, h) \subseteq R = K[x_1, \ldots, x_n]$ for $n \ge 5$ be a defect two ideal quadratic ideal. Then*

$$\dim_K I_3 \ge n^2 + 2n - 5.$$

*More precisely, $\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$ holds for homogeneous defect two quadratic ideals in $R$ for any $n \ge 5$.*

*Proof.* If the given defect two ideal satisfies Proposition 3.6 , then, by Chen's result, the theorem is proved.

Assume that $\dim_K \left(\mathfrak{f}_3 \cap g'R_1\right) \neq 2$ for any $g' \in Kg + Kh \setminus \{0\}$. If $\dim_K \left(\mathfrak{f}_3 \cap gR_1\right) = \dim_K \left(\mathfrak{f}_3 \cap hR_1\right) = 0$, by Proposition 3.12, we can always find another quadratic form $g' \in Kg + Kh \setminus \{0\}$ so that $g'$ has a linear annihilator in $A$. Then we can apply Corollary 3.16. If $\dim_K \left(\mathfrak{f}_3 \cap gR_1\right) = \dim_K \left(\mathfrak{f}_3 \cap hR_1\right) = 1$ and the same linear form annihilates both $g$ and $h$ in $A$, by Proposition 3.11. we have a situation contradicts our assumption. ∎

**Corollary 3.18.** $\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$ *holds for every defect two ideal containing a regular sequence of quadratic forms.*

*Proof.* This result follows from Lemma 2.6 and Theorem 3.17. ∎

## 4. The EGH conjecture when $n = 5$ and $a_1 = \cdots = a_5 = 2$

In this section $R = K[x_1, \dots, x_5]$ and $I = (f_1, \dots, f_5) + (g_1, \dots, g_\delta) = \mathfrak{f} + \mathfrak{g}$ is a homogeneous defect $\delta$ ideal in $R$, where $f_1, \dots, f_5$ is a regular sequence of quadrics and $\deg g_j \geq 2$ for $j = 1, \dots, \delta$. Throughout, we shall write $A := R/\mathfrak{f}$, which is a graded Gorenstein local Artin ring. We will show the existence of a lex-plus-powers ideal $\mathcal{L} \subseteq R$ containing $x_i^2$ for $i = 1, \dots, 5$ with the same Hilbert function as $I$ by proving the following main theorem.

**Theorem 4.1.** *The EGH conjecture holds for all homogeneous ideals containing a regular sequence of quadrics in $K[x_1, \dots, x_5]$.*

Lemma 2.8 of Caviglia-Maclagan tells us that $\mathrm{EGH}_{\underline{\mathbf{2}},5}(d)$ holds if and only if $\mathrm{EGH}_{\underline{\mathbf{2}},5}(5-d-1)$ holds. Thus it will be enough to show $\mathrm{EGH}_{\underline{\mathbf{2}},5}(d)$ when $d = 0, 1, 2$. By Remark 2.9 we know that $\mathrm{EGH}_{\underline{\mathbf{2}},5}(d)$ is true when $d = 0, 1$, therefore $\mathrm{EGH}_{\underline{\mathbf{2}},5}(3)$ and $\mathrm{EGH}_{\underline{\mathbf{2}},5}(4)$ both hold as well.

Our goal in this section is to prove $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ for any homogeneous ideal containing a regular sequence of quadrics: this will complete the proof of $\mathrm{EGH}_{\underline{\mathbf{2}},5}$. To achieve this, it suffices to understand $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ for quadratic ideals with arbitrary defect $\delta$ (but, of course, $\delta \leq 10$, since $\dim_K R_2 = 15$), by Lemma 2.6.

**Remark 4.2.** As a result of Corollary 3.18, we see that $\mathrm{EGH}_{\underline{\mathbf{2}},n}$ holds for any defect $\delta = 2$ quadratic ideal in $K[x_1, \dots, x_n]$ for $n = 5$.

To accomplish our goal we will prove $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ for defect $\delta \geq 3$ quadratic ideals. In the next subsection, we prove that if one knows the case where $\delta = 3$, on obtains all the cases for $\delta \geq 4$. In the final subsection we finish the proof by establishing $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ for $\delta = 3$.

