HOMOLOGICAL CONJECTURES, OLD AND NEW

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ABSTRACT. We discuss a number of the local homological conjectures, many of which are now theorems in equal characteristic and conjectures in mixed characteristic. One focus is the syzygy theorem of Evans and Griffith, and its connection with direct summand conjecture, the existence of big Cohen-Macaulay modules and algebras, and tight closure theory.

This paper is written to honor the many contributions of
Phil Griffith to commutative algebra

1. INTRODUCTION

All given rings in this paper are commutative, associative with identity, and Noetherian. Our objective is to discuss some conjectures and theorems related to the local homological conjectures — for example, almost all of the results have some connection with the direct summand conjecture. Some of the conjectures have been around for decades. Others are quite recent, and some have grown out of our understanding of the related subjects of tight closure theory and existence of big Cohen-Macaulay algebras.

In the Section 2 we discuss the syzygy theorem of Evans and Griffith, and several related intersection theorems. We are naturally led to consider the direct summand conjecture: other results, which either are equivalent to or imply the direct summand conjecture are considered in Section 3. Some of these other results involve the behavior of the local cohomolgy of the *absolute integral closure*, defined in Section 3. See [23], [26], and [21].

In Section 4 we give a characteristic p proof of the key lemma in the proof of the syzygy theorem that uses a little known variant of tight closure theory, followng [30, Section 10].

One of our reasons for giving the argument is that this particular variant of tight closure is almost entirely unexplored, despite the fact that it has important applications.

In Section 5 we describe recent progress in dimension 3: there was a spectacular breakthrough by Heitmann in [20], and the result of that paper was used in [25] to establish the existence of big Cohen-Macaulay algebras in dimension 3 in mixed characteristic.

Section 6 treats the strong direct summand conjecture, which was shown in [41] to be equivalent to a conjecture on vanishing of maps of Tor that can be proved in equal characteristic either using tight closure theory [33, Theorem 4.1] or the existence of big Cohen-Macaulay algebras in a weakly functorial sense as described in [34, Section 4].

In Section 7 we describe some recent results of G. Dietz [11] on when algebras over a local ring of characterisitc $p > 0$ can be mapped to a big Cohen-Macaulay algebra. Finally, in Section 8 we treat quite briefly some other homological questions about local rings that remain open.

2. The syzygy theorem and some intersection theorems

A remarkable theorem of Evans and Griffith [15] and [16, Corollary 3.16] asserts the following:

Theorem 2.1 (Syzygy Theorem). *Let R be a Cohen-Macaulay local ring that contains a field and let M be a finitely generated k th module of syzygies that has finite projective dimension over R. If M is not free, then M has rank at least k.*

This is not the most general result known: the condition that the ring be Cohen-Macaulay may be relaxed. However, the theorem is of great interest, and very hard to prove, even when the ring is regular! The original proof of Evans and Griffith used the existence of big Cohen-Macaulay modules [22], and while it is now known that the result can be deduced from *a priori* weaker statements, such as the direct summand conjecture [21, 23], it remains an open question in mixed characteristic, even when the ring is regular.

In fact, the theorem has the following amazing consequence [16, Theorem 4.4]:

Corollary 2.2. *Let R be a regular local ring of dimension at least three such that R contains a field. Let I be an ideal of R that is generated by three elements and is unmixed of height*

2. Then R/I is Cohen-Macaulay or, equivalently, the projective dimension of I over R is 1.

This result was so unexpected that I did not believe it. I tried repeatedly to give counterexamples, and on one occasion called Phil at an inconvenient time claiming that I had one. This turned out to be wrong, of course.

The method of proof of Evans and Griffith in essence showed that the theorem follows from an intersection theorem of the type introduced by Peskine and Szpiro [39, 40] and also studied by Roberts [42, 43, 44, 45, 46], Dutta [12, 13] and in my papers [22, 23]. Here is the result:

Theorem 2.3 (**Evans-Griffith improved new intersection theorem**). *Let (R, m) be a local ring of Krull dimension d that contains a field and let*

$$0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow 0$$

be a finite complex of finitely generated free modules such that

   (1) $H_i(G_\bullet)$ has finite length for $i \geq 1$, and
   (2) $H_0(G_\bullet)$ has a minimal generator that is killed by a power of m.

Then $d \leq n$.

The *new intersection theorem* of Peskine and Szpiro is the case where $H_0(G_\bullet)$ is nonzero of finite length. To see that this really is an intersection theorem, we deduce the Krull height theorem from it: consider a minimal prime P of an ideal generated by d elements, $x_1, \ldots, x_d$. We may pass to R_P, and we want to show that $\dim(R_P) \leq d$. This follows by applying the new intersection theorem to the Koszul complex $\mathcal{K}_\bullet(x_1, \ldots, x_d; R)$, which has finite length homology with nonzero $H_0\big(\mathcal{K}_\bullet(x_1, \ldots, x_d; R)\big) \cong R/(x_1, \ldots, x_d)R$.

By the main results of [23], the direct summand conjecture, which asserts that regular local rings are direct summands of their module-finite extensions, implies the improved new intersection theorem. In fact, this work of the author and the results of S. P. Dutta [13] combine to show that the direct summand conjecture, the improved new intersection

theorem, and the canonical element conjecture,[1] are equivalent. What we want to get across is that the direct summand conjecture appears to be central, and I conjecture that whatever method leads to its solution will also yield the existence of big Cohen-Macaulay algebras.

