Extensions of primes, flatness, and intersection flatness

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ABSTRACT. We study when $R \to S$ has the property that prime ideals of R extend to prime ideals or the unit ideal of S, and the situation where this property continues to hold after adjoining the same indeterminates to both rings. We prove that if R is reduced, every maximal ideal of R contains only finitely many minimal primes of R, and prime ideals of $R[X_1, \dots, X_n]$ extend to prime ideals of $S[X_1, \dots, X_n]$ for all n, then S is flat over R. We give a counterexample to flatness over a reduced quasilocal ring R with infinitely many minimal primes by constructing a non-flat R-module M such that $M = PM$ for every minimal prime P of R. We study the notion of intersection flatness and use it to prove that in certain graded cases it suffices to examine just one closed fiber to prove the stable prime extension property.

Dedicated to Roger and Sylvia Wiegand on the occasion of their 150th birthday

1. Introduction

All rings in this paper are assumed commutative, associative, with multiplicative identity. Although we were originally motivated in studying the Noetherian case, some of our results hold in much greater generality.

We say that an R-algebra S or that the homomorphism $R \to S$ has the *prime extension property* if for every prime P of R, PS is prime in S or the unit ideal of S. We say that the R-algebra S or the homomorphism $R \to S$ has the *stable prime extension property* if for every finite set of indeterminates $X_1, \dots, X_n$ over these rings, $R[X_1, \dots, X_n] \to S[X_1, \dots, X_n]$ has the prime extension property.

One of our results, Theorem 3.5, asserts the following.

THEOREM 1.1. *Let R be a reduced ring such that every maximal ideal contains only finitely many minimal primes; in particular, this holds if R is reduced and locally Noetherian. If $R \to S$ has the stable prime extension property, then S is flat over R.*

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This result surprised us. Note that the hypothesis that R be reduced is quite necessary, since $R \to R/\mathfrak{N}_R$, where $\mathfrak{N}_R$ is the ideal of nilpotent elements of R, also has the stable prime extension property. It is easy to see that the hypothesis of having the stable prime extension property cannot be weakened to having the prime extension property in Theorem 1.1; for example, since the surjection $\mathbb{C}[\![X,Y]\!]/(XY) \twoheadrightarrow \mathbb{C}[\![X]\!]$ sending $Y \mapsto 0$, satisfies the prime extension property, but is not flat. Note that in this example, the prime extension property is lost if we adjoin an indeterminate Z to both rings, since $(X, Y - Z^2)$ yields a prime in $\mathbb{C}[\![X,Y]\!][Z]/(XY)$ but not in $\mathbb{C}[\![X]\!][Z]$. The hypothesis on the minimal primes of R is also necessary: in Section 4, we construct an inclusion of quasilocal rings in which the source is reduced that satisfies the stable prime extension property, but is not flat. A more refined but more technical version of Theorem 1.1 is given in Theorem 3.6. Our results here are related to results of the first author on radical ideals in [**5**].

We note that there are significant examples where the stable prime extension property holds: in particular, the following result from §5, Theorem 5.10 generalizes [**1**, Corollary 2.9] (also cf. [**4**, Theorem 12.1(viii)]). This surprisingly enables one to deduce that the stable prime extension property holds by examining one closed fiber.

Theorem 1.2. *If K is an algebraically closed field, S is a $\mathbb{Z}$-graded K-algebra (but we are not assuming that S is Noetherian nor that $S_0 = K$) and $F_1, \dots, F_n$ are positive degree forms of S with coefficients in K that form a regular sequence and generate a prime ideal Q of S, then $K[F_1, \dots, F_n] \to S$ has the stable prime extension property.*

This and related results in §5 make use of the notion of *intersection flatness*. Some basic properties of this notion are established in §5, and applied to give sufficient criteria for the stable prime extension property. The results in §5 are inspired by the work of T. Ananyan and the first author in [**1**].

2. Basic properties

We collect some basic properties of the prime extension property and the stable prime extension property.

Proposition 2.1. *Let $R \to S$ be a ring homomorphism, and let $R_\lambda \to S_\lambda$ be a direct limit system of ring homomorphisms indexed by λ.*

- *(a) If $R \to S$ and $S \to T$ both have the prime extension property (respectively, stable prime extension property), then so does the composite map $R \to T$.*
- *(b) $R \to S$ has the prime extension property if and only if for every surjection $R \twoheadrightarrow D$, where D is a domain, $D \otimes_R S$ is a domain or zero.*
- *(c) $R \to S$ has the stable prime extension property if and only if for every map $R \to D$, where D is a domain finitely generated over R, $D \otimes_R S$ is a domain or zero.*
- *(d) If $R_\lambda \to S_\lambda$ has the prime extension property (respectively, stable prime extension property) for each λ, so does $\varinjlim_\lambda (R_\lambda \to S_\lambda)$.*
- *(e) $R \to S$ has the stable prime extension property if and only if for every map $R \to D$, where D is a domain, $D \otimes_R S$ is a domain or zero.*
- *(f) $R \to S$ has the stable prime extension property if and only if for every homomorphism $R \to R'$, $R' \to R' \otimes_R S$ has the prime extension property,*

and this holds if and only for every homomorphism $R \to R'$, $R' \to R' \otimes_R S$ has the stable prime extension property.

PROOF. Part (a) is clear, since $T/PT \cong T/(PS)T$, and (b) is a consequence of the fact that $D \cong R/P$. For (c), given a domain D finitely generated over R, take a surjection $R[X_1, \ldots, X_n] \twoheadrightarrow P$. Since $R[X_1, \ldots, X_n] \to S[X_1, \ldots, X_n]$ has the prime extension property, $S \otimes_R D \cong S[X_1, \ldots, X_n] \otimes_{R[X_1, \ldots, X_n]} D$ is a domain or zero by (b). Conversely, if $P \subseteq R[X_1, \ldots, X_n]$ is a prime for which $PS[X_1, \ldots, X_n]$ is neither prime nor the unit ideal, then setting $D = R[X_1, \ldots, X_n]/P$ yields $D \otimes_R S \cong S[X_1, \ldots, X_n]/PS[X_1, \ldots, X_n]$ which is neither a domain nor zero. For (d), note that if P is a prime of $\varinjlim_\lambda R_\lambda$ and P_λ is the contraction of P to R_λ, then $P = \varinjlim_\lambda P_\lambda$, and the result for the prime extension property follows from the fact that a direct limit of rings each of which is a domain or zero is itself a domain or zero. The result for stable prime extension property is then a consequence of the fact that direct limit commutes with adjoining the variables. Part (e) is a consequence of (c) and (d): given a map from R to a domain D, one may write $D = \varinjlim_\lambda D_\lambda$ for a directed system of domains that are finitely generated over R, and $S \otimes_R D \cong \varinjlim_\lambda (S \otimes_R D_\lambda)$. Part (f) follows at once from the characterization in (e) and the isomorphisms $(R' \otimes_R S) \otimes_{R'} D \cong S \otimes_R D$. □

The name "stable prime extension property" is partly motivated by statement (f) above. We will use the characterizations of the prime extension property in part (b) and the stable prime extension property in part (e) of the previous proposition repeatedly below.