### Quadratic ideals with defect $\delta \geq 4$.

**Lemma 4.3.** *If $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ holds for all defect three quadratic ideals, then it holds for all quadratic ideals with defect $\delta \geq 4$.*

*Proof.* Let $I = (f_1, \dots, f_5, g_1, g_2, g_3, g_4) = \mathfrak{f} + \mathfrak{g} \subseteq R$ be a defect 4 homogeneous ideal generated by quadrics, where $f_1, \dots, f_5$ form a regular sequence. By assumption the defect three quadratic ideal $J = \mathfrak{f} + (g_1, g_2, g_3) \subseteq I$ satisfies $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$, that is, $\dim_K J_3 \geq 31$.

Let $\mathcal{L} = (x_1^2, \dots, x_5^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5)$ be the LPP ideal with $\dim_K \mathcal{L}_2 = \dim_K I_2 = 9$. Then we get $\dim_K I_3 \geq \dim_K J_3 \geq 31 = \dim_K \mathcal{L}_3$, as we need for the case of defect $\delta = 4$.

Now assume $5 \leq \delta \leq 10$. Let ${}^{\delta}I$ denote an arbitrary defect $\delta$ quadratic ideal, and let ${}^{\delta}\mathcal{L}$ denote the lex-plus-power ideal with defect $\delta \geq 5$. More precisely, ${}^{\delta}\mathcal{L} := (x_1^2, \dots, x_5^2) + (m_1, \dots, m_\delta)$ where $m_i$ are the next greatest quadratic square-free monomials with respect to lexicographic order. We need to show that $\mathrm{Hilb}_{R/{}^{\delta}I}(3) \leq \mathrm{Hilb}_{R/{}^{\delta}\mathcal{L}}(3)$.

We assume that $\mathrm{Hilb}_{R/{}^{\delta}I}(3) \geq \mathrm{Hilb}_{R/{}^{\delta}\mathcal{L}}(3) + 1$, and we shall obtain a contradiction.

Using duality for Gorenstein rings, we know that for $0 \leq d \leq 5$ we have that

$$\mathrm{Hilb}_{R/{}^{\delta}I}(d) = \mathrm{Hilb}_{R/\mathfrak{f}}(d) - \mathrm{Hilb}_{R/(\mathfrak{f}:{}^{\delta}I)}(5-d).$$

Then, for $d = 3$, using the assumption we get

$$\begin{aligned} \mathrm{Hilb}_{R/(\mathfrak{f}:{}^{\delta}I)}(2) &= \mathrm{Hilb}_{R/\mathfrak{f}}(3) - \mathrm{Hilb}_{R/{}^{\delta}I}(3) \\ &\leq 9 - \mathrm{Hilb}_{R/{}^{\delta}\mathcal{L}}(3) = \begin{cases} 7 & \text{if } \delta = 5, \\ 8 & \text{if } \delta = 6, 7, \\ 9 & \text{if } \delta = 8, 9, 10. \end{cases} \end{aligned}$$

We next show that $\dim_K(\mathfrak{f} : {}^{\delta}I)_1 = 0$. If there is a nonzero linear form $\ell \in \mathfrak{f} : {}^{\delta}I$ then $\dim_K \mathrm{Ann}_{A_2} \ell A \geq \delta \geq 5$, so we get that $\dim_K A_3/\ell A_2 \geq 5$. On the other hand, we see that $A_3/\ell A_2 \cong [R/(\bar{f}_1, \dots, \bar{f}_4, \bar{f}_5, l)]_3$ where the $\bar{f}_i$ are the images of the $f_i$, and the dimension of $[R/(\bar{f}_1, \dots, \bar{f}_4, \bar{f}_5, l)]_3$ as a $K$-vector space is at most 4.