With the exception of the new intersection conjecture, these conjectures all remain open in mixed characteristic in dimension four or more. Progress in dimension 3 is discussed in Section 5. The new intersection conjecture was proved in complete generality by P. Roberts [45, 46, 47]. This work made use of an idea of S. P. Dutta [12] as well as techniques from intersection theory developed in [5]: an exposition of the latter material is given in [17]. For further background see also [50], and my review of [47] in [25]. Note that the results of [39] showed that an affirmative answer to Bass's question [2] (if a local ring has a finitely generated nonzero module of finite injective dimension, must it be Cohen-Macaulay) and M. Auslander's zerodivisor conjecture [1] (which asserts that a zerodivisor on a nonzero finitely generated module M of finite projective dimension over a local ring R must be a zerodivisor in the ring) both follow from the version of the intersection theorem stated in [39], and this follows easily from the new intersection theorem. Thus, Bass's question is affirmatively answered, and the zerodivisor conjecture is true, even in mixed characteristic.

3. The direct summand conjecture and local cohomology of R^+

We mentioned in Section 1 that the direct summand conjecture is equivalent to a host of other conjectures. In fact, even to give all the known forms of the conjecture would require a lengthy manuscript. In this section we want to discuss primarily one form, which is connected with the behavior of the local cohomology of rings of the form R^+, where this notation is defined in the next paragraph.

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[1]We shall not discuss this conjecture [23] in detail here, but here is one statement: a free resolution of the residue class field K of the local ring (R, m, K) may be truncated so as to end at a dth module of syzygies S of K, where $d = \dim(R)$: call the complex one obtains $G_\bullet$. If $x_1, \ldots, x_d$ is any system of parameters, the obvious surjection $R/(x_1, \ldots, x_d)R \twoheadrightarrow K$ lifts to a map of the Koszul complex $\mathcal{K}_\bullet(x_1, \ldots, x_d; R) \longrightarrow G_\bullet$ and so there is a map at the dth spots, $R \longrightarrow S$. The conjecture asserts that no matter what choices are made, the image of $1 \in R$ in S (and even in $S/(x_1, \ldots, x_d)S$) is not 0. In fact, it turns out to be equivalent to assert that even after one takes a direct limit as the system of parameters varies, one has that the image of 1 in $H_m^d(S)$ (this image is *the canonical element*) is not 0.

By the *absolute integral closure* of a domain R, denoted R^+, we mean the integral closure of R in an algebraic closure of its fraction field. The domain R^+ is unique up to non-unique isomorphism, since this is true of the algebraic closure of the fraction field. It is a maximal integral extension of R that is still a domain. Note that every monic polynomial over R (or R^+) factors into monic linear factors over R^+, and this characterizes R^+. Evidently, if $R \subseteq S$ are domains, we may identify R^+ with a subring of S^+, so that there is a commutative diagram

$$
\begin{array}{ccc}
R^+ & \longrightarrow & S^+ \\
\uparrow & & \uparrow \\
R & \longrightarrow & S
\end{array}
$$

where the horizontal arrows are inclusions. Likewise, given a surjection $S \longrightarrow T$ of domains, so that $T \cong S/Q$, there is a prime ideal Q' of S^+ lying over Q. Since S^+/Q' is an integral extension of T in which every monic polynomial factors into linear factors, we have that S^+/Q' may be identified with T^+, and so there is a commutative diagram

$$
\begin{array}{ccc}
S^+ & \longrightarrow & T^+ \\
\uparrow & & \uparrow \\
S & \longrightarrow & T
\end{array}
$$

where the horizontal arrows are now surjections. Since every homomorphism of domains $R \longrightarrow T$ is the composition of an injection and a surjection, it follows that there is a commutative diagram

$$
\begin{array}{ccc}
R^+ & \longrightarrow & T^+ \\
\uparrow & & \uparrow \\
R & \longrightarrow & T
\end{array}
$$

for *every* homomorphism $R \longrightarrow T$ of domains.

The rings R^+ have attracted great interest recently for several reasons. In [32] it was shown that when R is an excellent local domain of characteristic $p > 0$, R^+ is a big Cohen-Macaulay algebra over R, that is, every system of parameters of R is a regular sequence on R^+, and the maximal ideal of R expands to a proper ideal. This is false, in general, in equal characteristic 0. However, recent work of Heitmann [20] has used the properties of the local cohomology of these rings to prove the direct summand conjecture in dimension 3 in mixed characteristic: see Section 5.

The equivalence given in the following result on the direct summand conjecture in mixed characteristic is, I feel, somewhat surprising. It was first observed in [23, Theorem (6.1)], but does not seem to be well known except to experts on the problem.

Theorem 3.1. *Let V be a complete discrete valuation ring of mixed characteristic p, and let $A = V[[x_2, \ldots, x_d]]$. Let m be the maximal ideal of A. Then the direct summand conjecture holds for regular rings of dimension d in mixed characteristic if and only if for every such A, $H_m^d(A^+) \neq 0$.*

Before giving the proof, we remark that standard and relatively straightforward manipulations reduce the problem of proving the direct summand conjecture to the case where the regular ring A is a complete regular local ring. A subtler argument given in [23] makes the reduction to the case where A has the form above.

