QUASILENGTH, LATENT REGULAR SEQUENCES, AND CONTENT OF LOCAL COHOMOLOGY

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0. INTRODUCTION

We introduce the notion of *quasilength*. Let M be a finitely generated module over a ring R and let $\underline{x} = x_1, \dots, x_d$ be a sequence of elements of R. Neither R nor M needs to be Noetherian. Let $I = (\underline{x})R$. The I-quasilength of M is the least number of factors in a finite filtration of M by cyclic modules each of which is a homomorphic image of R/I.

We use the notion of quasilength to introduce a nonnegative real number $\mathfrak{h}_{\underline{x}}(M)$ that is intended heuristically as a "measure" of the local cohomology module $H^d_I(M)$. It is defined as a limit or lim inf of normalized quasilengths: see §2. When $\underline{x}$ is understood from context we shall also refer to this real number as the $\mathfrak{h}$-*content* of M. The positivity of $\mathfrak{h}_{\underline{x}}(R)$ gives a necessary condition for there to exist a map of R to a Noetherian ring such that the x_i map to generators of an ideal of height d.

We use these ideas to give conditions that may possibly characterize when a sequence $x_1, \dots, x_d$ of elements of a ring R has the property that there exists an R-algebra S such that $x_1, \dots, x_d$ is a regular sequence on S. We call such a sequence of elements a *latent regular sequence* in R. We also consider sequences such that there exists an R-module M on which the sequence is regular: we refer to these as *latent regular sequences for modules.* We do not know whether every latent regular sequences for modules is a latent regular sequence.

We also introduce two versions of the notion of a *q-sequence*: these sequences are defined in terms of quasilength and may characterize the latent regular sequences.

One motivation for our study is that these ideas ought to be useful in investigating the existence of big Cohen-Macaulay algebras over local rings, including the mixed characteristic case. Another is that results on $\mathfrak{h}$-content may well be helpful in studying the direct summand conjecture, and related conjectures.

Let Λ be either a field K or a discrete valuation domain (V, pV) of mixed characteristic. Let $X_1, \dots, X_d, Y_1, \dots, Y_d$ be indeterminates over Λ. We define

$$f = f_{d,t} = X_1^t \cdots X_d^t - \sum_{j=1}^{d} Y_j X_j^{t+1}.$$

Let

$$R = R_{d,t} = \Lambda[X_1, \dots, X_d, Y_1, \dots, Y_d]/(f_{d,t}).$$

(In mixed characteristic, one may also consider a variant definition by replacing R by $R/(X_1 - p)$.) The direct summand conjecture follows if one can prove that $\mathsf{h}_{\underline{x}}(R) = 0$. This is a weakening of the condition that $H^d_{\underline{x}}(R) = 0$.

1. QUASILENGTH

Let R be a ring, M an R-module, and I a finitely generated ideal of R. We define M to have *finite I-quasilength* if there is a finite filtration of M in which the factors are cyclic modules killed by I, so that the factors may be viewed as cyclic (R/I)-modules, The *I-quasilength* of M is then defined to be the minimum number of factors in such a filtration. If M does not have finite I-quasilength, we define its I-quasilength to be $+\infty$. We denote the I-quasilength of M over R as $\mathcal{L}_I^R(M)$. The ring R and/or the ideal I may be omitted from the terminology and notation if they are clear from context. We denote the least number of generators of M over R as $\nu_R(M)$ or simply $\nu(M)$, and the length of M over R as $\lambda_R(M)$ or simply $\lambda(M)$.

Here are some basic properties of I-quasilength.

Proposition 1.1. *Let R be a ring, I a finitely generated ideal of R, and M an R-module.*

(a) *M has finite I-quasilength if and only if M is finitely generated and killed by a power of I. In fact, $\nu(M) \le \mathcal{L}_I(M)$, and $I^{\mathcal{L}_I(M)}$ kills M.*

(b) *If M is killed by I, $\mathcal{L}_I(M) = \nu_R(M) = \nu_{R/I}(M)$.*

(c) *If I is maximal, then $\mathcal{L}_I(M)$ is finite if and only if M is killed by a power of I and has finite length as an R-module, and then $\mathcal{L}_I(M) = \lambda(M)$.*

(d) *Assume that $0 \to M' \to M \to M'' \to 0$ is exact. If M' and M'' have finite I-quasilength then so does M, and $\mathcal{L}(M) \le \mathcal{L}(M') + \mathcal{L}(M'')$. If M has finite I-quasilength then M'' does as well, and $\mathcal{L}(M'') \le \mathcal{L}(M)$. If M has finite I-quasilength, then M' has finite I-quasilength if and only if it is finitely generated.*