Then we can find a defect $\gamma$ quadratic ideal ${}^{\gamma}J \subseteq \mathfrak{f} : {}^{\delta}I$ for $\gamma = 3, 2, 1$ if the defect of ${}^{\delta}I$ is $\delta = 5$ or $\delta = 6, 7$ or $\delta = 8, 9, 10$, respectively. We then have the inequalities shown below, where the first is obvious and the second follows by comparison with Hilbert functions of quotients by LPP ideals in degree 3 and the fact that, by assumption, $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ holds for quadratic ideals with defect less than or equal to three.

$$\mathrm{Hilb}_{R/(\mathfrak{f}:{}^{\delta}I)}(3) \quad \le \quad \mathrm{Hilb}_{R/{}^{\gamma}J}(3) \le \begin{cases} 4 \ \text{ if } {}^{\gamma}J \text{ is a defect } \gamma = 3 \text{ quadratic ideal when } \delta = 5, \\ 5 \ \text{ if } {}^{\gamma}J \text{ is a defect } \gamma = 2 \text{ quadratic ideal when } 6 \le \delta \le 7, \\ 7 \ \text{ if } {}^{\gamma}J \text{ is a defect } \gamma = 1 \text{ quadratic ideal when } 8 \le \delta \le 10. \end{cases}$$

However, each of the cases above contradicts the following equality:

$$\mathrm{Hilb}_{R/(\mathfrak{f}:{}^{\delta}I)}(3) = \mathrm{Hilb}_{R/\mathfrak{f}}(2) - \mathrm{Hilb}_{R/{}^{\delta}I}(2) = \delta.$$

Thus, we get $\mathrm{Hilb}_{R/{}^{\delta}I}(3) \le \mathrm{Hilb}_{R/{}^{\delta}\mathcal{L}}(3)$ for any defect $\delta \ge 5$ quadratic ideal ${}^{\delta}I$ in $R$. $\square$

**Defect three quadratic ideals.**

**Lemma 4.4.** *Let $I = \mathfrak{f} + (g_1, g_2, g_3)$ be a defect three quadratic ideal in the polynomial ring $R$. Then, for any $1 \le i_1 < i_2 \le 3$,*

$$\dim_K(\mathfrak{f} : (g_{i_1}, g_{i_2}))_1 \le 1,$$

*and, furthermore,* $\dim_K(\mathfrak{f} : (g_1, g_2, g_3))_1 \le 1$.

*Proof.* Suppose that $\dim_K(\mathfrak{f} : (g_1, g_2))_1 \ge 2$, and assume there are $\ell_1, \ell_2 \in R_1$ such that $\ell_i g_1, \ell_i g_2 \in \mathfrak{f}$ for both $i = 1, 2$. Without loss of generality we assume that $\ell_1 = x_1$ and $\ell_2 = x_2$.

Therefore, we can write $(x_1, x_2, f_1, \ldots, f_5) \subseteq \mathfrak{f} : I$. Then

$$\begin{aligned} 2 = \mathrm{Hilb}_{(f_1, \ldots, f_5, g_1, g_2)/\mathfrak{f}}(2) &= \mathrm{Hilb}_{R/\left(\mathfrak{f}:(f_1, \ldots, f_5, g_1, g_2)\right)}(5-2), \ \text{(by duality)} \\ &\le \mathrm{Hilb}_{R/(x_1, x_2, f_1, \ldots, f_5)}(3) \\ &= \mathrm{Hilb}_{K[x_3, x_4, x_5]/(\bar{f}_1, \ldots, \bar{f}_5)}(3), \ \text{(where } \bar{f}_i \text{ is the image of } f_i \text{ in } K[x_3, x_4, x_5],) \\ &\le \binom{5-2}{3} = 1, \end{aligned}$$

which is a contradiction. $\square$

Hence, working in the graded Gorenstein Artin $K$-algebra $A = R/\mathfrak{f}$, we have from the lemma just above that $\mathrm{Ann}_{A_1}(g_1, g_2)$ is a $K$-vector space of dimension at most one, and, therefore

$$\dim_K \mathrm{Ann}_{A_1}(g_1, g_2, g_3) \le 1$$

since $\mathrm{Ann}_{A_1}(g_1, g_2, g_3) \subseteq \mathrm{Ann}_{A_1}(g_1, g_2)$.

**Remark 4.5.** By Remark 4.2 we know that for any defect two quadratic ideal $J$ in $R$, $\dim_K J_3$ is at least 30. Then $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ holds for the defect three quadratic ideals $I$ containing a defect two quadratic ideal $J$ with $\dim_K J_3 \ge 31$, as $\mathrm{Hilb}_{R/I}(3) \le \mathrm{Hilb}_{R/J} \le 4$.