Proof. The fact that the direct summand conjecture implies that $H_m^d(A^+) \neq 0$ is easy: it suffices to show that the map $H_m^d(A) \longrightarrow H_m^d(A^+)$ is injective, and since A^+ is the direct limit of module-finite extension domains B of A, it suffices to see that for each such B the map $H_M^d(A) \longrightarrow H_m^d(B)$ is injective. But this is immediate if $B = A \oplus_A W$ for some A-module W.

To prove the other direction we want to show that every ring A as above is a direct summand of every module-finite extension ring B, and it suffices to consider domains, for we may first kill a minimal prime P of B disjoint from $A - \{0\}$. (A splitting of $A \hookrightarrow B/P$ composed with $B \twoheadrightarrow B/P$ gives a splitting of $A \hookrightarrow B$.) In fact, we shall show that under the condition $H_m^d(A^+) \neq 0$ considered in Theorem 3.1, A is a direct summand of A^+.

We may identify $H_m^d(A^+) \cong H_m^d(A) \otimes_A A^+$, and $H_m^d(A) = E$ is the injective hull of $K = A/m$ over A. If $H_m^d(A^+) \neq 0$, we can conclude that $\operatorname{Hom}_A(H_m^d(A^+), E) \neq 0$, and this may be written

$$\operatorname{Hom}_A(A^+ \otimes_A E, E) \cong \operatorname{Hom}_A\big(A^+, \operatorname{Hom}_A(E, E)\big)$$

by the adjointness of tensor and Hom. We have that $A \cong \operatorname{Hom}_A(E, E)$ by Matlis duality, since A is complete. Thus, our hypothesis implies that there is a nonzero A-linear map $f : A^+ \longrightarrow A$. Let π be the generator of the maximal ideal of V. The image of this map

is a nonzero ideal of A, and we can write the image in the form $\pi^t J$ where t is a positive integer chosen as large as possible, so that $J \not\subseteq \pi A$. We may compose f with $\pi^t J \cong J \subseteq A$ to obtain a map $f_1 : A^+ \longrightarrow A$ whose image is J. Hence, for some $a \in A^+$ we know that $f_1(a) \notin \pi A$. Define $g : A^+ \longrightarrow A$ via $g(r) = f_1(ar)$ for all $r \in A^+$. Then g is an A-linear map $A^+ \longrightarrow A$ such that $g(1) \notin \pi A$. Since $\bigcap_{N=1}^{\infty} (\pi, x_2^N, \ldots, x_d^N)A = \pi A$, we can fix $N > 0$ such that $g(1) \notin (\pi, x_2^N, \ldots, x_d^N)A$.

Let $A_0 = V[[x_2^N, \ldots, x_d^N]] \subseteq A$, which has maximal ideal $m_0 = (\pi, x_2^N, \ldots, x_d^N)A_0$. Then A is A_0-free over A_0, and $g(1) \in A - m_0 A$ is part of a free basis. Thus, there is an A_0-linear map $h : A \longrightarrow A_0$ that sends $g(1)$ to 1. Then $h \circ g : A^+ \longrightarrow A_0$ is an A_0-linear map sending 1 to 1, and it follows that A_0 is a direct summand of A^+ as an A_0-module. Since A is module-finite over A_0, we have that A^+ is also A_0^+, and so $A_0 \hookrightarrow A_0^+$ splits over A_0. Since $A_0 \cong A$, it follows that $A \hookrightarrow A^+$ splits over A. $\qquad\square$

We next want to describe a conjecture, which we refer to as the *Galois conjecture*, that implies the direct summand conjecture in all characteristics. This observation was first made in [26].

The Galois conjecture, made explicit below, asserts that a certain module is "small" in a sense that we shall make precise. This is true both in equal characteristic $p > 0$ and in equal characteristic 0: in fact, in both equicharacteristic cases, the module is not just small — it is 0. It is striking that the reasons why it is zero in those two cases appear to be completely different.

Let V be a complete discrete valuation ring, which may be either equal characteristic or mixed characteristic. In the mixed characteristic case assume that the residual characteristic p is the generator of the maximal ideal. In either case, denote the generator of the maximal ideal by $x = x_1$. Let $A = V[[x_2, \ldots, x_d]]$ be a formal power series ring over V. Let $\mathcal{F}$ denote the fraction field of A, and then the fraction field of A^+ is an algebraic closure $\overline{\mathcal{F}}$ of $\mathcal{F}$. Let G be the Galois group of $\mathcal{F}$-automorphisms of $\overline{\mathcal{F}}$, which also acts on A^+. Note that $A^{+G} = A$ when $\mathcal{F}$ has characteristic zero, which includes the case where A has equal characteristic zero and the case where A has mixed characteristic.

We let $E = H_m^d(A)$, the highest (in fact, the only) nonzero local cohomology module of A with support in $m = m_A$, since it is also an injective hull $E_A(K)$ for the residue field

$K = A/m$ of A over A. We write $M^\vee$ for $\mathrm{Hom}_A(M, E)$. If (B, n, L) is any complete local ring, we shall call a B-module Q *small* if $E_B(L)$, the injective hull of $L = B/n$ over B, cannot be injected into Q. Note that if $E_B(L)$ is a submodule of Q, then it is actually a direct summand of Q, since $E_B(L)$ is an injective B-module. Thus, the condition that a module be small does not seem unduly restrictive.

Theorem 3.1 above implies that in order to prove the direct summand conjecture, it suffices to show that the modules $H_m^d(A^+)$ are not zero.

