STRENGTH CONDITIONS, SMALL SUBALGEBRAS, AND STILLMAN BOUNDS IN DEGREE ≤ 4

TIGRAN ANANYAN AND MELVIN HOCHSTER[1]

Abstract. In [2], the authors prove Stillman's conjecture in all characteristics and all degrees by showing that, independent of the algebraically closed field K or the number of variables, n forms of degree at most d in a polynomial ring R over K are contained in a polynomial subalgebra of R generated by a regular sequence consisting of at most ${}^{\eta}B(n,d)$ forms of degree at most d: we refer to these informally as "small" subalgebras. Moreover, these forms can be chosen so that the ideal generated by any subset defines a ring satisfying the Serre condition R_η. A critical element in the proof is to show that there are functions ${}^{\eta}A(n,d)$ with the following property: in a graded n-dimensional K-vector subspace V of R spanned by forms of degree at most d, if no nonzero form in V is in an ideal generated by ${}^{\eta}A(n,d)$ forms of strictly lower degree (we call this a *strength* condition), then any homogeneous basis for V is an R_η sequence. The methods of [2] are not constructive. In this paper, we use related but different ideas that emphasize the notion of a *key function* to obtain the functions ${}^{\eta}A(n,d)$ in degrees 2, 3, and 4 (in degree 4 we must restrict to characteristic not 2, 3). We give bounds in closed form for the key functions and the ${}^{\eta}A$ functions, and explicit recursions that determine the functions ${}^{\eta}B$ from the ${}^{\eta}A$ functions. In degree 2, we obtain an explicit value for ${}^{\eta}B(n,2)$ that gives the best known bound in Stillman's conjecture for quadrics when there is no restriction on n. In particular, for an ideal I generated by n quadrics, the projective dimension R/I is at most $2^{n+1}(n-2)+4$.

1. Introduction

Throughout this paper, let R denote a polynomial ring over an arbitrary field K: say $R = K[x_1, \dots, x_N]$. We will denote the projective dimension of the R-module M over R by $\mathrm{pd}_R(M)$. The following theorem was conjectured by M. Stillman in [32] and proved, in a strengthened form (for submodules with a specified number of generators of free modules with a specified number of generators, and without the assumption of homogeneity) in Theorem C of [2].

Theorem 1.1. *There is an upper bound, independent of N, on $\mathrm{pd}_R(R/I)$, where I is any ideal of R generated by n homogeneous polynomials $F_1, \dots, F_n$ of given degrees $d_1, \dots, d_n$.*

We refer to such bounds, which are now known to exist, as *Stillman bounds.*

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Descriptions of earlier work related to this problem [5, 6, 7, 9, 15, 16, 17, 28, 30] are given in the introductions of [1, 2] and in [31]. [12, 13, 18, 19, 20, 21] contain recent work, some of which utilizes the results of [2], on Stillman's conjecture and related questions, including Noetherianity problems.

For n quadratic forms generating an ideal I of height h, there are examples where $\mathrm{pd}_R(R/I)$ is $h(n-h+1)$: see [30]. The projective dimension cannot be larger when $h = 2$ (see [25]) but when $h \geq 3$ and $n \geq 5$ it is an open question whether $h(n-h+1)$ is a bound. When $n = 4$ and $h = 3$ it has recently been shown that the largest possible projective dimension is 6: see [26].

The proof of the existence of Stillman bounds in [2] depends critically on proving auxiliary bounds ${}^{\eta}\!A$, ${}^{\eta}\!B$. Our focus here is on giving explicit bounds for ${}^{\eta}\!A$ in degrees 2, 3, 4, and on ${}^{\eta}\!B$ when the degree is 2. Our methods in degree 4 are vastly different from those in [2].

Because Stillman's conjecture is unaffected by a base change on the field, we shall assume from now on that the base field K is algebraically closed, which is needed for many of our theorems about the functions ${}^{\eta}\!A$, ${}^{\eta}\!B$. Moreover, at many points there is a restriction that the characteristic of K is either 0 or larger than some given integer. However, all assumptions of this type will be made specific.

Our main results are stated below, but we must first recall some definitions from [2]. We write $\mathbb{N}$ for the nonnegative integers and $\mathbb{N}_+$ for the positive integers. A *graded* K-algebra R will always be a finitely generated K-algebra graded by $\mathbb{N}$ such that $R_0 = K$. We write R_d for the graded component of degree d, typically thought of as a finite dimensional vector space over K. Unless otherwise specified, polynomial rings always have the standard grading in which all variables have degree one.