PROPOSITION 2.2. *Let $R \to S$ be any ring homomorphism.*

- *(a) If $R \to S$ has the prime extension property or the stable prime extension property and W is a multiplicative system in R (respectively, V is a multiplicative system in S) then $W^{-1}R \to W^{-1}S$ (respectively, $R \to V^{-1}S$) has the same property.*
- *(b) If $\mathfrak{A}$ is an ideal consisting of nilpotent elements of R, then $R \to S$ has the prime extension property (respectively, the stable prime extension property) if and only if $R/\mathfrak{A} \to S/\mathfrak{A}S$ has that property.*
- *(c) The map $R/PR \to S/PS$ has the prime extension property (respectively, the stable prime extension property) for every minimal prime P of R if and only if $R \to S$ has that property.*
- *(d) If S is a polynomial ring in any family of variables over R, then $R \to S$ has the stable prime extension property.*
- *(e) If R is Noetherian, a formal power series ring in any family of variables over R has the prime extension property.*

PROOF. For part (a), suppose that $R \to S$ has the prime extension property and $W^{-1}R \twoheadrightarrow D$ for some domain D. Since every prime of $W^{-1}R$ is expanded from R, we can write $D \cong W^{-1}D'$ for some domain D' with $R \twoheadrightarrow D'$. Then, by Proposition 2.1(b), $S \otimes_R D'$ is either a domain or zero, so

$$W^{-1}S \otimes_{W^{-1}R} D \cong W^{-1}S \otimes_{W^{-1}R} W^{-1}D' \cong W^{-1}(S \otimes_R D')$$

is either a domain or zero, so $W^{-1}R \to W^{-1}S$ has the prime extension property. The argument for stable prime extension property in the same, except replacing surjections to domains with general maps to domains, and using Proposition 2.1(e). The case with V in place of W is similar.

Part (b) follows from the fact all maps from R to a domain D must have $\mathfrak{A}$ in their kernel, so $S \otimes_R D \cong S/\mathfrak{A}S \otimes_{R/\mathfrak{A}} D$; thus the condition of Proposition 2.1(b) holds or fails simultaneously for the two given maps, and likewise for Proposition 2.1(e).

For the forward implication of (c), a map $R \to D$ to a domain D must contain a minimal prime P in the kernel, so $S \otimes_R D \cong S/PS \otimes_{R/P} D$ for that prime P, and the conclusion follows from Proposition 2.1(b) and (e). The reverse follows from Proposition 2.1(f).

Parts (d) and (e) follow from the fact that S/PS may be identified with the corresponding polynomial or power series ring over R/P, since for any ideal (respectively, finitely generated ideal) I, the expansion IS is the same as the ideal of polynomials (respectively power series) all of whose coefficients are in I. Moreover, in the polynomial case, the hypothesis continues to hold after adjoining indeterminates to both rings. □

3. Flatness

We say that T is a *geometrically reduced and irreducible* algebra over the field K if for every field extension $K \subseteq L$, $L \otimes_K T$ is a domain. If $D \subseteq L$ is a domain, then $D \otimes_K T \subseteq L \otimes_K T$. Thus $K \to T$ is geometrically reduced and irreducible if and only if it has the stable prime extension property.

Observe that a direct limit of geometrically reduced and irreducible algebras is again geometrically reduced and irreducible, since tensor products commute with direct limits. If $T_0 \subseteq T$ is a K-subalgebra, then $L \otimes_K T_0 \subseteq L \otimes_K T$, so that T is geometrically reduced and irreducible if and only if all K-subalgebras are geometrically reduced and irreducible, which happens if and only if all finitely generated K-subalgebras are geometrically reduced and irreducible.

If P is a prime ideal of R, we let $\kappa_P = R_P/PR_P$, which is canonically isomorphic with the field of fractions of R/P. The *fiber* of $R \to S$ over a prime ideal P of R is $\kappa_P \otimes_R S$.

PROPOSITION 3.1. *If $R \to S$ has stable prime extension property, then for all primes P of R, the fiber $\kappa_P \otimes_R S$ is a geometrically reduced and irreducible κ_P-algebra. If, moreover, $R \to S$ is flat, the converse holds, i.e. if $R \to S$ is flat, then $R \to S$ has the stable prime extension property if and only if $\kappa_P \otimes_R S$ is a geometrically reduced and irreducible κ_P-algebra for all primes P of R.*

PROOF. Assume that $R \to S$ has the stable prime extension property. If L is any extension field of κ_P, we have a composite map $R \to \kappa_P \to L$, and

$$L \otimes_{\kappa_P} (\kappa_P \otimes_R S) \cong L \otimes_R S$$

is a domain by Proposition 2.1(e).

Now assume that $R \to S$ is flat and that all fibers are geometrically reduced and irreducible. Suppose that we have a homomorphism $R \to D$, where D is a domain, and let P be the kernel. Let L be the field of fractions of D. Then $D \otimes_R S$ is flat over D. Hence, the elements of $D \setminus \{0\}$ are nonzerodivisions, and to show that $D \otimes_R S$ is a domain it suffices to show that $L \otimes_R S$ is a domain. Since we have an injection $\kappa_P \hookrightarrow L$, we have the identification $L \otimes_{\kappa_P} (\kappa_P \otimes_R S) \cong L \otimes_R S$, and this ring is a domain by the hypothesis that $\kappa_P \otimes_R S$ is geometrically reduced and irreducible. □

Our next goal is to show that under mild conditions on the reduced ring R, the condition that $R \to S$ has the stable prime extension property forces the flatness of S over R. See Theorem 3.5 below. We need some preliminary results.

The result that intersecting two ideals commutes with extension from R to S is often stated for the case where S is flat over R. We prove that the result holds under a much weaker assumption: that for one of the ideals $\mathfrak{A}$, $S/\mathfrak{A}S$ is flat over $R/\mathfrak{A}$. In fact, we assume even less:

Lemma 3.2. *If $R \to S$ is any ring homomorphism and $\mathfrak{A}, \mathfrak{B} \subseteq R$ are ideals such that $\mathrm{Tor}_1^{R/\mathfrak{A}}(R/(\mathfrak{A}+\mathfrak{B}),\, S/\mathfrak{A}S) = 0$, which holds, for example, if $S/\mathfrak{A}S$ is flat over $R/\mathfrak{A}$, then $\mathfrak{A}S \cap \mathfrak{B}S = (\mathfrak{A}\cap\mathfrak{B})S$. Hence, if $\mathfrak{A}_1, \ldots, \mathfrak{A}_h$ are ideals of R such that $S/\mathfrak{A}_iS$ is flat over $R/\mathfrak{A}_i$ for $1 \le i \le h-1$, then $\bigcap_i(\mathfrak{A}_iS) = (\bigcap_i \mathfrak{A}_i)S$.*

Proof. We use an overline to indicate images of elements in $R/\mathfrak{A}$ or $S/\mathfrak{A}S$: context should make it clear which is meant. Let $u \in \mathfrak{A}S \cap \mathfrak{B}S$. Let $\{g_\lambda \mid \lambda \in \Lambda\}$ be a generating set for $\mathfrak{B}$, and take a free resolution

$$\cdots \to (R/\mathfrak{A})^{\oplus\Gamma} \xrightarrow{d_1} (R/\mathfrak{A})^{\oplus\Lambda} \xrightarrow{d_0} R/\mathfrak{A} \to 0$$

for $R/(\mathfrak{A}+\mathfrak{B})$ as an $(R/\mathfrak{A})$-module, where d_0 maps the generator indexed by λ to $\overline{g_\lambda}$. We can write

$$u = \sum_{j=1}^{k} g_{\lambda_j} s_{\lambda_j} \in \mathfrak{A}S$$

with $s_{\lambda_1}, \ldots, s_{\lambda_k} \in S$ and $\lambda_1, \ldots, \lambda_k \in \Lambda$. Let $\boldsymbol{s}$ be the vector with coordinate s_{λ_j} in index λ_j, and all other coordinates zero. The relation $\overline{u} = 0$ above specifies an element in $\ker(S/\mathfrak{A}S \otimes_{R/\mathfrak{A}} d_0)$, namely, the vector $\overline{\boldsymbol{s}}$ with coordinate $\overline{s_{\lambda_j}}$ in index λ_j, and all other coordinates zero.