Now $x = x_1$ is a regular parameter in A, and we have a short exact sequence

$$0 \longrightarrow A^+ \stackrel{x}{\longrightarrow} A^+ \longrightarrow A^+/xA^+ \longrightarrow 0.$$

If we contradict the direct summand conjecture and assume that $H_m^d(A^+) = 0$, part of the corresponding long exact sequence for local cohomology gives:

$$\cdots \longrightarrow H_m^{d-1}(A^+) \stackrel{x}{\longrightarrow} H_m^{d-1}(A^+) \longrightarrow H_m^{d-1}(A^+/xA^+) \longrightarrow 0.$$

This implies an isomorphism

$$H_m^{d-1}(A^+/xA^+) \cong H_m^{d-1}(A^+)/xH_m^{d-1}(A^+).$$

The regular ring A/xA injects into A^+/xA^+ (because A is normal, the principal ideal xA is contracted from A^+). Suppose that A provides a counterexample to the direct summand conjecture of smallest dimension or that A has mixed characteristic, provides a counterexample, and $x = p$. Under either hypothesis, A provides a counterexample, but the direct summand conjecture holds for the regular ring A/xA. Then A/xA is a direct summand of A^+/xA^+ as an (A/xA)-module, and it follows that $H_m^{d-1}(A/xA)$ injects into $H_m^{d-1}(A^+/xA^+)$. Evidently, since G acts on A^+, m is contained in the ring of invariants of this action, and the element x is an invariant, we have that G acts on $H_m^{d-1}(A^+)/xH_m^{d-1}(A^+)$, and it is clear that $H_m^{d-1}(A/xA)$ injects into

$$\left(H_m^{d-1}(A^+)/xH_m^{d-1}(A^+)\right)^G \subseteq H_m^{d-1}(A^+)/xH_m^{d-1}(A^+).$$

We therefore will have a contradiction that establishes the direct summand conjecture if we can prove the following:

Conjecture 3.2 (Galois Conjecture). *Let (A, m, K) be a complete regular local ring of dimension d with fraction field $\mathcal{F}$, let G be the automorphism group of the algebraic closure $\overline{\mathcal{F}}$ over $\mathcal{F}$, and let x be a regular parameter in A. Then $\left(H_m^{d-1}(A^+)/xH_m^{d-1}(A^+)\right)^G$ is a small (A/xA)-module.*

Theorem 3.3. *The Galois Conjecture holds if $\dim A \leq 2$ or if A contains a field. In fact, in all of these cases $\left(H_m^{d-1}(A^+)/xH_m^{d-1}(A^+)\right)^G = 0$.*

Proof. The explanation when A contains a field is quite different depending on whether the field has characteristic 0 or positive characteristic. In the first case, it turns out that ^{G} is an exact functor here, so that what we have is $\left(H_m^{d-1}(A^{+^G})/xH_m^{d-1}(A^{+^G})\right)$, and since $A^{+^G} = A$, this is $H_m^{d-1}(A)/xH_m^{d-1}(A)$, and $H_m^{d-1}(A) = 0$. In the positive characteristic case we know from the main result of [32] that A^+ is a big Cohen-Macaulay algebra, so that $H_m^{d-1}(A^+) = 0$, and the result follows again. The same argument shows that the conjecture is true when A has dimension at most two. $\qquad\square$

From the discussion above, we have the following:

Theorem 3.4. *If the Galois Conjecture is true whenever A is a formal power series ring $V[[x_2, \ldots, x_d]]$ over a complete discrete valuation domain (V, pV, K) of mixed characteristic and residual characteristic $p > 0$, and $x = p$, then the direct summand conjecture is true.* $\quad\square$

4. A "TIGHT CLOSURE" PROOF OF THE SYZYGY THEOREM IN CHARACTERISTIC p

In this section, we give a "tight closure" proof of a key lemma in the proof of the syzygy theorem. The usual notion of tight closure in a Noetherian ring R of characteristic $p > 0$ uses a fixed multiplier c that is required not to be in any minimal prime of R. The reader is referred to [30, 29, 31, 35, 37, 28], and [8] for background. The argument below, based on a treatment in [30, Section 10], utilizes a variant notion of tight closure in which the only restriction placed on c is that it be nonzero. This has certain disadvantages. The operation one gets, when iterated, typically produces a larger closure. However, the usefulness of this variant notion in the argument below argues that it deserves further study. One of our main motivations in presenting the argument here is to encourage investigation of this variant notion.

We first present a family of variant notions of tight closure:

Consider a non-empty family of nonzero ideals $\mathcal{C}$ in a Noetherian ring R of characteristic $p > 0$ with the property: $(*)$ if C, $C' \in \mathcal{C}$ then there exists $C'' \in \mathcal{C}$ such that $C'' \subseteq C \cap C'$. Then we can define the tight closure with respect to $\mathcal{C}$: an element $u \in N \subseteq M$ is in the *tight closure with respect to $\mathcal{C}$* of N in M if there exists an ideal $C \in \mathcal{C}$ such that $Cu^q \in N^{[q]}$ for all $q = p^e \gg 0$.[2] We can also define the *small tight closure* of N in M with respect to $\mathcal{C}$: for this we require that for some $C \in \mathcal{C}$, $Cu^q \in N^{[q]}$ for all q (which includes $q = 1$). The property $(*)$ is needed so that the tight closure of N will be closed under addition.