Given a finite-dimensional $\mathbb{N}_+$-graded K-vector space V over a field K such that $\dim_K V_t = n_t$ for $t \geq 1$, we refer to $\delta = \delta(V) = n_1, n_2, n_3, \ldots, n_t, \ldots$ as the *dimension sequence* of V. It should cause no confusion when we also write $\delta(V) = (n_1, \ldots, n_d)$ to mean that $n_1, \ldots, n_d$ constitute the first d terms of the dimension sequence of V, and that the other terms in the sequence are 0.

For any finite-dimensional vector space V over an algebraically closed field K we denote by $\mathbb{P}(V)$ the projective space whose points correspond to the lines through the origin in V. If $\dim(V) = d$, $\mathbb{P}(V) \cong \mathbb{P}^{d-1}_K$.

Discussion 1.2. Given an $\mathbb{N}$-graded K-algebra and $k \in \mathbb{N}$ we shall say that a form F has a *k-collapse* if it is a graded linear combination of k or fewer forms of strictly smaller positive degree. Note that only the 0 element has a 0-collapse. Nonzero scalars and nonzero linear forms cannot have a k-collapse for any k. With this terminology, a form is *k-strong* if and only if it has no k-collapse. Given a graded K-algebra S, a finite-dimensional $\mathbb{N}_+$-graded K-vector subspace $V \subseteq S$ with dimension sequence $(n_1, \ldots, n_d)$, and a d-tuple of non-negative integers $\kappa = (k_1, \ldots, k_d)$, we call V *κ-strong* if there is no nonzero form in V_i, the graded component of V in degree i, that has a k_i-collapse. We shall say that a sequence of forms of positive degree is κ-strong if the forms are linearly independent over K and the graded K-vector space they span is κ-strong. This means that if $F_{j_1}, \ldots, F_{j_s}$ are elements of the sequence of the same degree i with mutually distinct indices and $c_1, \ldots, c_s \in K$ are not all zero, then $\sum_{t=1}^{s} c_t F_{j_t}$ is nonzero and has no k_i-collapse. Hence, a graded K-vector space V is κ-strong if and only if every sequence of independent forms in

V is κ-strong. If all entries of κ are equal to the same integer k, we may use the term k-strong instead of κ-strong.

We call a prime ideal of a polynomial ring k-*linear* if it is generated by at most k linear forms. In a polynomial ring over a field, a form F of degree two or three has a k-collapse if and only if it is contained in a k-linear prime ideal. This is not true for forms of degree 4 or higher: for example, a degree 4 form may have a collapse in which some of the summands are the product of two quadrics.

We define the *strength* of a nonzero form F of positive degree as the largest integer k such that F is k-strong. If F is a nonzero 1-form, we make the convention that its strength is $+\infty$. We define the strength of a nonzero vector space V consisting of forms of the same degree as the smallest strength of a nonzero element of V. For nonzero vector spaces V of linear forms, the strength of V is $+\infty$. Thus F (respectively, V) is k-strong if and only if its strength is at least k.

We note that notions closely related to strength (q-rank, slice rank) have been considered independently in [10] and in [4], which was inspired by [35]. E.g., the notion of q-rank for cubic forms utilized in [10] is the same as the strength of the cubic form minus one.

Discussion 1.3. Recall that a Noetherian ring R satisfies the Serre condition R_i if R_P is regular for every prime P of height $\leq i$. If the singular (i.e., non-regular) locus in R is closed with defining ideal J, this means that J has height at least $i+1$. (If R is regular, $J = R$ and has height $+\infty$.) In the sequel, the rings that we are studying are standard graded algebras over a field K: they are finitely generated $\mathbb{N}$-graded rings R over K such that $R_0 = K$ and R_1 generates R as a K-algebra. If such a ring is normal, it is a domain, since it is contained in its localization at the homogeneous maximal ideal. We will know inductively that the rings we are studying are complete intersections. As just noted, R_1 implies normal domain.

By a theorem of A. Grothendieck, if a homogeneous complete intersection in a polynomial ring is R_3 then it is a UFD. This is Corollary 1.5 below.

Theorem 1.4. *Let R be a local ring that is a complete intersection whose localization at any prime ideal of height 3 or less is a UFD. Then R is a UFD. Hence, if R is a local ring that is a complete intersection in which the singular locus has codimension 4 or more, then R is a UFD.*

We need this in a graded version (cf. [33], Proposition 7.4):

Corollary 1.5. *Let S be an $\mathbb{N}$-graded algebra finitely generated over a field K that is a complete intersection (the quotient of a polynomial ring over K by the ideal generated by a regular sequence of forms of positive degree). Suppose that the singular locus in S has codimension at least 4. Then S is a UFD.*

Theorem 1.4 was conjectured by P. Samuel in the local case that was proved by A. Grothendieck. An algebraic proof is given in [8].