The vanishing of

$$\mathrm{Tor}_1^{R/\mathfrak{A}}(R/(\mathfrak{A}+\mathfrak{B}),\, S/\mathfrak{A}S)$$

implies that $\overline{\boldsymbol{s}} \in \mathrm{im}(S/\mathfrak{A}S \otimes_{R/\mathfrak{A}} d_1)$, i.e., the vector $\overline{\boldsymbol{s}}$ of coefficents is an $(S/\mathfrak{A}S)$-linear combination of finitely many, say h, vectors $\overline{\boldsymbol{r_1}}, \ldots, \overline{\boldsymbol{r_h}}$ with coefficients in $R/\mathfrak{A}$ that each give relations on the $\overline{g_\lambda}$ in $R/\mathfrak{A}$. We lift each $\overline{r_i}$ to a vector r_i with coefficients in R by choosing arbitrary preimages for the nonzero coordinates, and zero in the zero coordinates. Note that each $\overline{\boldsymbol{r_i}}$ and hence each r_i, has finite support, so there is a finite subset $\Lambda_0 = \{\lambda_1, \ldots, \lambda_w\}$ that contains the support of all of the vectors $\boldsymbol{s}, \boldsymbol{r_1}, \ldots, \boldsymbol{r_h}$. Then, we have

(1) $\boldsymbol{s} \equiv \sum_{i=1}^h u_i \boldsymbol{r_i}$ modulo $\mathfrak{A}S$ (as vectors, coordinatewise) for some elements $u_i \in S$, and
(2) for every i, $\sum_{\lambda\in\Lambda} \boldsymbol{r_{i\lambda}} g_\lambda = \sum_{\lambda\in\Lambda_0} \boldsymbol{r_{i\lambda}} g_\lambda \in \mathfrak{A}$, and hence $\mathfrak{A}\cap\mathfrak{B}$, because of the presence of the g_λ.

From (1) we have

(3) for $\lambda \in \Lambda_0$, $s_\lambda = \sum_{i=1}^h u_i \boldsymbol{r_{i\lambda}} + f_\lambda$ for some $f_\lambda \in \mathfrak{A}S$.

Then

$$u = \sum_{\lambda\in\Lambda_0} g_\lambda s_\lambda = \sum_{\lambda\in\Lambda_0} g_\lambda \left(\sum_{i=1}^h u_i \boldsymbol{r_{i\lambda}} + f_\lambda\right) = \sum_{i=1}^h u_i \left(\sum_{\lambda\in\Lambda_0} \boldsymbol{r_{i\lambda}} g_\lambda\right) + \sum_{\lambda\in\Lambda_0} f_\lambda g_\lambda$$

and the result follows because the terms in the first sum on the right are in $(\mathfrak{A}\cap\mathfrak{B})S$ by (2) and each $f_\lambda g_\lambda \in (\mathfrak{A}\mathfrak{B})S$. The final statement follows by a straightforward induction on h. □

PROPOSITION 3.3. *Let $R \to S$ have the stable prime extension property.*

(a) If $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a local homomorphism of quasilocal rings, then the kernel is contained in the nilradical of R. Thus, if R is reduced, then the map is injective.

(b) For every prime ideal P of R, either $PS = S$ or PS contracts to P in R.

PROOF. For part (a), suppose that $a \in R$ is in the kernel and not nilpotent. By forming the quotient by a prime of R not containing a and the extension of this prime to S, we obtain an example where R and S are domains and a is a nonzero element in the kernel, using Proposition 2.1(f). Adjoin two indeterminates X, Y to both rings. Then $X^2 - aY$ is prime in $R[X, Y]$: this is true in $\mathcal{F}(Y)[X]$, where $\mathcal{F}$ is the fraction field of R, and the contraction of $(X^2 - aY)\mathcal{F}(Y)[X]$ to $(R[Y])[X]$ is generated by $X^2 - aY$, by the division algorithm for monic polynomials. The expansion of the prime $(X^2 - aY)R[X, Y]$ to $S[X, Y]$ is $X^2S[X, Y]$, a contradiction.

For part (b), suppose $Q' = PS \neq S$ contracts to Q in R. Then the map $(R/P)_Q \to (S/PS)_{Q'}$ has the stable prime extension property by Proposition 2.2(a), and so must be injective by part (a). But $(Q/P)R_Q$ is in the kernel, so that $Q = P$. □

We need one more small observation for the proof of the main theorem.

LEMMA 3.4. *Let A be an $\mathbb{N}$-graded ring with $(A_0, \mathfrak{m}_0)$ quasilocal. Denote by $\mathfrak{A} = \mathfrak{m}_0 + A_{>0}$ the unique maximal homogeneous ideal. Then every element of $A \smallsetminus \mathfrak{A}$ is a nonzerodivisor on A and on every $\mathbb{Z}$-graded A-module.*

PROOF. If $f \in A \smallsetminus \mathfrak{A}$, write $f = f_0 + f'$, with $f_0 \in A_0 \smallsetminus \mathfrak{m}_0$ and $f' \in A_{>0}$. There is some $h \in A_0$ with $hf_0 = 1$. If $fu = 0$ with $u \neq 0$, let v be the lowest degree term of u. Then $0 = hfu = v +$ higher degree terms, and so $v = 0$, a contradiction. □

THEOREM 3.5. *Let R be a reduced ring such that every maximal ideal contains only finitely many minimal primes. If $R \to S$ has the stable prime extension property, then S is flat over R.*

PROOF. First, flatness is local on the maximal ideals of S and their contractions to R. Hence, by Proposition 2.2(a), we may assume that $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is an injective local homomorphism of quasilocal rings that has the stable prime extension property, and that $(R, \mathfrak{m})$ is reduced with finitely many minimal primes $P_1, \ldots, P_h$. We proceed by induction on h.

We need only show that $\mathrm{Tor}_1^R(R/I, S) = 0$ for all ideals I of R, which is equivalent to the injectivity of $I \otimes S \to S$. Consider $\alpha : R[It] \otimes_R S \twoheadrightarrow S[It]$. It will suffice to prove the injectivity of the map α: the fact that α is injective in degree one is exactly what we need. If $h = 1$, both rings are domains containing S and, therefore, R, and so the (non)injectivity of the map α is unaffected by localizing at $R \smallsetminus \{0\}$. But then the isomorphism is clear, since I becomes either the zero ideal or the unit ideal, so α identifies with $R \otimes_R S \xrightarrow{\cong} S$ or $R[t] \otimes_R S \xrightarrow{\cong} S[t]$.

Now assume that $k \geq 2$. We shall prove $W = R \smallsetminus \bigcup_j P_j$ consists of nonzerodivisors on $R[It] \otimes_R S$. Assuming this, by Proposition 2.2(a) we may localize both rings at W preserving the stable prime extension property without affecting (non)injecivity of α and so reduce to the case where R becomes a finite product of fields, and S becomes a product of algebras over these fields. The injectivity of α is local on the prime ideals of R. But after localization, I becomes either the zero ideal or the unit ideal, and we have injectivity in either case, as above.

It remains to show that if $r \in W$, then r is not a zerodivisor on $T = R[It] \otimes_R S$. We consider the $\mathbb{N}$-grading on T induced by the usual grading on $R[It]$ (by giving R degree zero and t degree one) and giving S degree zero. Under this grading, $T_0 \cong S$ is quasilocal and thus T has a unique maximal homogeneous ideal $\mathfrak{N}$. If r is a zerodivisor on T, then we have a nonzero form ϕ in the annihilator of r in T.