If we take the family $\mathcal{C}$ to consist of all principal ideals generated by an element of R°, we obtain the usual notion of tight closure.

If the family consists of only the unit ideal R, tight closure with respect to this family is Frobenius closure, while the small tight closure of N is the submodule N itself.

If R has a test element, tight closure with respect to the family consisting of the single ideal it generates gives ordinary tight closure, as does small tight closure with respect to the family consisting of the single ideal it generates.

We note that iterating one of these variant tight closure operations may give a larger result than performing it once. One can show that iterating the operation gives the same result if the family of ideals has the property that for all C, $C' \in \mathcal{C}$, there exists $C'' \in \mathcal{C}$ such that $C'' \subseteq CC'$. To see this, suppose that the tight closure Q of N with respect to $\mathcal{C}$ has generators u_i. For every i, we can choose $C_i \in \mathcal{C}$ such that $C_i u^q \in N^{[q]}$ for all $q \gg 0$. By property $(*)$ we can choose $C \subseteq \bigcap_i C_i$ with $C \in \mathcal{C}$. It follows that $CQ^{[q]} \subseteq N^{[q]}$ for all $q \gg 0$. If v is in the tight closure of Q with respect to $\mathcal{C}$, then we can choose C' such that $C'v^q \subseteq Q^{[q]}$ for all $q \gg 0$. Then $C'Cv^q \subseteq N^{[q]}$ for all $q \gg 0$, and so if $C'' \in \mathcal{C}$ and $C'' \subseteq C'C$, then $C''v^q \subseteq N^{[q]}$ for all $q \gg 0$. An entirely similar argument establishes the corresponding fact for small tight closure.

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[2]Let R be Noetherian of prime characteristic $p > 0$. We abbreviate $q = p^e$, and let $\mathbf{F}^e$ denote the eth iteration of the Frobenius functor: this is the base change functor $S_e \otimes_R \_$, where S_e is the R-algebra whose structural homomorphism is $F^e : R \longrightarrow R$, with $F^e(r) = r^q$ for all $r \in R$. If $N \subseteq M$, let $N^{[q]}$ be the image of $\mathbf{F}^e(N) \longrightarrow \mathbf{F}^e(M)$. If $u \in M$, let u^q denote $1 \otimes u$ in $\mathbf{F}^e(M)$. Then $N^{[q]}$ is the R-span within M of the elements $\{u^q : u \in N\}$. In particular, for ideals $I \subseteq R$, $I^{[q]} = (i^q : i \in I)R$.

We now want to show how one of these variant notions of tight closure can be used to prove the Evans-Griffith syzygy theorem. We want to make two remarks. First, it is immediate from the definition that $u \in M$ is in the tight closure (respectively, small tight closure) with respect to $\mathcal{C}$ of N in M if and only if the image of u in M/N is in the tight closure (respectively, small tight closure) of 0 in M/N with respect to $\mathcal{C}$. The second remark we state as:

Lemma 4.1. *If (R, m, K) is local, $\mathcal{C}$ is a non-empty family of nonzero ideals of R, and x is a minimal generator of a finitely generated module M, then x is not in the tight closure (nor in the small tight closure) of 0 in M with respect to $\mathcal{C}$.*

Proof. If u is in the tight closure of 0 in M we have that $Cx^q = 0$ in $F^e(M)$ for all $q \gg 0$. We can map $M \twoheadrightarrow K$ so that $x \mapsto 1$. We get an induced surjection $F^e(M) \longrightarrow R/m^{[q]}$. It follows that $C \subseteq m^{[q]}$ for all $q \gg 0$, which implies that $C = (0)$, a contradiction. $\square$

We shall need to make use of the notion of order ideal.

Definition 4.2. Let x be an element of M, a finitely generated module over a Noetherian ring R. We define the *order ideal* $\mathfrak{O}_M(x) = \mathfrak{O}(x)$ to be $\{f(x) : f \in \operatorname{Hom}_R(M, R)\}$.

For finitely generated modules over a Noetherian ring R, the formation of the order ideal commutes with localization. The map $R \longrightarrow M$ sending $1 \mapsto x$ evidently splits if and only if $\mathfrak{O}_M(x) = R$.

Also note that for any finitely generated free R-module G, any R-linear map $M \longrightarrow G$ takes x into $\mathfrak{O}_M(x)G$.

The Evans-Griffith syzygy theorem asserts that a kth module of syzygies over a regular local ring, if not free, has rank at least k. They prove more general statements, in which the conditions on the ring are weakened but the module is assumed to have finite projective dimension. However, the key point in their proof is the following:

Theorem 4.3 (Evans-Griffith)). *Let R be a local ring such that R contains a field, let M be a kth module of syzygies of a finitely generated module of finite projective dimension, and suppose that M_P is R_P-free for every prime P of R except the maximal ideal, i.e., M*

is locally free on the punctured spectrum of R. Let $x \in M$ be a minimal generator. Then $\mathfrak{D}(x)$ is either the unit ideal or else has height at least k.

In fact, they show that this is true by using the fact that the improved new intersection theorem is true when R contains a field, which they deduce from the existence of big Cohen-Macaulay modules in the equal characteristic case. We shall not duplicate their argument here. But we shall prove a better result in characteristic p, with depth replacing height and without the assumption that M is locally free on the punctured spectrum. This is where we use a variant notion of tight closure. The following is [30, Theorem 10.8, p. 103].