The following result is proved in [2]. Our main goal here is to give specific bounds for the functions A and ${}^{\eta}A$ in degrees 2, 3, and 4.

Theorem 1.6. *There is a function $\underline{A}$ (respectively, for every $\eta \geq 1$ there is a function ${}^{\eta}\underline{A}$) from dimension sequences $\delta = (n_1, \ldots, n_d)$ to $\mathbb{N}^d$ with the following property. If V is a finite-dimensional $\mathbb{N}_+$-graded subspace of a polynomial ring*

R over an algebraically closed field K with dimension sequence δ that is $\underline{A}(\delta)$-strong(respectively, ${}^\eta\underline{A}(\delta)$-strong) then if $F_1, \dots, F_h$ are K-linearly independent forms in V, they form a regular sequence (respectively, a regular sequence such that $R/(F_1, \dots, F_h)$ satisfies the Serre condition R_η). If $\eta = 1$, then $R/(F_1, \dots, F_h)$ is a normal domain and if $\eta = 3$ then $R/(F_1, \dots, F_h)$ is a UFD.

Remark 1.7. We write $A_i(\delta)$ (respectively, ${}^\eta A_i(\delta)$) for the i th entry of $\underline{A}(\delta)$ (respectively, ${}^\eta\underline{A}(\delta)$).

In case the vector space V has only one graded component, whose degree is d, so that $\delta = (0, \dots, 0, n)$, we may write $A_d(n)$ (respectively, ${}^\eta A_d(n)$) instead of $A_d(\delta)$ (respectively, ${}^\eta A_d(\delta)$).

Remark 1.8. We may take $\underline{A}$ to be any ${}^\eta\underline{A}$ for $\eta \geq 1$. However, it is frequently the case that a smaller choice will yield regular sequences. For example, consider an n-dimensional vector space of forms of the same degree d. If one has a level of strength at least 1 that guarantees that any $n-2$ linearly independent elements form a regular sequence such that the quotient by the ideal they generate is a UFD, that level of strength guarantees that any n independent forms in the vector space give a regular sequence: the first $n-2$ define a UFD by assumption, the next therefore gives a domain by Proposition 3.2 (g), and, consequently, the last is a nonzerodivisor. The level of strength ${}^3A_d(n-2)$ will always suffice for this argument, by Corollary 1.5.

Theorem 1.6 above implies the existence of small subalgebras (the precise statement is given below): this is Theorem B in [2]. We give a brief discussion of the argument in §2. It should be noted that the proof of this result gives an explicit recursive formula for obtaining the functions ${}^\eta B$ from the functions ${}^\eta A$.

We call a function of several integer variables *ascending* if it is nondecreasing as a function of each variable when the other variables are held fixed.

We say that a sequence of elements $G_1, \dots, G_s$ in a Noetherian ring R is an R_η-sequence if it is a regular sequence and for $0 \leq i \leq s$, $R/(G_1, \dots, G_i)$ satisfies the Serre condition R_η.

Theorem 1.9. *There is an ascending function B from dimension sequences $\delta = (n_1, \dots, n_d)$ to $\mathbb{N}_+$ with the following property. If K is an algebraically closed field and V is a finite-dimensional $\mathbb{N}_+$-graded K-vector subspace of a polynomial ring R over K with dimension sequence δ, then $K[V]$ is contained in a K-subalgebra of R generated by a regular sequence $G_1, \dots, G_s$ of forms of degree at most d, where $s \leq B(\delta)$.*

Moreover, for every $\eta \geq 1$ there is a function ${}^\eta B$ with the same properties as above such that, in addition, every sequence consisting of linearly independent homogeneous linear combinations of the elements in $G_1, \dots, G_s$ is an R_η-sequence.

In §2, we explain how Theorem 1.9 follows from Theorem 1.6, and also how Theorem 1.9 yields bounds on projective dimension that prove Stillman's conjecture.

Our main results here are as follows.