Let $Q_i := P_i R[t] \cap R[It]$. Clearly, the intersection of the Q_i is also zero, since it is contained in $\bigcap_i (P_i R[t]) = (\bigcap_i P_i) R[t] = 0$. Now, $R[It] \to S \otimes_R R[It] = T$ has the stable prime extension property by Proposition 2.1(f), and since r is not in any $Q_i T$ and these ideals are prime, it follows that $\phi \in \bigcap_i Q_i T$, and it suffices to show that this intersection is zero. To prove this, we may localize at the respective homogeneous maximal ideals $\mathfrak{M}$, $\mathfrak{M}'$ of $R[It]$ and T: by Lemma 3.4, the multiplicative systems that become inverted do not contain any zerodivisors, so $\bigcap_i Q_i T$ injects into its localization at $\mathfrak{M}'$. But we may then apply Lemma 3.2 to the homomorphism $R[It]_{\mathfrak{M}} \to T_{\mathfrak{M}'}$, which has the stable prime extension property by Proposition 2.2(a), and to the ideals $Q_i R[It]_{\mathfrak{M}}$. We have

$$\bigcap_i Q_i T \subseteq \bigcap_i Q_i T_{\mathfrak{M}'} = \Big(\bigcap_i Q_{i\mathfrak{M}}\Big) T_{\mathfrak{M}'} = \Big(\bigcap_i Q_i\Big)_{\mathfrak{M}} T_{\mathfrak{M}'} = 0,$$

as required. □

THEOREM 3.6. *Let $h : R \to S$ be a ring homomorphism. If h has the stable prime extension property, then all fibers are geometrically reduced and irreducible. Under the hypothesis that all fibers are geometrically reduced and irreducible, the following are equivalent:*

(i) $R \to S$ has the stable prime extension property.
(ii) $R/\mathfrak{N}_R \to S/\mathfrak{N}_R S$ has the stable prime extension property, where $\mathfrak{N}_R$ is the nilradical of R.
(iii) For every minimal prime P of R, $R/P \to S/PS$ is flat.
(iv) For every finite intersection J of primes of R, $R/J \to S/JS$ is flat.

PROOF. The first statement was already shown in Proposition 3.1. The equivalence of (i) and (ii) is Proposition 2.2(b). If R has the stable prime extension property, then for any ideal $J \subseteq R$ that is a finite intersection of primes, R/J has finitely many minimal primes, and the map $R/J \to S/JS$ has the stable prime extension property by Proposition 2.1(f), and hence is flat by Theorem 3.5. Thus, (i) implies (iv). The implication (iv) implies (iii) is trivial. If (iii) holds, then $R/P \to S/PS$ has the stable prime extension property for every minimal prime P, and, hence, (i) follows by Proposition 2.2(c). □

4. A counterexample when a quasilocal ring has infinitely many minimal primes

We construct $R \hookrightarrow S$ local with $(R, \mathfrak{m}, K)$, R, S quasilocal, R reduced, and S not flat over R such that $R \hookrightarrow S$ has the stable prime extension property. To do this, we will construct a non-flat module satisfying $PM = M$ for every minimal prime P of R, and take S to be the Nagata idealizer $R \ltimes_R M$ of M, which is defined to be $R \oplus M$ with multiplication $(r \oplus m)(r' \oplus m') = rr' \oplus (rm' + r'm)$, so that $M^2 = 0$.

We first construct an example where R is $\mathbb{N}$-graded over a field K and S is $\mathbb{Z}$-graded. We may then localize. Let $(\Sigma, \preceq)$ denote a partially ordered set with the following properties:

(1) Σ is nonempty.
(2) For all $\sigma \in \Sigma$, the set $\{\tau \in \Sigma : \sigma \preceq \tau\}$ is finite and totally ordered. This implies that for every element τ, there is a unique minimal element τ_0 of Σ with $\tau_0 \preceq \tau$, and that every element $\tau \in \Sigma$ that is not minimal has a unique immediate predecessor, which we denote τ_-. We also write $\sigma \prec_{\mathrm{im}} \tau$ to mean that $\sigma = \tau_-$.
(3) For every $\sigma \in \Sigma$, there exist incomparable elements τ, τ' such that $\sigma = \tau_- = \tau'_-$.

The *height* $\mathfrak{h}(\sigma)$ of σ is the length of a maximal chain of elements descending from σ and is one less than the cardinality of $\{\tau \in \Sigma : \sigma \preceq \tau\}$, since this set is such a chain. We let Σ_+ denote the set of nonminimal elements of Σ. Let $(\Sigma, \preceq)$ be a partially ordered set satisfying (1), (2), and (3) above. Let K be a field. Let $\{X_\sigma : \sigma \in \Sigma_+\}$ be indeterminates over K. Let $R = K[X_\sigma : \sigma \in \Sigma_+]/\mathfrak{I}$, where $\mathfrak{I}$ is the ideal generated by the products $X_\sigma X_\tau$ where σ and τ are incomparable. We let x_σ denote the image of X_σ in R. Note that the indices of variables not in a given prime ideal Q must be linearly ordered (if two were incomparable, their product is $0 \in Q$, and so at least one of them is in Q). It follows at once that the minimal primes of R correspond bijectively to the maximal chains Γ in Σ, where Γ corresponds to $P_\Gamma = (\{x_\gamma : \gamma \notin \Gamma\}) \subset R$.

There is a K-basis for R consisting of products of powers of variables whose indices form a chain in Σ.

Example 4.1. If S is a set with two or more elements, the set Σ of finite sequences (including the empty sequence) of elements of S with the relation that $\sigma \preceq \tau$ when σ is an initial segment of τ is an example of such a partially ordered set. In this example, the empty sequence is the unique minimal element. If the empty sequence is omitted, the one element sequences are minimal. In this example, σ_- is the initial segment of σ that omits the last term of σ, and the height of σ is its length as a sequence. The minimal primes of $\mathfrak{I}$ are in bijection with $\mathbb{N}$-indexed sequences of elements of S.

In fact, if S has two elements, the poset Σ of this example is a subposet (up to relabeling) of any poset satisfying the three conditions above.

Let $\{U_\sigma : \sigma \in \Sigma\}$ be a free basis for a free R-module $\mathcal{M}$, and let M denote the quotient of $\mathcal{M}$ by the submodule spanned by the set of elements

$$\{\, U_\sigma - x_\tau U_\tau \,:\, \tau \in \Sigma \text{ and } \sigma = \tau_- \,\}.$$

Let u_σ denote the image of U_σ in M.

We shall show that R and M, suitably localized, have the required properties. We first explore what happens in the graded case. We introduce a "multigrading" as follows: the index set will be $\mathbb{Z}^{\oplus\Sigma}$, the free abelian group on the elements of Σ. The degree of x_σ is σ. The degree of u_σ is $-\sum_{\tau \preceq \sigma} \tau$. Since the defining relations $X_\sigma X_\tau$ for σ, τ incomparable and $U_{\tau_-} - x_\tau U_\tau$ for all τ are multihomogeneous, we obtain compatible gradings on R and M. These gradings yield an $\mathbb{N}$-grading on R, with $R_0 = K$, and a $\mathbb{Z}$-grading on S by summimg the components of the respective multidegrees.

We first give a concrete description of M.

PROPOSITION 4.2. *With the notations introduced above, we have the following:*

(a) $x_\sigma u_\tau = 0$ *in* M *unless* $\sigma \preceq \tau$, *and* $x_\sigma^2 u_\sigma = 0$.