Theorem 4.4. *Let (R, m, K) be a local ring of prime characteristic $p > 0$ and let N be a finitely generated module of finite projective dimension over R. Let M be a finitely generated k th module of syzygies of N, and let $x \in M$ be a minimal generator of M. Let $I = \mathfrak{D}_M(x)$. Then either $I = R$ or else $\operatorname{depth}_I R \geq k$.*

Proof. If not, let $y_1, \ldots, y_d$ be a maximal regular sequence in the proper ideal I, where $d < k$, and let $J = (y_1, \ldots, y_d)R$. Then we can choose $c \in R - J$ such that $cI \subseteq J$. Let c' denote the image of c in $R' = R/J$. Let $G_\bullet$ be a resolution of N by finitely generated free modules over R such that $G_k \longrightarrow G_{k-1}$ factors $G_k \twoheadrightarrow M \hookrightarrow G_{k-1}$, which we know exists because M is k th module of syzygies of N over R. Let B denote the image of G_{k+1} in G_k. Let G'_j denote $R' \otimes_R G_j$, while M' denotes $R' \otimes_R M$ and B' denotes the image of $R' \otimes_R B$ in G'_k. Choose an element z of G_k that maps onto $x \in M$. We shall obtain a contradiction by showing that the image z' of z in G'_k is in the tight closure of B' in G'_k with respect to the family $\{c'R'\}$. This implies that the image x' of x in M' is in the tight closure of 0 in M' with respect to the family $\{c'R'\}$, a contradiction using the Lemma 4.1 above, because x' is a minimal generator of M'.

To see this, note that $F_R^e(G_\bullet)$ remains acyclic for all e: the determinantal ranks of the maps and the depths of the ideals of minors do not change. Thus, this is a free resolution of $F_R^e(N)$, and it follows that $R' \otimes_R F_R^e(G_\bullet)$ has homology $\operatorname{Tor}_\bullet^R\big(R', F_R^e(N)\big)$. Since $\operatorname{pd}_R R' = d < k$, we have that $\operatorname{Tor}_k^R\big(R', F_R^e(N)\big) = 0$. But the complex $R' \otimes_R F_R^e(G_\bullet)$ may be identified with $F_{R'}^e(G'_\bullet)$. Let δ denote the map $G'_k \longrightarrow G'_{k-1}$. Now consider the value of the R-linear map $F_{R'}^e(\delta)$ evaluated on $c'(z')^q$. This is evidently $c'F_{R'}^e(\delta)\big((z')^q\big)$. Since the

map $G_k \longrightarrow G_{k-1}$ factors through M, the image of z, which maps to $x \in M$, is in IG_{k-1}. It follows that the image of z' under δ in IG'_{k-1}, and, hence, that the image of $(z')^q$ under $F^e(\delta)$ is in

$$I^{[q]} F^e(G'_{k-1}) \subseteq IF^e_{R'}(G'_{k-1}).$$

Since $cI \subseteq J$ and J becomes 0 in R', we have that $F^e_{R'}(\delta)\big(c'(z')^q\big) = 0$. Since $c'(z')^q$ is a cycle and the homology at this spot is 0, it follows that $c'(z')^q$ is a boundary, which means that it is in the image $(B')^{[q]}$ of $F^e_{R'}(B')$. Thus, z' is in the tight closure with respect to the family $\{c'R'\}$ of B' in G'_k, and this means that x' is in the tight closure with respect to $\{cR'\}$ of 0 in M'. Since x is a minimal generator of M and $J \subseteq m$, it follows that x' is a minimal generator of M', and we have obtained the contradiction of Lemma 4.1 mentioned earlier. $\qquad\square$

It is quite easy to deduce the syzygy theorem from Theorem 4.3: for details see, for example, [30, Cor. 10.10, p. 105].

Finally, note that the equal characteristic 0 cases of both Theorem 4.3 and of the syzygy theorem can be deduced from the equal characteristic $p > 0$ case by standard methods of reduction to characteristic p.

5. Recent progress in dimension 3

Ray Heitmann [20] recently proved the direct summand conjecture in dimension 3. The argument is very difficult. The main result of Heitmann's paper (although stated differently from the version in [20]) is:

Theorem 5.1 (Heitmann). *Let (R, m, K) be a complete local domain of mixed characteristic p and dimension 3. Then every element of $H^m(R^+)$ is killed by arbitrarily small powers of p, i.e., by $p^{1/N}$ for arbitrarily large values of the positive integer N.*

This result can be expressed more concretely as follows: let x, y, z be a system of parameters for a three dimensional complete local domain R of mixed characteristic p, and suppose that $zu \in (x, y)R$. Then for every positive integer N there exists a module-finite extension domain S of R such that $p^{1/N} \in S$ and $p^{1/N}u \in (x, y)S$.

We make the curious observation that, in general, the direct summand conjecture seems to follow if we can show that, for all complete local domains R of dimension d, $H_m^d(R^+)$ is "big" (in fact, simply nonzero) or if we can show that for all such R of dimension d, $H_m^{d-1}(R^+)$ is "small" (in one of several senses that are rather technical).