Theorem 1.10. *Let V be a vector space of quadratic forms in R of dimension n over K. If every element of $V - \{0\}$ is $(n-1)$-strong, every sequence of linearly independent elements of V is a regular sequence. If $\eta \geq 1$ and every element of $V - \{0\}$ is $(n-1+\lceil\frac{\eta}{2}\rceil)$-strong, then the quotient by the ideal generated by any subset* *of V satisfies the Serre condition R_η. Hence, every sequence of linearly independent elements of V is an R_η-sequence.*

Note that this is equivalent saying that one may take $A_2(n) = n-1$ and ${}^\eta A_2(n) = (n-1+\lceil \frac{\eta}{2} \rceil)$. See Theorem 4.14, Theorem 4.20, and Corollary 4.21.

Theorem 1.11. *A vector space of quadrics in the polynomials ring R that has dimension n is contained in a polynomial subring generated by a regular sequence consisting of at most $2^{n+1}(n-2)+4$ linear and quadratic forms. Hence, the projective dimension of R/I, where I is the ideal generated by these forms, is at most $2^{n+1}(n-2)+4$.*

See Theorem 4.22.

Theorem 1.12. *Let $b = 2(n_2+n_3)+\eta+1$ if $n_2 \neq 0$, and $2(n_2+n_3)+\eta$ if $n_2 = 0$. Let $\mathfrak{J}(b) = (2b+1)(b-1)$ if the characteristic is not 2 or 3, $\mathfrak{J}(b) = 2(2b+1)(b-1)$ if $\mathrm{char}(K)$ is 2, and $\mathfrak{J}(b) = 2b^2 - b$ if $\mathrm{char}(K)$ is 3. Then we may take*

$$ {}^\eta\underline{A}(n_1, n_2, n_3) = \Big(0, \lceil \frac{b}{2} \rceil + n_1, \mathfrak{J}(b) + n_1\Big). $$

See Theorem 6.4.

Our construction of the functions $\underline{A}, {}^\eta\underline{A}$ in degrees 3 and 4 will depend on proving the existence of certain key functions $\mathfrak{K}_i(k)$. Note that by Proposition 3.3, one element of a vector space of forms of degree h is k-strong if and only if a Zariski open subset of the vector space consists of forms that are k-strong: the set of k-strong forms of a given degree h is Zariski open in R_h.

Definition 1.13. If F is a form in a polynomial ring $K[X_1, \ldots, X_N]$ over K, we write $\mathcal{D}F$ for the K-vector space spanned by the partial derivatives of F. One may also think of $\mathcal{D}F$ as the image of F under all K-derivations from R to R that are homogeneous of degree -1, and so it is unaffected by K-linear changes of coordinates in R.

We say $\mathfrak{K}_i$ is a *key* function for degree $i \geq 3$ and for a specified set $\mathcal{S}$ of characteristics if, independent of the algebraically closed field with a characteristic in $\mathcal{S}$ and the number of variables N, for any form F of degree i, if F is $\mathfrak{K}_i(k)$-strong, then the elements in a Zariski dense open subset of $\mathcal{D}F$ are k-strong (equivalently, one element in $\mathcal{D}F$ is k-strong).

Our primary goal in the sequel is to construct key functions for degree 3 in all characteristics and for degree 4 in characteristic not 2 or 3.

We shall not need the notion of key function for degree 2 or smaller in our proofs of the existence of ${}^\eta A$ or ${}^\eta B$. One can make the same definition if $i = 1$ or $i = 2$. The restriction one needs on the one-form when $i = 1$ is that it not be 0: this implies that some partial has infinite strength. Thus, we may take $\mathfrak{K}_1(k) = 0$ for all k. If $i = 2$, the condition needed on the quadratic form if the characteristic is not 2 is that it not be zero, for then some partial derivative does not vanish and it has infinite strength. If the characteristic is 2, we need that the form not be a square, which follows if it has strength 1. Thus, we may take $\mathfrak{K}_2(k) = 0$ for all k if the characteristic is not 2, and $\mathfrak{K}_2(k) = 1$ for all k if the characteristic is 2.

The following result plays an important role in our treatment of $\mathfrak{K}_3$ in characteristic not 2 or 3.

Theorem 1.14. *Let K be an algebraically closed field of characteristic $\neq 2$. Let V be a vector space of quadratic forms that is not contained in an ideal generated by $2k$ or fewer linear forms. Then there exists a Zariski dense open subset of V consisting of quadratic forms with no k-collapse.*

See Theorem 4.8.

Theorem 1.15. *For any algebraically closed field K of characteristic $\neq 2, 3$, we may take $\mathfrak{K}_3(k) = 2k$.*

Theorem 1.15 is immediate from Theorem 6.2 below in the case $b = 1$.