We henceforth focus on defect three quadratic ideals $I = \mathfrak{f} + (g_1, g_2, g_3)$ in $R$ such that every defect two quadratic ideal $J \subseteq I$ containing $\mathfrak{f}$ has $\dim_K J_3 = 30$.

For such defect three quadratic ideals, we observe the following.

**Remark 4.6.** Consider the ideal $\mathcal{I} = (g_1, g_2, g_3)A$ in the Gorenstein ring $A$ such that any ideal $(g_{i_1}, g_{i_2})A$ contained in $\mathcal{I}$ has degree three component of dimension $\dim_K (g_{i_1}, g_{i_2})A_1 = 5$. Assuming that $\dim_K \mathrm{Ann}_{A_1}(g_1) = 1$, we have that $\mathrm{Ann}_{A_1}(g_1, g_2, g_3) = \mathrm{Ann}_{A_1}(g_1)$.

Furthermore, if $g_1 A_1$ is 5-dimensional, that is, there is no linear form that annihilates $g_1$ in $A$, then for any quadric $g$ in $Kg_1 + Kg_2 + Kg_3$ the vector space $gA_1 \subseteq A_3$ is either 3 or 5 dimensional.

*Proof.* Let $\dim_K \operatorname{Ann}_{A_1}(g_1) = 1$, and let the linear form $L$ annihilate $g_1$ but not some form $g' \in Kg_2 + Kg_3$ in $A$. We define a defect two quadratic ideal

$$J = (f_1, \ldots, f_5, g_1, g') \subseteq \mathfrak{f} + (g_1, g_2, g_3)$$

in $R$. Hence, by Corollary 3.16, we know already that $\dim_K J_3 \geq 31$, which means that $\dim_K (g_1, g')A_1 = 6$. This contradicts our assumption. Thus, $L$ must be in $\operatorname{Ann}_{A_1}(g_1, g_2, g_3)$. $\square$

Recall that the following holds, by Proposition 3.14, when $\delta = 3$.

**Proposition 4.7.** *Let $I = \mathfrak{f} + (g_1, g_2, g_3) \subseteq K[x_1, ..., x_5]$ be a defect 3 quadratic ideal.*
*As usual, let $A = R/\mathfrak{f}$. If there is a linear form $L \in \operatorname{Ann}_A(g_1, g_2)$ such that $L \notin \operatorname{Ann}_A(g_3)$, then*

$$\dim_K \big((\mathfrak{f} + (g_1, g_2))_3 \cap g_3 R_1\big) \leq 3.$$

$\square$

When a defect three quadratic ideal $I$ satisfies the condition of the above proposition, we notice a sharp bound for $\mathrm{Hilb}_{R/I}(3)$.

**Corollary 4.8.** *Given a defect three quadratic ideal $I = \mathfrak{f} + (g_1, g_2, g_3)$ in $R = K[x_1, \ldots, x_5]$, and, as usual, let $A = R/\mathfrak{f}$, which is a graded Gorenstein Artin ring. If $\dim_K \operatorname{Ann}_{A_1}(g_1, g_2) = 1$ and $\operatorname{Ann}_{A_1}(g_1, g_2, g_3) = 0$ then*

$$\dim_K I_3 \geq 32 > \dim_K \mathcal{L}_3,$$

*where $\mathcal{L} = (x_1^2, \ldots, x_5^2, x_1x_2, x_1x_3, x_1x_4)$.*

*Proof.* By assumption there is a linear form in $\operatorname{Ann}_A(g_1, g_2)$, say $L$, such that $L$ does not annihilate $g_3$. Hence, Proposition 4.7 gives us $\dim_K \big((\mathfrak{f} + (g_1, g_2))_3 \cap g_3 R_1\big) \leq 3$. Then we get

$$\begin{aligned} \dim_K(\mathfrak{f} + (g_1, g_2, g_3))_3 &= \dim_K(\mathfrak{f} + (g_1, g_2))_3 + \dim_K g_3 R_1 \\ &\quad - \dim_K \big((\mathfrak{f} + (g_1, g_2))_3 \cap g_3 R_1\big) \\ &\geq 30 + 5 - 3 = 32 > 31 = \dim_K \mathcal{L}_3\,. \end{aligned}$$

$\square$

**Proposition 4.9.** *Suppose that for all quadratic forms $g$ in $Kg_1 + Kg_2$, the subspace $gA_1$ of $A_3$ is a 3-dimensional. If $\dim_K (g_1, g_2)A_1 = 5$, then $\dim_K \operatorname{Ann}_{A_1}(g_1, g_2) = 1$.*

We first state the following observation in a linear algebra setting, which will be useful for the proof Proposition 4.9.