(b) If

$$\tau_{k-1} \prec_{\mathrm{im}} \tau_k \prec_{\mathrm{im}} \tau_{k+1} \prec_{\mathrm{im}} \cdots \prec_{\mathrm{im}} \tau_{n-1} \prec_{\mathrm{im}} \tau_n$$

is a strict saturated chain of elements of Σ *and* a_i *is a positive integer for all* i *with* $k \leq i \leq n$, *then*

$$x_{\tau_k}^{a_k} x_{\tau_{k+1}}^{a_{k+1}} \cdots x_{\tau_n}^{a_n} u_{\tau_n} = 0$$

if any $a_i \geq 2$, *and is equal to* $u_{\tau_{k-1}}$ *if* $a_k = a_{k+1} = \cdots = a_n = 1$.

(c) The annihilator in R *of* $u_\tau \in M$ *is the monomial ideal generated by all* x_σ *such* $\sigma \npreceq \tau$, *and all products* $x_{\tau_k}^2 x_{\tau_{k+1}} \cdots x_{\tau_{n-1}} x_{\tau_n}$ *with*

$$\tau_k \prec_{\mathrm{im}} \tau_{k+1} \prec_{\mathrm{im}} \cdots \prec_{\mathrm{im}} \tau_{n-1} \prec_{\mathrm{im}} \tau_n$$

in Σ_+.

(d) A K*-basis for* M *is given by products* μu_τ, *where* μ *is a monomial in the variables* x_σ *such that* $\sigma \npreceq \tau$. *Moreover, the basis elements have mutually distinct multidegrees.*

(e) For all minimal primes P *of* R, $M = PM$. *Moreover, for any* $\sigma \in \Sigma_+$, $(\mathrm{Ann}_R\ (x_\sigma))M = M$.

(f) M *is not* R*-flat.*

PROOF. For part (a), note that if $\sigma \not\preceq \tau$, then either σ, τ are incomparable or $\tau \npreceq \sigma$. If σ, τ are incomparable, and γ is an immediate successor of τ, then σ, γ are incomparable. Then $x_\sigma u_\tau = x_\sigma(x_\gamma u_\gamma) = 0$. Similarly, if $\tau \npreceq \sigma$, there is an immediate successor of τ that is not comparable with σ, and the relation follows in the same way. This justifies the first statement. From this, we have $x_\sigma^2 u_\sigma = x_\sigma u_{\sigma_-} = 0$.

For part (b), we use induction on $n - k$. If $k = n$, the result follows at once from the final statement of part (a), and this is also true if $a_n \geq 2$. If $a_n = 1$, we may replace final part of the product, consisting of $x_{\tau_n} u_{\tau_n}$, by τ_{n-1}, and then the result follows from the induction hypothesis.

We prove (c) and (d) simultaneously. Note that the monomials specified in (c) kill u_τ by part (b). It is easy to see from part (a) that M is spanned as a K-vector space by the terms μu_τ such that all variables occurring in μ having subscripts strictly less than τ; we will call the set of such expressions $\mathcal{B}$. For a fixed $\omega \in \Sigma$, we will write $\mathcal{B}(\omega)$ for the elements $\mu u_\tau \in \mathcal{B}$ with $\tau \preceq \omega$. It is easy to recover an element of $\mathcal{B}$ in M from its multidegree: τ will be the unique largest element with a negative coefficient in the multidegree, and the exponent on each variable x_σ occurring in μ will be one greater than the coefficient of σ in the multidegree. Thus, each multigraded component of M is at most one-dimensional as a K-vector space. However, it remains to see that the elements in $\mathcal{B}$ are all nonzero in M.

Let V be the "formal" K-vector space spanned by the terms μu_τ in $\mathcal{B}$. We shall give an R-module structure to V by specifying a K-linear endomorphism θ_τ of V for every $\tau \in \Sigma_+$ such that the following two conditions hold:

(†) For all $\sigma, \tau \in \Sigma_+$, $\theta_\sigma \circ \theta_\tau = \theta_\tau \circ \theta_\sigma$,
(‡) For all $\sigma, \tau \in \Sigma_+$, if σ and τ are incomparable, $\theta_\sigma \circ \theta_\tau = \theta_\tau \circ \theta_\sigma = 0$.

These conditions give V the structure of an R-module. We then verify $V \cong M$ as R-modules in such a way that the formal basis element μu_τ corresponds to the term μu_τ in V.

Let $\mathfrak{h}(\tau) = n$, and let $\tau_0 \prec_{\mathrm{im}} \tau_1 \prec_{\mathrm{im}} \cdots \prec_{\mathrm{im}} \tau_{n-1} \prec_{\mathrm{im}} \tau_n = \tau$ be the chain of elements that are $\preceq \tau$. For each monomial μ in the x_σ for $\sigma \precneqq \tau$, let k be one plus the largest index j such that x_{τ_j} does not occur with positive degree in μ, which we take to be 0 if all the variables occur; note that k depends on μ, but we omit it from the notation for ease of reading. Then μ can be written uniquely in the form $\nu_\mu \gamma_\mu$ where

$$\begin{array}{ll} \gamma_\mu & \text{is a monomial in the variables indexed by} \quad \tau_k,\, \tau_{k+1}, \ldots,\, \tau_{n-1} \\ & \text{and each such variable divides} \ \ \gamma_\mu, \end{array}$$

and

$$\nu_\mu \quad \text{involves only variables with subscripts} \quad \tau_1,\, \ldots, \tau_{k-2}.$$

Note that if $x_{\tau_{n-1}}$ does not occur with a positive exponent, then $\nu_\mu = \mu$ and $\gamma_\nu = 1$, while if for $1 \le j \le n$ each x_{τ_j} occurs with positive exponent then $\nu_\mu = 1$ and $\gamma_\mu = \mu$. For $\sigma \in \Sigma_+$, let θ_σ be the K-linear endomorphism of V specified on the basis $\mathcal{B}$ by

$$\theta_\sigma(\mu u_\tau) = \begin{cases} x_\sigma \mu u_\tau & \text{if } \sigma \precneqq \tau, \\ \nu_\mu u_{\tau_{k-1}} & \text{if } \sigma = \tau \text{ and all exponents in } \gamma_\mu \text{ are } 1, \\ 0 & \text{if } \sigma = \tau \text{ and some exponent in } \gamma_\mu \text{ is } \ \ge 2, \\ 0 & \text{if } \sigma \not\preceq \tau. \end{cases}$$

We need to check that (†) and (‡) hold. Fix $\tau = \tau_n$ of height n. Note that θ_σ stabilizes the K-span of $\mathcal{B}(\tau)$, which we denote $W(\tau)$, and that all x_σ kill these elements unless $\sigma \preceq \tau$. It therefore suffices to consider only the variables x_{τ_i} for $1 \le i \le n$, and the u_{τ_i} for $0 \le i \le n$. To simplify notation, we shall write i instead of τ_i, so that we have $0 \prec_{\mathrm{im}} 1 \prec_{\mathrm{im}} \cdots \prec_{\mathrm{im}} n$. To verify (†) and (‡) consider the R-module $R/\mathfrak{A}$ where $\mathfrak{A}$ is the monomial ideal generated by all variables whose subscripts are not $\preceq n$ and all monomials of the form $x_k^2 x_{k+1} \cdots x_n$. The quotient has a K-basis consisting of all monomials of the form $\nu\gamma_k$ for $0 \le k \le n$ where $\gamma_k = \prod_{j=k+1}^n x_j$ and ν is a monomial in $x_1, \ldots, x_{k-1}$.

There is a vector space isomorphism between $R/\mathfrak{A}$ and $W(\tau)$ that maps $\nu\gamma_k$ to $\nu\tau_k$. It is straightforward to verify that for $1 \le j \le n$, $x_j \nu \gamma_k$ corresponds to $\theta_j(\nu\tau_k)$. It follows that (†) and (‡) hold for the θ_j acting on $W(\tau)$, and thus (†) and (‡) hold on $V = \varinjlim_\tau W(\tau)$. This gives V the structure of an R-module.