We also consider functions $\mathfrak{J}_d$, which we call *J-rank functions*, such that if a form F of degree d has strength at least $\mathfrak{J}_d(k)$, then the height of the ideal generated by F and $\mathcal{D}F$ is at least k. This is the same as requiring that $V(F)$ be R_{k-2} for $k \geq 2$. The functions $\mathfrak{J}_d$ coincide with the function ${}^{\eta}\!A$ for a vector space spanned by one form of degree d, with a shift in the index. See Discussion 5.5 and Definition 5.6 below. J-rank functions can be used to construct the functions ${}^{\eta}\!A$ in general: see Theorem 5.7. When the characteristic is greater than d, one can use key functions to construct J-rank functions, and this is the basis for our theorems on the existence of ${}^{\eta}\!A$ in degrees 3 and 4 when the characteristic is not 2 or 3. See Theorem 5.9 and Corollary 5.10.

For degree 3, if the algebraically closed field has characteristic 2 or 3, we instead use a direct construction to obtain our main result on the existence of $\mathfrak{J}_3(k)$:

Theorem 1.16. *Let R be a polynomial ring over an algebraically closed field K. If K has characteristic 2, then we may take $\mathfrak{J}_3(k) = 2(k-1)(2k+1)$. If K has characteristic 3, we may take $\mathfrak{J}_3(k) = 2k^2 - k$.*

See Theorem 6.3. Note that these expressions for $\mathfrak{J}_3$ are not much worse than the result one can obtain for characteristic not 2, 3 using Theorem 1.15, namely $\mathfrak{J}_3(k) = (2k+1)(k-1)$. See also Theorem 6.1.

The following is one of our main results here: it shows that one can construct the functions ${}^{\eta}\!A$ explicitly (and, hence, the functions ${}^{\eta}\!B$) up through degree d whenever one can construct the functions $\mathfrak{K}_i$ for $i \leq d$.

Theorem 1.17. *Let K be algebraically closed field of characteristic 0 or $> d$, where $d \geq 3$, and let $R = K[x_1, \ldots, x_N]$. Suppose that for $2 \leq i \leq d-1$, we have a function $A_i : \mathbb{N}_+ \to \mathbb{N}_+$ such that if a vector space of forms of degree i of dimension n is $A_i(n)$-strong, then every sequence of linearly independent forms of V is a regular sequence. Suppose also that for every i, $2 \leq i \leq d$, we have a key function $\mathfrak{K}_i$ that is also nondecreasing. Let $\delta = (n_1, \ldots, n_d)$ be a dimension sequence. Let h be the number of nonzero elements among $n_2, \ldots, n_d$, and let $n' = n_2 + \cdots + n_d$. Let $b := h - 1 + 2n' + \eta$. Then we may define ${}^{\eta}\!\underline{A}(\delta) = \big({}^{\eta}\!A_1(\delta), \ldots, {}^{\eta}\!A_d(\delta)\big)$ as follows.*

(1) ${}^{\eta}\!A_1(\delta) := 0$.

(2) ${}^{\eta}\!A_2(\delta) := \lceil \frac{b}{2} \rceil + n_1$.

(3) *For* $3 \leq i \leq d$, ${}^{\eta}\!A_i(\delta) := \mathfrak{K}_i\big(bA_{i-1}(b)\big) + b - 1 + n_1$.

Then whenever a vector space V of forms in R with dimension sequence δ is ${}^{\eta}\!\underline{A}(\delta)$-strong, every sequence of linearly independent forms in V is an R_η-sequence.

Theorem 1.17 is proved in §5, following the proof of Corollary 5.10, after a substantial number of preliminary results are established.

Corollary 1.18. *With hypothesis as in Theorem 1.17, if $\delta = (0, \ldots, 0, n)$, corresponding to an n-dimensional vector space whose nonzero forms all have degree $d \geq 3$, then we may take ${}^{\eta}A_d(\delta) = \mathfrak{K}_d\big((2n+\eta)A_{d-1}(2n+\eta)\big) + 2n + \eta - 1$.*

Proof. This is immediate from Theorem 1.17, since in this case $h = 1$, $n' = n$, and $b = 2n + \eta$. □

Our main result in degree 4 is to give an explicit formula for $\mathfrak{K}_4$: coupled with Theorem 1.17, one obtains the functions A, ${}^{\eta}A$ for degree up to 4. However, for this result we must exclude characteristics 2 and 3. See Theorem 10.2 and Corollary 10.4.