(e) *If M has a finite filtration in which every factor has finite I-quasilength then M has*

(f) *If M has finite I-quasilength, $\mathcal{L}(M) \leq \sum_j \nu(I^j M/I^{j+1}M)$ (the sum is finite since $I^j M = 0$ for all $j \gg 0$).*

(g) *If S is an R-algebra and M has finite I-quasilength, then $\mathcal{L}^S_{IS}(S \otimes_R M) \leq \mathcal{L}^R_I(M)$.*

(h) *$\mathcal{L}_I(M) = 0$ if and only if $M = 0$.*

(i) *If $I = P$ is prime, $\mathcal{L}_P(M)$ is at least the length of M_P as an R_P-module.*

Proof. Given filtrations of M' and M'', the filtration of M' together with the inverse image of the filtration of M'' in M yields a filtraton of M whose factors are the union of the sets of factors from the filtrations of M' and M''. This proves the first statement in (d). The second statement follows from the fact that a filtration of length h on M whose factors are cyclic R/I-modules induces a quotient filtration on M'' of at most the same length whose factors are also cyclic R/I-modules. We postpone the proof of the third statement until we have proved part (a).

Part (e) follows from the first statement in part (d) by an immediate induction on the length of the filtration.

To prove (b), note that if $u_1, \ldots, u_h$ generate M, then the submodules $Ru_1 + \cdots + Ru_j$ give a filtration of M whose whose factors are cyclic modules killed by I. Therefore, $\mathcal{L}(M) \leq \nu(M)$.

If M has finite quasilength then, since the factors are all cyclic modules, lifitngs of the generators of the factors to M generate M. This shows that, in general, $\nu(M) \leq \mathcal{L}(M)$. It follows that $\nu(M) = \mathcal{L}(M)$ when I kills M.

If $0 \to Q' \to Q \to Q'' \to 0$ is exact, $\mathfrak{A}$ kills Q', and $\mathfrak{B}$ kills Q'', then $\mathfrak{A}Q \subseteq Q'$ and so $\mathfrak{A}\mathfrak{B}$ kills Q. It follows that the product of the annihilators of the factors in a finite filtration of M kills M. If $\mathcal{L}_I(M)$ is finite, we therefore have that $I^{\mathcal{L}_I(M)}$ kills M. On the other hand $\nu(M) \leq \mathcal{L}_I(M)$. Part (a) is now proved except for the "if" part. But if M is finitely generated and killed by I^h, then every $I^j M$ is finitely generated (since I and, hence, each I^j is). From part (e),

$$\mathcal{L}_I(M) \leq \sum_{j=0}^{h} \mathcal{L}_I(I^j M/I^{j+1}M) = \sum_{j=0}^{h} \nu(I^j M/I^{j+1}M)$$

by part (b), and this completes the proofs of both (a) and (f). The third statement in part (d) also follows, because whatever power of I kills M also kills M'.

If M is finitely generated and killed by a power of I, then each $I^j M$ is finitely generated, Part (c) is clear, because when $I = m$ is maximal, the only nonzero cyclic (R/I)-module is R/m.

Part (g) is clear because given any finite filtration of M by modules M_j such that every M_j/M_{j-1} is cyclic and killed by I, we may use the images of the $S \otimes_R M_j$ to give a filtration of $S \otimes_R M$ whose factors are cyclic S-modules killed by IS, and its length is at most the length of the original filtration.

Part (h) is obvious. Part (i) follows from parts (g) and (c) by choosing $S = R_P$. $\square$

Let $\{R_\lambda\}_{\lambda \in \Lambda}$ be a direct limit system of rings with direct limit R. Let $\underline{x}_\lambda$ by a sequence of n elements in R_λ and suppose that under every direct limit system map $R_\lambda \to R_\mu$, $\underline{x}_\lambda$ maps to $\underline{x}_\mu$. Then there is a correspondingly indexed sequence $\underline{x}$ in the direct limit R, and if every $\underline{x}_\lambda$ is a regular sequence on R_λ, then the $\underline{x}$ is a regular sequence on R. We shall say that $\underline{x}$ arises as a direct limit of regular sequences in this situation.

Suppose that we also have a direct limit system of abelian groups $\{M_\lambda\}_{\lambda \in \Lambda}$ with direct limit M, where every M_λ is an R_λ-module and, if $\lambda \leq \mu$, then $M_\lambda \to M_\mu$ is R_λ-linear when M_μ is given an $R_|ambda$-module structure via restriction of scalars. Then M is an R-module, and if $\underline{x}_|ambda$ is a regular sequence on M_λ for all λ, then $\underline{x}$ is a possibly improper regular sequence on M. If $(\underline{x})M \neq M$ we say that $\underline{x}$ arises as a direct limit of regular sequences in this situation as well.