In [27] Heitmann's result is used to prove that every complete local domain of dimension at most 3 has a big Cohen-Macaulay algebra. In characteristic p, one has a "weakly functorial" version of the existence of big Cohen-Macaulay algebras: if $R \longrightarrow S$ is a local homomorphism of complete local domains, one has a commutative diagram:

$$
\begin{array}{ccc}
R^+ & \longrightarrow & S^+ \\
\uparrow & & \uparrow \\
R & \longrightarrow & S
\end{array}
$$

and, hence, a commutative diagram:

$$
\begin{array}{ccc}
B & \longrightarrow & C \\
\uparrow & & \uparrow \\
R & \longrightarrow & S
\end{array}
$$

where B and C are big Cohen-Macaulay algebras. A corresponding result for the equal characteristic 0 case is proved in [34] by reduction to characteristic p. It is shown in [27] that one has:

Theorem 5.2 (weakly functorial big Cohen-Macaulay algebras in dimension at most 3). *Let $R \longrightarrow S$ be a local homomorphism of complete local domains both of mixed characteristic, and both of dimension at most 3. Then there exists a commutative diagram*

$$
\begin{array}{ccc}
B & \longrightarrow & C \\
\uparrow & & \uparrow \\
R & \longrightarrow & S
\end{array}
$$

where B is a big Cohen-Macaulay algebra over R and C is a big Cohen-Macaulay algebra over S.

I conjecture that a sufficiently good result on the weakly functorial existence of big Cohen-Macaulay algebras in mixed characteristic is equivalent to the existence of tight closure theory in mixed characteristic. This is an admittedly vague statement. A much

more precise version of this statement is made in [11]. Some of the results of [11] are discussed in Section 7.

We note that the existence of big Cohen-Macaulay algebras in a weakly functorial sense implies the vanishing conjecture for maps of Tor discusssed in Section 6 just below.

6. The strong direct summand conjecture and vanishing of maps of Tor

In this section we discuss one of the new homological conjectures, the vanishing conjecture for maps of Tor. It is a theorem in equal characteristic but remains open in mixed characteristic. In a reasonably non-technical version, the conjecture may be phrased as follows:

Conjecture 6.1 (**Vanishing conjecture for maps of Tor**). *Suppose that $A \longrightarrow R \longrightarrow T$ be homomorphisms of Noetherian rings such that*

  (1) A is regular.
  (2) R is a module-finite and torsion-free extension of A.
  (3) T is regular.

  Then for every A module M and $i \geq 1$, the map

$$\mathrm{Tor}_i^A(M,\, R) \longrightarrow \mathrm{Tor}_i^A(M,\, T)$$

is 0.

It is reasonably straightforward to reduce to the case where M is finitely generated, $i = 1$, T and A are complete regular local rings, and R is a complete local domain. As mentioned earlier, proofs for the equicharacteristic case are given in [33, Section 4] and [32, Theorem (4.1)].

This conjecture, when it is true, is a very powerful tool. For example, it implies the direct summand conjecture [33, Section 4] and the statement that a ring R that is a direct summand of a regular ring T is Cohen-Macaulay. The latter statement was originally proved for certain rings of invariants [36], and stronger statements are true in equal characteristic 0 [7].

The proof of the statement that direct summands of regular rings are Cohen-Macaulay may be sketched as follows. One can come down to the case where R is complete local and

module-finite over a regular local ring (A, m, K). But then the maps

$$\operatorname{Tor}_i^A(K, R) \longrightarrow \operatorname{Tor}_i^A(K, T),$$

which are injective because R is a direct summand of T, are 0 for $i \geq 1$ by the Vanishing Conjecture (6.1), and so the modules $\operatorname{Tor}_i(K, R) = 0$ for $i \geq 1$, which implies that R is Cohen-Macaulay.

Although Conjecture 6.1 does not appear to be purely a result about "splitting," N. Ranganathan proved that it is equivalent to the following conjecture in [41]:

Conjecture 6.2 (Strong direct summand conjecture). *Let $A \longrightarrow R$ be a local homomorphism where A is a regular local ring and R is a domain module-finite over A. Let Q be a height one prime ideal of R lying over xA, where x is a regular parameter for A (i.e., A/xA is again a regular local ring). Then the inclusion map $xA \longrightarrow Q$ splits as a map of A-modules.*

Note that this result implies the direct summand conjecture, for $xA \subseteq xR \subseteq Q$, and so xA is a direct summand of xR, which means that A is a direct summand of R. The direct summand conjecture has an easy proof in equal characteristic 0, using a trace argument, but Conjecture 6.2 appears subtle and difficult in all characteristics.

7. Algebras that map to big Cohen-Macaulay algebras

A central and challenging open question is this: given an algebra S over a complete local domain R, when can S be mapped to a big Cohen-Macaulay algebra over R? Following [11], we call such an R-algebra S a *seed* over R. Some remarkable results about seeds in characteristic $p > 0$ are obtained in [11]. Whether there are corresponding results in equal characteristic 0 is an open question: in these instances, standard methods of reduction to characteristic p do not succeed.

Theorem 7.1 (G. Dietz). *Let (R, m) be a complete local domain of characteristic $p > 0$. Then every seed S over R maps to a seed T with all of the following properties:*

    (1) T is a domain.
    (2) T is absolutely integrally closed, i.e., $T = T^+$.

(3) T is m-adically complete and separated.