The R-linear map from the free module $\mathcal{M}$ to V such that $U_\tau \mapsto 1 \cdot u_\tau$ kills the relations defining M, and so induces an R-linear surjection $M \twoheadrightarrow V$. Since the elements in M corresponding to $\mathcal{B}$ span M, these elements are also a K-basis for M: if they were linearly dependent, their images in V would be as well. It follows that the map $M \twoheadrightarrow V$ is an R-isomorphism. This proves part (d), while part (c) now follows from the fact that $Ru_\tau \cong W(\tau) \cong R/\mathfrak{A}$.

Part (e) is clear, since every $\sigma \in \Sigma$ has at least two incomparable immediate successors τ, ω, and given any minimal prime $P = P_\Gamma$, at least one of x_τ or x_ω is in P_Γ, say x_τ, and then $u_\sigma = x_\tau u_\tau \in PM$.

It remains only to prove (f). Fix $\sigma \in \Sigma_+$, and consider the exact sequence

$$0 \longrightarrow \mathrm{Ann}_R(x_\sigma) \longrightarrow R \xrightarrow{x_\sigma} R\cdot$$

If M were R-flat then tensoring with M would yield that

$$\mathrm{Ann}_M(x_\sigma) = (\mathrm{Ann}_R(x_\sigma))M.$$

We have

$$\mathrm{Ann}_R(x_\sigma) = \big(\{x_\tau : \tau \in \Sigma_+ \text{ and } \sigma, \tau \text{ are incomparable }\}\big).$$

Suppose $\sigma \prec_{\mathrm{im}} \sigma_1$. Then $x_\sigma u_{\sigma_1}$ is a nonzero element of M in $\mathrm{Ann}_M(x_\sigma)$. Hence, if M were R-flat we would have that $x_\sigma u_{\sigma_1} = \sum_j x_{\tau_j} \mu_j u_{\omega_j}$ with τ_j incomparable to σ and μ_j a monomial for each j. Using the $\mathbb{Z}^{\oplus \Sigma}$ grading, and the fact that each piece is a one-dimensional K-vector space, there must be an equality of the form $x_\sigma u_{\sigma_1} = x_\tau \mu u_\omega$, with σ, τ incomparable, and $\tau \preceq \omega$, for otherwise the right-hand side would be zero. Using the module structure as computed above, we can write $x_\tau \mu u_\omega = \mu_1 u_{\omega_0}$ with $\omega_0 \preceq \omega$. Thus we have $x_\sigma u_{\sigma_1} = \mu_1 u_{\omega_0}$ in M. Since the elements of $\mathcal{B}$ form a basis, we must have $\sigma_1 = \omega_0$. But then, we have $\sigma \preceq \sigma_1 \preceq \omega$, and $\tau \preceq \omega$, so σ and τ are comparable, a contradiction. □

THEOREM 4.3. *Let R and M be as above. Let $\mathfrak{m}$ be the homogeneous maximal ideal of R. Then $R_\mathfrak{m} \to (R \ltimes M)_{\mathfrak{m}(R\ltimes M)}$ is a local homomorphism of quasilocal rings, with source $R_\mathfrak{m}$ reduced, that satisfies the stable prime extension property, but is not flat.*

PROOF. Since R is reduced, $R_\mathfrak{m}$ is as well. Observe that $R \to R \ltimes M$ has the stable prime extension property by Proposition 2.2(c), since for any minimal prime P, we have $(R \ltimes M)/P(R \ltimes M) \cong R/P$. Then, by Proposition 2.2(a), the map $R_\mathfrak{m} \to (R \ltimes M)_{\mathfrak{m}(R\ltimes M)}$ has the stable prime extension property as well. It remains only to show that $M_\mathfrak{m}$ is not flat over $R_\mathfrak{m}$. We can argue in the same way: if $M_\mathfrak{m}$ were flat, we would have for every $\sigma \in \Sigma_+$ that $\mathrm{Ann}_{M_\mathfrak{m}}(x_\sigma)/(\mathrm{Ann}_R(x_\sigma))M_\mathfrak{m} = 0$, and since localization commutes both with taking the annihilator of an element and with forming quotients, this would imply $\big(\mathrm{Ann}_M(x_\sigma)/(\mathrm{Ann}_R(x_\sigma))M\big)_\mathfrak{m} = 0$. To see that this does not happen we may apply Lemma 3.4, using the $\mathbb{N}$-grading on R and the $\mathbb{Z}$-grading on M introduced before the statement of Proposition 4.2. □

5. Intersection Flatness

Recall (cf. [**6**, p. 41]) where the terms "intersection-flat" and "∩-flat are used) that an R-module S is *intersection flat* if S is R-flat and for any finitely generated R-module M,

(#) for every family of submodules $\{M_\lambda\}_{\lambda\in\Lambda}$ of M we have $S \otimes_R (\bigcap_\lambda M_\lambda) = \bigcap_\lambda (S \otimes_R M_\lambda)$ when both are identified with their images in $S \otimes_R M$.

Note that, quite generally, there is an obvious injective map from the first module to the second. Here, S will usually be an R-algebra in which case we also say the homomorphism $R \to S$ is *intersection flat*. A flat homomorphism satisfies the property (#) for any module M whenever Λ is finite.

In particular, if S is intersection flat, then for every family of ideals $\{I_\lambda : \lambda \in \Lambda\}$ of R the equality $(\bigcap_\lambda I_\lambda)S = \bigcap_\lambda (I_\lambda S)$ holds. We shall say that S (or $R \to S$ in the algebra case) is *intersection flat for ideals* if this condition holds, i.e., if (#) holds when $M = R$. We caution the reader that the definition of "intersection flat" given in [**1**] is the notion that we call "intersection flat for ideals" here. Note that when $M = R$, we may identify $I \otimes_R S$ with IS.

For the most part, it is intersection flatness for ideals that we need here for results on the stable prime extension property (and, as mentioned above, is the definition used in [**1**]). As is shown in Proposition 5.5(b) below, if $\phi : R \to S$ is module-free (i.e., S is a free R-module) then ϕ is intersection flat [**6**, p. 41]. Proposition 5.5 below collects some facts about intersection flatness. Note that intersection flatness for the Frobenius endomorphism mapping a regular ring to itself is studied in [**7**, 5.3] and in [**8**, §9].

Example 5.1. Let K be a field and let $R = K[x]_{\mathfrak{m}}$ where $\mathfrak{m} = xK[x]$. Let f be a formal power series in x that is not in the fraction field $K(x)$ of $K[x]$. Then $K[\![x]\!]$ is intersection flat for ideals over R, since the intersection of any infinite family of ideals is (0) in both rings. However, $K[\![x]\!]$ is *not* intersection flat over R. To see this, let f_n denote unique the polynomial of degree at most n that agrees with f modulo $x^{n+1}K[\![x]\!]$. Let $M = R^{\oplus 2}$, and let $M_n = R(1, f_n) + R(0, x^{n+1})$. When we tensor with $\widehat{R} = K[\![x]\!]$, then $\widehat{R} \otimes_R M_n$ may be identified with the submodule of $\widehat{R}^{\oplus 2}$ spanned by $(1, f)$ and $(0, x^{n+1})$. The intersection of these is $\widehat{R}(1, f)$. But the intersection of the M_n in R^2 is 0: if $(g, h) \in M_n$, then $h - fg \in \mathfrak{m}^n$, so if (g, h) were a nonzero element of the intersection, we would have that $f = g/h$ would be rational over $K[x]$.