Proposition 1.2. *Let R be a ring, let $I = (x_1, \ldots, x_d)$ be an ideal of R, and let M and N be generated R-modules. Let $I_t = (x_1^t, \ldots, x_d^t)$*

(a) *If N is finitely generated and killed by I_t, then $\mathcal{L}_I(N) \leq \nu(N/IN)d^t$. In fact, N has a filtration by d^t modules such that every factor is a homomorphic image of N/IN.*

(b) *If $x_1, \ldots, x_d$ is a regular sequence on M and $N = M/I_tM$, then N has a filtration by d^t modules each of which is isomorphic to $N/IN \cong M/IM$.*

(c) *If R is a Noetherian ring in which $x_1, \ldots, x_d$ is a regular sequence on M, or if $\underline{x}$ arises as a direct limit of regular sequences in Noetherian rings, then $\mathcal{L}_I(R) = d^t$.*

Proof. (a) N has a filtration by d^t modules each of which is a homomorphic image of N/IN. To see this, note that N has a filtration

$$N \supseteq x_1N \supseteq x_1^2N \supseteq \cdots \supseteq x_1^{t-1}N \supseteq x_1^tN = 0$$

with t factors, each of which is a homomorphic image of N/x_1N, since there is a surjection $N/x_1N \twoheadrightarrow x_1^jN/x_1^{j+1}N$ induced by multiplication by x_1^j on the numerators. We may use induction on t to complete the proof: each of these factors will have a filtration with d^{t-1} factors killed by $(x_2, \ldots, x_d)R$ as well as x_1, and each of these factors will be of a homomorphic image of N/x_1N and therefore of N/IN. The result now follws from parts

(b) With x_1 not a zerodivisor, the surjection $M/x_1M \twoheadrightarrow x_1^{t-1}M/x_1^tM$ induced by multiplication by x_1^{t-1} is an isomorphism. This yields a filtration of M/x_1^tM by factors each isomorphic to M/x_1M. The result now follows from by induction on d and the fact that $x_2, \ldots, x_d$ is a regular sequence on each of these factors.

(c) If R is Noetherian case and $\mathcal{L}_I(R) < t^d$, we shall show that R cannot have a module M on which $x_1, \ldots, x_d$ is a regular sequence. First, we may localize at a minimal prime of $\mathrm{Ann}_R M + I$. Second, we may replace R by $R/Ann_R M$. Thus, we may assume that M is faithful and $M/(x_1, \ldots, x_d)M$ has finite length. Tensor the bad filtration for R with M to get a bad filtration of M. This gives an incorrect length estimate. Finish. $\square$

Bring in the I^{lim} condition here.

2. A HEURISTIC MEASURE OF LOCAL COHOMOLOGY

Suppose that M is a finitely generated module over the ring R, $\underline{x} = x_1, \ldots, x_d$, and $I = (x_1, \ldots, x_n)R$. We define $I_t = (x_1^t, \ldots, x_d^t)$,

$$M_t = \frac{M}{\bigcup_k \big((I_{t+k}M) :_M (x_1 \cdots x_d)^k}\big),$$

and

$$\mathsf{h}_{\underline{x}}(M) = \lim_{t\to\infty} \inf\{\frac{\mathcal{L}_I(M_s)}{s^d} : s \geq t\}.$$

Give comments that the sequence of infs is bounded above and non-decreasing. Thus, the limit exists. Also discuss a limit, the same as the lim inf, when t gets large in a multiplicative sense. Is there a limit without the multiplicative restriction?

Let's change this with variable t_i. So the definition is the limit if M/I_t^{lim} where t is a d-tuple and approaches infinity in a multiplicative sense.

Hope: replacing one x be a power does not change this version of content. This appears OK.

Hope: two sets of generators with the same number of elements and same radical give the same answer.

Heuristically, we view this number as a numerical measure of how small the local cohomology $H^d_(\underline{x})(M)$ is. However, *a priori* it depends on knowing the explicit generators $x_1, \ldots, x_d$ and M. We do not know what happens, for example, when we change the generators of the ideal $(x_1, \ldots, x_d)R$.