**Lemma 4.10.** *Let $S$, $T$ be linear transformations from $V$ to $W$, both $n$-dimensional vector spaces over $K$, such that $\operatorname{rank}(S) = \operatorname{rank}(T) = \operatorname{rank}(S - T) = r$, and the kernels of $S$, $T$ are disjoint. Then the images of $S$ and $T$ are contained in the same $(3r - n)$-dimensional subspace of $W$.*

*Proof.* $V_0 = \ker(S - T)$ is $(n - r)$-dimensional. $S$ and $T$ are injective on $V_0$, since for $v \in V_0$, $S(v) = 0$ iff $T(v) = 0$, and $\operatorname{Ker}(S) \cap \operatorname{Ker}(T) = 0$. Thus, $S(V_0) = T(V_0)$ is an $(n - r)$-dimensional space in $S(V) \cap T(V)$. Since $S(V), T(V)$ are $r$-dimensional and overlap in a space of dimension at least $n - r$, $S(V) + T(V)$ has dimension at most $r + r - (n - r) = 3r - n$. $\square$

*Proof of Proposition 4.9.* Assume that $\dim_K \operatorname{Ann}_{A_1}(g_1, g_2) = 0$. Since all quadratic forms $g$ in $Kg_1 + Kg_2$ are such that $gA_1 \subseteq A_3$ has vector space dimension 3, we have from Lemma 4.10 with $n = 5, r = 3$, that $(Kg_1 + Kg_2)A_1 \subseteq A_3$ is at most 4-dimensional. Consequently,

$$\dim_K [A/(g_1, g_2)A]_3 = \dim_K [R/\mathfrak{f} + (g_1, g_2)]_3 \geq 6,$$

contradicting $\mathrm{EGH}_{\underline{2},5}(2)$ for defect 2 quadratic ideals. Hence, $\dim_K \operatorname{Ann}_{A_1}(g_1, g_2) = 1$. $\square$

**Proposition 4.11.** *Let $I = \mathfrak{f} + (g_1, g_2, g_3)$ be a defect three quadratic ideal in $R = K[x_1, \ldots, x_5]$. If $\dim_K \operatorname{Ann}_{A_1}(g_1, g_2, g_3) = 0$ then $\mathrm{Hilb}_{R/I}(3) \leq 4$.*

*Proof.* First, by Remark 4.5 we note that it suffices to consider any defect two quadratic ideal $J \subseteq I$ with $\mathrm{Hilb}_{R/J}(3) = 5$.

Suppose that $\dim_K \mathrm{Ann}_{A_1}(g_1, g_2, g_3) = 0$. Then, clearly, no $g_i$, for $i = 1, 2, 3$ has a 1-dimensional linear annihilator space in $A$, since, otherwise, by Remark 4.6, we obtain that $\dim_K \mathrm{Ann}_{A_1}(g_1, g_2, g_3) = 1$, which contradicts our assumption. Thus, for the rest of the proof we may assume that each $g_i A_1$, $i = 1, 2, 3$, is either 3 or 5 dimensional.

If all forms $g$ in $Kg_1 + Kg_2 + Kg_3$ are such that $\dim_K gA_1 = 3$ then we can find two independent quadratic forms whose linear annihilator spaces intersect in 1-dimensional space, and the result follows from Corollary 4.8.

Let $g_1 A_1$ be a 5-dimensional subspace of $A_3$ and for every $g \in Kg_2 + Kg_3$, $gA_1$ has dimension either 3 or 5.

We complete the proof by obtaining a contradiction. We assume that $\mathrm{Hilb}_{R/I}(3) = 5$. In other words, the space $W = (Kg_1 + Kg_2 + Kg_3)A_1 \subseteq A_3$ is 5-dimensional. Then we get $W = g_1 A_1 = (Kg_2 + Kg_3)A_1$.