Example 5.2. $K[\![x, y]\!]$ is not even intersection flat for ideals over $R = K[x, y]_{\mathfrak{m}}$, where $\mathfrak{m}$ is the maximal ideal (x, y) in the polynomial ring $K[x, y]$. Let $f \in K[\![x]\!]$ be transcendental over $K[x]$ and let f_n be as in Example 5.1. Then the ideals $I_n = (y - f_n, x^{n+1})R$ when extended to $\widehat{R} = K[\![x, y]\!]$ agree with $(y - f, x^{n+1})\widehat{R}$, and so their intersection is $(y - f)\widehat{R}$. But their intersection in R is 0, for if $0 \neq G(x, y) \in R$ were in the intersection we may clear the denominator and assume that $G \in K[x, y]$, and then we would have $G(x, f) = 0$, contradicting the transcendence of f.

Discussion 5.3. Note that a family of R-submodules of M/N has the form M_λ/N where the M_λ are submodules of M containing N, and we have

$$\bigcap_\lambda (M_\lambda/N) = \big(\bigcap_\lambda M_\lambda\big)/N \quad \text{and so}$$

$$\bigcap_\lambda \big(S \otimes_R (M_\lambda/N)\big) = \bigcap_\lambda \big((S \otimes_R M_\lambda)/(S \otimes_R N)\big) = \Big(\bigcap_\lambda (S \otimes_R M_\lambda)\Big)/(S \otimes_R N).$$

Under the assumption that $R \to S$ is intersection flat, this identifies with

$$\left(S \otimes_R \Big(\bigcap_\lambda M_\lambda\Big)\right)\Big/(S \otimes_R N) = S \otimes_R \left(\Big(\bigcap_\lambda M_\lambda\Big)\Big/N\right) = S \otimes_R \big(\bigcap_\lambda (M_\lambda/N)\big),$$

where the equalities are canonical identifications of submodules of $S \otimes_R M$ or of $S \otimes_R (M/N)$.

PROPOSITION 5.4. *Let R be a ring, and S be an R-module. The following are equivalent.*

(i) S is an intersection flat R-module.
(ii) For every finitely generated R-module M, the property (#) holds.
(iii) For every finitely generated free R-module M, the property (#) holds.
(iv) For every finitely generated R-module M and any family of submodules $\{M_\lambda : \lambda \in \Lambda\}$ such that $\bigcap_\lambda M_\lambda = 0$, we have $\bigcap_\lambda (S \otimes_R M_\lambda) = 0$.

PROOF. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear. The equivalence between (ii) and (iv) is immediate from Discussion 5.3 by replacing M by M/N and every M_λ by M_λ/N. Similarly, for (iii) $\Rightarrow$ (ii), given a module M and a family of submodules, we map a free module $G \twoheadrightarrow M$ and work with the family consisting of the inverse images of the M_λ in G. We may then apply Discussion 5.3. It remains to show (ii) $\Rightarrow$ (i); i.e., that (ii) implies that S is flat.

It suffices to show that for every finitely generated ideal $I = (f_1, \dots, f_n)R$ of R, we have that $\mathrm{Tor}_1^R(R/I, S) = 0$. This follows if every relation $\sum_{i=1}^n f_i s_i = 0$ is an S-linear combination of relations on $f_1, \dots, f_n$ over R. We prove this by induction on n. If $n = 1$ and $f_1 s_1 = 0$, consider the submodules spanned by $(1, 0)$ and $(1, f_1)$ in R^2. Their intersection is $\mathrm{Ann}_R(f_1) \times 0$. If we expand to S, the intersection is $\mathrm{Ann}_S(f_1) \times 0$, while the expansion of the intersection is $(\mathrm{Ann}_R(f_1))S \times 0$, from which the result follows. Now suppose we know the result for $f_1, \dots, f_{n-1}$, $n \geq 2$. The relation $\sum_{i=1}^n f_i s_i = 0$ implies that

$$(\star) \qquad -f_n s_n = \sum_{i=1}^{n-1} f_i s_i$$

and this is element is in $(f_1, \dots, f_{n-1})S \cap f_n S = \big((f_1, \dots, f_{n-1})R \cap f_n R\big)S$, and so the equation in $(\star)$ arises as an S-linear combination, say with coefficients $t_j \in S$, $1 \leq j \leq h$, of equations

$$(\star) \qquad -f_n r_{nj} = \sum_{i=1}^{n-1} f_i r_{ij}$$

where the $r_{ij} \in R$. These may be rewritten as relations on $f_1, \dots, f_n$ with coefficients r_{ij} in R. After multiplying these relations by the t_j, adding, and subtracting the sum from the original relation, we get a new relation on $f_1, \dots, f_n$ with coefficients in S, say $\sum_{i=1}^{n-1} f_i u_i + f_n u_n = 0$, in which both $\sum_{i=1}^{n-1} f_i u_i = 0$ and $f_n u_n = 0$. Using the induction hypothesis for the former and the case $n = 1$ for the latter, we see that this relation is an S-linear combination of relations over R. $\square$

PROPOSITION 5.5. *Let R be a ring.*

(a) If $R \to S$ is intersection flat (respectively, intersection flat for ideals) and T is an S-module that is intersection flat (respectively, intersection flat for ideals) over S, then T is intersection flat (respectively, intersection flat for ideals) over R.
(b) Direct sums and direct summands of modules that are intersection flat (respectively, intersection flat for ideals) are intersection flat (respectively, intersection flat for ideals).

(c) *If R is Noetherian, arbitrary products of modules that are intersection flat (respectively, intersection flat for ideals) are intersection flat (respectively, intersection flat for ideals).*
(d) *The formal power series ring in finitely many variables over a Noetherian ring is intersection flat.*
(e) *Let R be a complete local Noetherian ring and S an R-flat algebra. If S/IS is $\mathfrak{m}$-adically separated for every ideal I of R, then S is intersection flat for ideals. If $S \otimes_R M$ is $\mathfrak{m}$-adically separated for all finitely generated R-modules M then $R \to S$ is intersection flat. In particular, if S is Noetherian and $\mathfrak{m}S$ is contained in the Jacobson radical of S, then $R \to S$ is intersection flat.*
(f) *If S is intersection flat (respectively, intersection flat for ideals) over R and W is a multiplicative system in S such that no element of W is a zerodivisor on S/IS for any ideal I of R, then $W^{-1}S$ is intersection flat (respectively, intersection flat for ideals) over R. In particular, if we take S to be a polynomial ring in an arbitrary set of variables over R and W to consist of a set of polynomials, each of which has a set of coefficients that generates the unit ideal of R, then $W^{-1}S$ is intersection flat over R.*

Proof. The arguments dealing with the case with the property "intersection flat for ideals" are identical with those for the module case, and are not given separately, except for a remark in the proof of part (e).

Part (a) is immediate from the definition, and part (b) is a completely straightforward consequence of the fact that for a direct sum $S = \bigoplus_\mu S_\mu$, we have

$$(\bigcap M_\lambda) \otimes_R (\bigoplus_\mu S_\mu) \cong \bigoplus_\mu (\bigcap_\lambda M_\lambda) \otimes_R S_\mu,$$

which will be equal to

$$\bigcap_\lambda (M_\lambda \otimes_R (\bigoplus_\mu S_\mu)) \cong \bigcap_\lambda (\bigoplus_\mu (M_\lambda \otimes_R S_\mu)) \cong \bigoplus_\mu (\bigcap_\lambda (M_\lambda \otimes_R S_\mu)).$$

To prove (c), note that over a Noetherian ring, an arbitrary product of flat modules is flat, by Chase's theorem [**2**, Theorem 2.1]. It suffices to prove the property (#) for an arbitrary finitely generated R-module M (or when $M = R$ in the ideal case). This follows from the observation that for a finitely presented R-module $N = M_\lambda$, the map $N \otimes_R (\prod_\mu S_\mu) \to \prod_\mu (N \otimes_R S_\mu)$ is an isomorphism. (Using the finite presentation of N, we reduce to the case where $N = \bigoplus_{i=1}^k Re_i$ is free. If an element of $\prod_\mu (N \otimes_R S_\mu)$ has μ-coordinate $\sum_{i=1}^k s_{i,\mu} e_i$, we let it correspond to $\sum_{i=1}^k \sigma_i e_i$ in $N \otimes_R \prod_\mu S_i$, where σ_i has μ-coordinate $s_{i,\mu}$.)