Theorem 7.2 (G. Dietz). *If $R \longrightarrow R'$ is a local homomorphism of complete local domains and S is a seed over R then $R' \otimes_R S$ is a seed over R'. That is, for every big Cohen-Macaulay algebra B over R there is a big Cohen-Macaulay C over R' and a commutative diagram*

$$
\begin{array}{ccc}
B & \longrightarrow & C \\
\uparrow & & \uparrow \\
R & \longrightarrow & R'
\end{array}
$$

Remark 7.3. Note that before the work of [11], it was known that there exist big Cohen-Macaulay algebras B and C such that the diagram commutes: one can take, for example, $B = R^+$ and $C = S^+$. But it was not known that one can construct such a diagram for every given B.

Theorem 7.4 (G. Dietz). *Let R be a complete local domain and let S and S' be seeds over R. Then $S \otimes_R S'$ is a seed over R. Equivalently, given big Cohen-Macaulay algebras B and B' over R there exists a big Cohen-Macaulay algebra C and a commutative diagram*

$$
\begin{array}{ccc}
B & \longrightarrow & C \\
\uparrow & & \uparrow \\
R & \longrightarrow & B'
\end{array}
$$

Of course, in mixed characteristic in dimension 4 or more, we do not even know whether the complete local domain R is itself a seed over R: this question is the existence of big Cohen-Macaulay algebras.

Even in characteristic p, we do not fully understand how to characterize seeds. First, we recall:

Definition 7.5. If R is a domain, an R algebra S is called *solid* if there exists a nonzero R-linear module homomorphism $f : S \longrightarrow R$.

If (R, m) is a complete local domain of dimension d, it turns out that R is solid if and only if $H_m^d(R) \neq 0$ [24, Corollary (2.4)]. We refer the reader to [24] for a detailed treatment.

It is an open question in characteristic p whether an R-algebra S is solid if and only if it is a seed. However, this is known to be false in equal characteristic 0.[3]

8. OTHER QUESTIONS

In this final section we want to make some brief comments on other homological questions. The conjecture of Buchsbaum and Eisenbud [3], reported in [4] as a question by Horrocks, that the ith Betti number of a module of finite length over a local ring is at least $\binom{n}{i}$ remains open in dimension 5 or more.

Serre's conjecture [49] that if (R, m) is regular local of Krull dimension d and M, N are finitely generated nonzero modules such that $M \otimes_R N$ has finite length then $\chi(M, N) = \sum_{i=0}^{d} \ell(\operatorname{Tor}_i^R(M, N))$ is nonnegative and vanishes if and only if $\dim(M) + \dim(N) < \dim(R)$ (Serre showed that one must have $\dim(M) + \dim(N) \leq \dim(R)$) remains open. The conjecture was already proved by Serre if $\widehat{R}$ is formal power series over a field or discrete valuation ring. That $\chi(M, N) = 0$ if $\dim(M) + \dim(N) < \dim(R)$ was proved by P. Roberts [44] and Gillet-Soulé [18] independently. See also [47, Corollary 13.1.1]. Nonnegativity in the remaining cases was proved by O. Gabber (there is an exposition in [6]) using results of De Jong [10] on alterations. Strict positivity in the mixed characteristic case when $\dim(M) + \dim(N) = \dim(R)$ remains an open question.

This intersection multiplicity is defined more generally when the local ring is not necessarily regular, but M, N are finitely generated modules such that $M \otimes_R N$ has finite length and one of them, say M, has finite projective dimension. But in this generality $\chi(M, N)$ can be negative [14], although [40] has an affirmative result for the graded equicharacteristic case.

---

[3]For example, let $X = (x_{ij})$ be a 2×3 matrix of indeterminates over a field K of characteristic 0, and let $\Delta_1, \Delta_2, \Delta_3$ be the 2×2 minors. Then $R = K[\Delta_1, \Delta_2, \Delta_3] \subseteq K[x_{ij}] = S$ splits, since $R = S^G$ where $G = \mathrm{SL}(2, K)$ acting so that $\gamma \in G$ maps the entries of X to the corresponding entries of γX. The splitting is a consequence of the fact that G is reductive: these ideas originate in [51], but see also [36, p. 119]. We still have the splitting after applying $\widehat{R} \otimes_R \_$. Thus, $T = \widehat{R} \otimes_R S$ is solid over $\widehat{R}$, but T does not map to a big-Cohen Macaulay algebra over $\widehat{R}$. In any such algebra the relations $\sum_{j=1}^{3} (-1)^{j-1} x_{ij} \Delta_j = 0$ will force each x_{ij} into the ideal J generated by the Δ_j, which, since each Δ_j is a quadratic form in the x_{ij}, implies that $\Delta_j \in J^2$, so that $J = J^2$. This is not possible when J is generated by a proper regular sequence.

With the same hypotheses as in the preceding paragraph, one may ask whether it must be true that $\dim(M) + \dim(N) \leq \dim(R)$. See [39, 40]. This is very much an open question. In fact, Peskine and Szpiro [39] even raised the following question: under the same hypotheses, with $I = \mathrm{Ann}_R M$, must it be true that $\dim(N) \leq \mathrm{depth}_I R$ (which is always $\leq \dim(R) - \dim(M)$). This remains open.

A conjecture of M. Auslander on the rigidity of Tor for modules of finite projective dimension was disproved by Ray Heitmann [19] using "generic" modules of projective dimension two introduced in [22].

Finally, we mention that [9] has results over local complete intersections connecting a version of rigidity and the dimension inequality

$$\dim(M) + \dim(N) \leq \dim(R).$$

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