Consider the multiplication maps by $g_1, g_2$ and $g_3$ from $A_1$ to the subspace $W$ of $A_3$. By adjusting the bases of $A_1$ and $W$ we can assume the matrix of $g_1$ is the identity matrix $\mathbb{I}_5$ of size 5. Denote the matrices of $g_2$ and $g_3$ by $\alpha$ and $\beta$, respectively. We can assume that $\alpha$ and $\beta$ are both singular, and so have rank 3, by subtracting the suitable multiples of $\mathbb{I}_5$ from them if they are not singular.

We see that all matrices $z\mathbb{I}_5 + x\alpha + y\beta$ must have at most two eigenvalues, otherwise we can form a linear combination whose kernel is 1-dimensional, which corresponds to a quadratic form with 1-dimensional linear annihilator space. Then there are two main cases: one is that every matrix in the space spanned by $\mathbb{I}_5, \alpha$ and $\beta$ has one eigenvalue. The other is that almost all matrices in the form $z\mathbb{I}_5 + x\alpha + y\beta$ have two eigenvalues, since the subset with at most one eigenvalue is Zariski closed.

Define $D(x, y, z) = \det(z\mathbb{I}_5 - x\alpha - y\beta)$, a homogeneous polynomial in $x, y, z$ of degree 5 that is monic in $z$. Note that $D$ is also the characteristic polynomial, in $z$, of $x\alpha + y\beta$. Notice that the singular matrices in the subspace of $5 \times 5$ matrices spanned by $\mathbb{I}, \alpha$ and $\beta$ are defined by the vanishing of $D$.

If the determinant $D$ is square-free (as the characteristic polynomial in $z$), then the ideal $(D)$ is a radical ideal and it cannot contain a nonzero polynomial of degree less than 5, which contradicts the fact that all size 4 minors of a singular matrix must vanish, since in our situation these singular matrices have rank 3. Therefore the size 4 minors, whose degrees are at most 4, are in the radical $(D)$.

If the determinant $D$ is not square-free, then its squared factor must be linear or quadratic: in the latter case the other factor is linear, so that in either case $D$ has a linear factor, say $z - ax - by$.

Consider the independent matrices $\alpha' = a\mathbb{I}_5 - \alpha$, $\beta' = b\mathbb{I}_5 - \beta$. Then we think of any linear combination of them, say $r\alpha' + s\beta' = r(a\mathbb{I}_5 - \alpha) + s(b\mathbb{I}_5 - \beta) = (ar + bs)\mathbb{I}_5 - r\alpha - s\beta$. As $z - ax - by$ is a factor of $D(x, y, z)$, and hence, $D$ vanishes for $x = r, y = s, z = ar + bs$. This means that every linear combination of $\alpha'$ and $\beta'$ is singular. Therefore, we can replace $\alpha, \beta$ by $\alpha'$ and $\beta'$ and so we can assume that we are in the case where every linear combination of the two non-identity matrices is singular, and, if not 0, of rank 3. By Lemma 4.10, this implies that the kernels of $\alpha'$ and $\beta'$ cannot be disjoint, so we are done by Proposition 4.9 and Corollary 4.8. $\square$

In order to prove $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ for every defect three quadratic ideal $I = \mathfrak{f} + (g_1, g_2, g_3)$ in $R = K[x_1, \ldots, x_5]$ we must also discuss the cases when there is a nonzero linear form $L \in \mathrm{Ann}_A(g_1, g_2, g_3)$.

**Proposition 4.12.** *Let $I = \mathfrak{f} + (g_1, g_2, g_3)$ be a defect three quadratic ideal in $R$. If $\mathrm{Ann}_{A_1}(g_1, g_2, g_3)$ is a 1-dimensional $K$-subspace of $A_1$, say $KL$, then*

$$\mathrm{Hilb}_{R/I}(3) = 4.$$

**Claim 4.13.** *One of the quadratic forms $f_i$ in the regular sequence has the linear factor $L$.*

*Proof of claim.* As $g_1, g_2, g_3 \in \operatorname{Ann}_{A_2}(L) \subseteq A_2$ for $L \in \operatorname{Ann}_{A_1}(g_1, g_2, g_3)$ we know that

$$\dim_K \operatorname{Ann}_{A_2}(L) \geq 3.$$

This tells us that $\dim_K LA_2 \leq 7$, which implies

$$\dim_K(A_3/LA_2) = \dim_K[A/LA]_3 \geq 3 \tag{4}$$

as $\dim_K A_3 = 10$.