Lemma 2.1. *Let (R, m, K) be a local ring of dimension r and let c be an element of R such that $\dim(R/cR) = r-1$, i.e., c is part of a system of parameters for R. Then there exists a constant b such that $\ell(\mathrm{Ann}_{R/m^t}c) < bt^{r-1}$ for all $t \geq 1$.*

Proof. Let $S = R/m^t$ and $J = \mathrm{Ann}_S c$. Then $S/J \cong cS$ and so $\ell(J) = \ell(S) - \ell(cS) = \ell(S/cS)$, which, as a function of t, is the Hilbert function of R/cR. $\square$

State result on colon-killers: (2.2).

Theorem 2.3. *Let R be any ring and $x_1, \dots, x_d$ elements of R. Suppose that for some map of R to a Noetherian ring S, $(x_1, \dots, x_d)S$ is a proper ideal of height d. Then $\mathfrak{h}_{\underline{x}}(R) > 0$. (One can say that this is at last the multiplicity of the system of parameters modulo the length of their quotient under mild conditions on R.)*

Proof. If $(x_1, \dots, x_d)S$ has height d we may localize at a minimal prime of this ideal of height d, complete, and then kill a minimal prime so as to obtain a complete local domain of dimension d for which the images of $x_1, \dots, x_d$ are a system of parameters. Then the length of S/I_tS is asymptotic to at^d for some positive real constant a. Also, there is a fixed nonzero $c \in S$ such that $cJ_tS \subseteq I_tS$ for all t (S has a colon-killer). By the lemma, this bounds $\ell(J_tS/I_tS)$ by a constant times t^{d-1}.

The filtration of R/J_tR with $\mathcal{L}_I(R/J_t)$ factors induces a filtration of S/J_tS with the same number of factors, each a homomorphic image of S/I. This shows that $\ell(S/J_tS) \leq C\mathcal{L}_I(R/J_t)$ where $C = \ell\big(S/I\big)$. But then $\ell(S/I_tS) - b_1t^{d-1} \leq C\mathcal{L}_P(R/J_t)$, from which the result follows. $\square$

Theorem 2.4. *For an equicharacteristic local ring of dimension d, $h^d_{\underline{x}}(R) = 1$.*

Proof. The equal characterisitic 0 case follows from the equal characteristic p case by standard methods.

In characteristic p: we may map a counterexample to one which is a complete local domain. As in (2.3), we may ignore $I_t^{\lim}/I_t$ in the limit.

Suppose $x_1, \dots, x_d$ is a system of parameters of a local ring of characteristic p. We want to show that $\mathcal{L}(R/x_1^t, \dots, x_d^t)) \geq t^d$. Suppose, to the contrary, that it is strictly smaller, i.e., $\leq t^d - 1$. We can write the length of $R/(x_1^N, \dots, x_d^N)$ as $N^d(\mu + \epsilon_N)$ where μ is the mulltiplicity of the system of parameters $x_1, \dots, x_d$ and $\epsilon_N \to 0$ as $N \to \infty$.

Consider the bad filtration with at most $t^d - 1$ factors, each of which is a homomorphic image of $R/(x_1, \dots, x_d)$, and apply iterated Frobenius, F^e. Then we obtain a filtration of $R/x_1^{qt}, \dots, x_d^{qt})^{\lim}$ with $t^d - 1$ factors, where each factor is a homomorphic image of $R/(x_1^q, \dots, x_d^q)$. The length of each factor is bounded by $q^d(\mu + \epsilon_q)$ where $\epsilon_q \to 0$ as

$(x_1^{qt}, \dots, x_d^{qt})^{\rm lim}/(x_1^{qt}, \dots, x_d^{qt})$ is bounded by Bq^{d-1} for a fixed $B > 0$ by the usual tight closure colon-killer argument. This gives a bound for the length of $R/(x_1^{qt}, \dots, x_d^{qt})$ of the form $(t^d-1)(q^d)(\mu+\epsilon_q)+Bq^{d-1}$. But this length is $t^dq^d(\mu+\epsilon_{qt}) \leq (t^d-1)q^d(\mu+\epsilon_q)+Bq^{d-1}$. Divide by q^d and take the limit as $q \to \infty$. This yields $t^d\mu \leq (t^d-1)\mu$, a contradiction. □

What happens for the local cohomology of the hypersurfaces which map to a counterexample to the monomial conjecture?

2. LATENT REGULAR SEQUENCES AND q-SEQUENCES

Raise the question as to whether a latent regular sequence is the same as a q-sequence, where the latter means, equivalent that $h^n_{\underline{x}}(R) = 1$ or that the quasilength of every $R/I_t^{\rm lim}$ is t^n.

Is it true that if $x_1, \dots, x_n$ is a regular sequence on S then the quasilength of every S/I_tS is t^n. Include what is known ($n = 1$, $n = 2, t = 2$, some refinements, and Noetherian cases).

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