Part (d) is immediate from part (c), since these power series rings, as R-modules, are products of copies of R.

To prove part (e), we note that since flatness implies that extension commutes with finite intersection, we may assume the family M_λ is closed under finite intersection. Let N denote the intersection of the family. Chevalley's lemma implies that for every $C \in \mathbb{N}$, there exists $\lambda_C \in \Lambda$ such that $M_{\lambda_C} \subseteq N + \mathfrak{m}^C M$. It follows that

$$S \otimes_R M_{\lambda_C} \subseteq S \otimes_R N + S \otimes_R \mathfrak{m}^C M \subseteq S \otimes_R N + \mathfrak{m}^C (S \otimes_R M).$$

Since $S \otimes_R (M/N) \cong (S \otimes_R M)/(S \otimes_R N)$ is $\mathfrak{m}$-adically separated by hypothesis, the intersection of the modules on the right-hand side as C varies is $S \otimes_R N$, so the intersection of $\{S \otimes_R M_\lambda\}$ is $S \otimes_R N$. In the ideal case, we only need that for every ideal I of R, S/IS is $\mathfrak{m}$-adically separated. For the last statement of (e), we simply recall that every finitely generated S-module is separated with respect to the Jacobson radical of S.

For part (f), note that no element of W is a zerodivisor on any module of the form $S \otimes_R N$, where N is an R-module: it suffices to consider finitely generated R-modules N, and these have finite filtrations in which the factors have the form R/I.

Now consider an element of $\bigcap_\lambda (W^{-1}S \otimes_R M_\lambda)$ in $W^{-1}S \otimes_R M$: after multiplying by a unit from the image of W, we may assume this element is the image of an element $u \in S \otimes_R M$. Then $u/1$ is in every $W^{-1}(S \otimes_R M_\lambda)$, and since W consists of nonzerodivisors on $S \otimes_R (M/M_\lambda)$, we have that u is in every $S \otimes_R M_\lambda$. By the hypothesis on S, this implies that $u \in \big(S \otimes_R (\bigcap_\lambda M_\lambda)\big)$, and so $u/1 \in W^{-1}S \otimes_R (\bigcap_\lambda M_\lambda)$, as required. □

Note that part (e) of Proposition 5.5 considerably strengthens the result of [**7**, Proposition 5.3].

Proposition 5.6. *If $R \to S$ is intersection flat for ideals and P is the intersection of a family of primes $\{Q_\lambda \ : \ \lambda \in \Lambda\}$ in R such that $S/(Q_\lambda S)$ is zero or a domain for each λ, then S/PS a domain.*

Proof. If $fg \in PS$, then for each λ, we have either $f \in Q_\lambda S$ or $g \in Q_\lambda S$. Set $\Lambda = \Lambda_f \cup \Lambda_g$, where $f \in Q_\lambda S$ for $\lambda \in \Lambda_f$ and $g \in Q_\lambda S$ for $\lambda \in \Lambda_g$. Then, if $A = \bigcap_{\lambda \in \Lambda_f} Q_\lambda$ and $B = \bigcap_{\lambda \in \Lambda_g} Q_\lambda$ we have radical ideals A, B for which $A \cap B = P$. We have $f \in AS$ and $g \in BS$. But one of A, B must be P, so $f \in PS$ or $g \in PS$. □

We recall that a *Hilbert ring* is a ring in which every prime ideal is an intersection of maximal ideals.

Proposition 5.7. *If R is a Hilbert ring, $R \to S$ is intersection flat for ideals, and for every maximal ideal $\mathfrak{m}$ of R, $S/\mathfrak{m}S$ is zero or a domain, then R has the prime extension property.*

Proof. This is immediate from Proposition 5.6, since every prime is an intersection of maximal ideals by definition. □

Using this, we can give some versions of Proposition 3.1 that refer only to closed fibers.

Proposition 5.8. *Let R be a Hilbert ring, $R \to S$ be module-free, and suppose that for every maximal ideal $\mathfrak{m}$ of R, the fiber $\kappa_\mathfrak{m} \otimes_R S$ is a geometrically reduced and irreducible $\kappa_\mathfrak{m}$-algebra. Then $R \to S$ has the stable prime extension property.*

Proof. Since $R \to S$ is module-free, by base change

$$R[X_1, \ldots, X_n] \to S \otimes_R R[X_1, \ldots, X_n] \cong S[X_1, \ldots, X_n]$$

is module-free as well, and hence intersection flat. By the Hilbert hypothesis, any maximal ideal $\mathfrak{M}$ contracts to a maximal ideal $\mathfrak{m}$ of R. We then have that

$$\frac{S[X_1, \ldots, X_n]}{\mathfrak{M}S[X_1, \ldots, X_n]} \cong \kappa_\mathfrak{M} \otimes_R S \cong (\kappa_\mathfrak{m} \otimes_R S) \otimes_{\kappa_\mathfrak{m}} \kappa_\mathfrak{M}$$

and by the hypothesis on the fibers, this quotient is either a domain or zero. By Proposition 5.7, $R[X_1, \dots, X_n] \to S[X_1, \dots, X_n]$ then has the prime extension property, as required. □

Corollary 5.9. *Let R be a finitely generated algebra over an algebraically closed field, and let S be a nonzero module-free R-algebra. Suppose that for every maximal ideal $\mathfrak{m}$ of R, we have $S/\mathfrak{m}S$ is a domain. Then $R \to S$ has the stable prime extension property.*

Proof. In this case R is a Hilbert ring, and every residue field is algebraically closed. Thus, if the fiber $\kappa_{\mathfrak{m}} \otimes_R S$ is a domain or zero, then it is a geometrically reduced and irreducible $\kappa_{\mathfrak{m}}$-algebra, by, e.g., [**3**, Propositions 4.5.1 and 4.6.1]. Proposition 5.8 then applies. □

Theorem 5.10. *Let K be an algebraically closed field and let S be an $\mathbb{N}$-graded K-algebra that is a domain. Let $F_1, \dots, F_n$ be a regular sequence of forms in S generating a prime ideal Q of S. Then the K-algebra map of the polynomial ring $R = K[X_1, \dots, X_n] \to S$ such that $X_i \mapsto F_i$, $1 \le i \le n$, has the stable prime extension property.*

Proof. The hypothesis is stable under adjoining finitely many indeterminates to both rings: we may use these to enlarge the sequence $F_1, \dots, F_n$. Thus, it suffices to prove the prime extension property under the given hypotheses. The hypothesis implies that S is free over R. Indeed, fix a homogeneous basis for S/QS over K. These elements will span S over R by the graded version of Nakayama's lemma. They have no relations by induction on n: given a nonzero relation we may factor out the highest power of F_1 occurring in all coefficients, since F_1 is a nonzerodivisor in S, and then we obtain a nonzero relation on the images of these generators working over S/F_1S and R/X_1R. Since every maximal ideal of R has the form $(X_1 - c_1, \dots, X_n - c_n)$ for $c_1, \dots, c_n \in K$, we know that the expansion of any maximal ideal of R to S has the form $(F_1 - c_1, \dots, F_n - c_n)$. The result now follows from Corollary 5.9 and [**1**, Proposition 2.8]. □

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