Assume that $L = x_5$ and let $\bar{f}_i$ be the image of $f_i$ modulo $x_5$ .

Suppose that $\bar{\mathfrak{f}} = (\bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4, \bar{f}_5)$ is an almost complete intersection in $K[x_1, x_2, x_3, x_4]$. Thus,

$$A/LA \cong \frac{K[x_1, \ldots, x_5]}{\mathfrak{f} + (x_5)} \cong \frac{K[x_1, x_2, x_3, x_4]}{\bar{\mathfrak{f}}}.$$

However, using the Francisco's result for almost complete intersections [8], we know that

$$\dim_K \Big[\frac{K[x_1, x_2, x_3, x_4]}{\bar{\mathfrak{f}}}\Big]_3 \leq 2 = \dim_K \Big[\frac{K[x_1, x_2, x_3, x_4]}{(x_1^2,\, x_2^2,\, x_3^2,\, x_4^2,\, x_1x_2)}\Big]_3.$$

This contradicts (4).

Hence the images of $f_i$ modulo $L$ form a regular sequence in $K[x_1, \ldots, x_4]$, that is, one of them has a linear factor $x_5$. □

As a result of the claim, after a suitable change of variables, we may assume that the linear annihilator is $L = x_5$ and may consider $I$ in two possible forms: either $I$ is in the form of (5) in *Case 1* below, where $f_1,\, f_2,\, f_3,\, f_4,\, x_1x_5$ is the regular sequence, or $I$ is as in (6) in *Case 2* below, where $f_1,\, f_2,\, f_3,\, f_4,\, x_5^2$ form a quadratic regular sequence in $I$.

*Case 1.* Suppose that $f_5 = x_1x_5$. Then we can assume that $g_1 = x_1x_2$, $g_2 = x_1x_3$, $g_3 = x_1x_4$. Furthermore, after we alter the $f_i$ by getting rid of all the terms containing $x_1$ except $x_1^2$, we may assume that the defect three quadratic ideal $I$ looks like

$$I = (f_1,\, f_2,\, f_3,\, f_4 + cx_1^2,\, x_1x_5,\, x_1x_2,\, x_1x_3,\, x_1x_4), \tag{5}$$

where $f_1, f_2, f_3, f_4$ form a regular sequence in $K[x_2, x_3, x_4, x_5]$ and $c \in K$.

**Proposition 4.14.** *Let $I = (f_1,\, f_2,\, f_3,\, f_4 + cx_1^2,\, x_1x_5,\, x_1x_2,\, x_1x_3,\, x_1x_4)$ be a defect three quadratic ideal in $R$ where $f_1, f_2, f_3, f_4$ is an $K[x_2, x_3, x_4, x_5]$-sequence. Then*

$$\operatorname{Hilb}_{R/I}(3) = 4 = \operatorname{Hilb}_{R/\mathcal{L}}(3)$$

*where $\mathcal{L} = (x_1^2, \ldots, x_5^2, x_1x_2, x_1x_3, x_1x_4)$.*

*Proof.* One can easily see that $I$ contains all cubic monomials divisible by $x_1$ since $x_1x_i \in I$ for all $i = 2, 3, 4, 5$ and $f_4$ is a quadratic form in $K[x_2, x_3, x_4, x_5]$, therefore $x_1f_4 \in I$ and so is $x_1^3$. Thus, the Hilbert functions of $R/I$ and $k[x_2, x_3, x_4, x_5]/I \cap K[x_2, x_3, x_4, x_5]$ agrees in degree 3. So $\operatorname{Hilb}_{R/I}(3) = \operatorname{Hilb}_{K[x_2,x_3,x_4,x_5]/I\cap K[x_2,x_3,x_4,x_5]}(3) = \operatorname{Hilb}_{K[x_2,x_3,x_4,x_5]/(f_1,\, f_2,\, f_3,\, f_4)}(3) = 4$ □

*Case 2.* Suppose that $f_5 = x_5^2$ by altering the variables and generators, and then we can assume that $g_1 = x_1x_5$, $g_2 = x_2x_5$, $g_3 = x_3x_5$. As we did in the case above, we get rid of all the terms containing $x_5$ except $x_4x_5$ in the $f_i$, and so the defect three quadratic ideal can be written as follows:

$$I = (f_1,\, f_2,\, f_3,\, f_4 + cx_4x_5,\, x_5^2,\, x_1x_5,\, x_2x_5,\, x_3x_5), \tag{6}$$

where $f_1,\, f_2,\, f_3,\, f_4$ form a regular sequence in $K[x_1, x_2, x_3, x_4]$ and $c \in K$.

**Lemma 4.15.** *Let $\mathfrak{a} = (f_1,\, f_2,\, f_3,\, f_4 + x_4x_5,\, x_5^2) : (x_1x_5,\, x_2x_5,\, x_3x_5)$ be the colon ideal in $R$. Then we have $\operatorname{Hilb}_{R/\mathfrak{a}}(2) = 6$.*

*Proof.* It suffices to show $\dim_K \mathfrak{a}_2 = 9$.

We know that $x_1x_5,\ x_2x_5,\ x_3x_5,\ x_4x_5,\ x_5^2$ are all in $\mathfrak{a}_2$, and $f_1,\ f_2,\ f_3,\ f_4 \in \mathfrak{a}_2$ as well. Thus we see that $\dim_K \mathfrak{a}_2 \geq 9$.

If there is another independent quadratic form in $\mathfrak{a}$, it must be in $R[\check{x}_5]$, as we have all quadratic monomials containing $x_5$, so call it $Q$ in $R[\check{x}_5]$. Then we consider the cubic form $H = x_5Q$. Clearly $H$ is not in the $R_1$-span of $f_1,\ f_2,\ f_3,\ f_4,\ x_5^2$, therefore we can define the ideal $J = (f_1,\ f_2,\ f_3,\ f_4,\ x_5^2,\ H)$, which is an almost complete intersection in $R$. Then we get $\dim_K \left((f_1,\ f_2,\ f_3,\ f_4, x_5^2)_4 \cap HR_1\right) \geq 4$ as $x_1H, x_2H, x_3H$ and $x_5H$ are in $(f_1,\ f_2,\ f_3,\ f_4, x_5^2)_4$, but by Corollary 2.4 this dimension must be at most 3. This proves that there cannot be such a quadratic form $Q$ in $\mathfrak{a}$. □

**Proposition 4.16.** *Let $I = (f_1,\ f_2,\ f_3,\ f_4 + x_4x_5,\ x_5^2,\ x_1x_5,\ x_2x_5,\ x_3x_5)$ be a defect three quadratic ideal in $R$ where $f_1, f_2, f_3, f_4$ is an $R[\check{x}_5]$-sequence. Then*

$$\mathrm{Hilb}_{R/I}(3) = 4 = \mathrm{Hilb}_{R/\mathcal{L}}(3)$$

*where $\mathcal{L} = (x_1^2,\ \ldots,\ x_5^2,\ x_1x_2,\ x_1x_3,\ x_1x_4)$.*

*Proof.* Using the duality of Gorenstein algebras, again we can obtain

$$\mathrm{Hilb}_{R/I}(3) = \mathrm{Hilb}_{R/(f_1,\, f_2,\, f_3,\, f_4 + x_4x_5,\, x_5^2)}(3) - \mathrm{Hilb}_{R/\mathfrak{a}}(5-3),$$

where $\mathfrak{a}$ is the colon ideal $(f_1,\ f_2,\ f_3,\ f_4 + x_4x_5,\ x_5^2) : I$.

Then proof is done, since $\mathrm{Hilb}_{R/(f_1,\, f_2,\, f_3,\, f_4 + x_4x_5,\, x_5^2)}(3) = 10$ and $\mathrm{Hilb}_{R/\mathfrak{a}}(2) = 6$ by the above lemma. □

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