Theorem 1.19. *Let K be an algebraically closed field of characteristic not 2 or 3. Then $\mathfrak{K}_4(k) = 6k(k+1)4^{k(k+1)} + (k+1)^2$*

2. Deriving Stillman's Conjecture and the functions ${}^{\eta}B$ from the functions ${}^{\eta}A$

We describe first the algorithm for obtaining ${}^{\eta}B$ for ${}^{\eta}A$ (and B from A, which is the same argument) from [2].

One linearly orders the dimension sequences $\delta = (\delta_1, \ldots, \delta_d)$ so that $\delta < \delta'$ precisely if $\delta_i < \delta'_i$ for the largest value of i for which the two are different. This is a well-ordering. Assume that ${}^{\eta}B$ is known for all values all predecessors of δ. If the vector space is ${}^{\eta}A(\delta)$-strong, it satisfies R_η and we are done. If not, for some i an element of V_i has an ${}^{\eta}A_i(\delta)$-collapse, and we can express the element in terms of at most $2 \cdot {}^{\eta}A_i(\delta)$ forms of lower degree. This enables us to form a new vector space in which δ_j remains the same for $j > i$, δ_i decreases by 1, and the δ_h for $h < i$ increase by a total of $2 \cdot {}^{\eta}A_i(\delta)$. If we let δ' run through all dimension sequences with the properties just described (these automatically precede δ in the well-ordering), we may take ${}^{\eta}B(\delta) = \max_{\delta'}\{{}^{\eta}B(\delta')\}$.

Once we have the functions ${}^{\eta}B$, it is easy to use them to give bounds on projective dimension independent of the number of variables. Suppose one wants to give a bound $C(r, s, d)$ the projective dimension of the cokernel of an $r \times s$ matrix over a polynomial ring R whose entries (which need not be homogeneous) have degree at most d. Each entry is the sum of a most d forms of positive degree and, possibly, a scalar. Hence, all of the entries of the matrix are in a ring generated by rsd forms of positive degree at most d. The function ${}^{\eta}B$ places all of these form in a subalgebra of R generated by, say, B forms of degree at most d that form a regular sequence, where B depends on r, s, and d but not in the number of variables. The regular sequence generates a polynomial ring A such that R is flat (even free) over A. (We may extend the regular sequence to a homogeneous system of parameters, and R is free over the ring the parameters generate). The minimal free resolution over A has length at most B, and this is preserved by the flat base change to R.

3. Basic results on strength and safety

Then notions of k-collapse and strength were discussed briefly earlier: see Discussion 1.2. In this section we introduce the additional notions of *collective collapse* and *safety*, and systematically study all of these properties.

Definition 3.1. Collective collapse and safety. A set $\mathcal{F}$ of forms of a graded K-algebra R is *k-safe* if it is not contained in an ideal generated by k forms of positive degree. The statement for the set of forms $\mathcal{F}$ is evidently equivalent to the same statement for the ideal the forms in $\mathcal{F}$ generate. *Note that in this definition there is no restriction on the degrees of the k forms*, which is an important difference between safety and strength conditions. The degrees of the generators of the ideal can always be taken to be at most the highest degree of a form in the set $\mathcal{F}$. If the set of forms $\mathcal{F}$ is not k-safe, we shall say that it has a *collective k-collapse.* In the discussion below, we refine this terminology to keep track of how many of the k forms that give the collapse have the highest degree of an element of the set $\mathcal{F}$.

If the highest degree of a form in the set $\mathcal{F}$ is d, we shall say that the set of forms $\mathcal{F}$ has a *collective $(k-h,h)$-collapse*, where $0 \le h \le k$, if it is contained in an ideal generated by at most $k-h$ forms of degree at most $d-1$, which we call the *auxiliary ideal* for the collective $(k-h,h)$-collapse, and at most h forms of degree d. We refer to the K-vector space generated by these h d-forms as the *auxiliary* vector space for $(k-h,h)$-collapse.

We shall refer to a collective $(k,0)$-collapse as a *strict collective* k-collapse. In the case of a single form f, a collective $(k,0)$-collapse is the same as a k-collapse, defined earlier. However, we may also use the term *strict k-collapse*, to emphasize that the degrees of the generators are strictly smaller than $\deg(f)$.

If a set of forms does not have a collective $(k-h,h)$-collapse we say that it is *$(k-h,h)$-safe.* Thus, a set of forms $\mathcal{F}$ is k-safe if and only if there do not exist integers $k_0, h \in \mathbb{N}$ such that $k_0 + h \le k$ and $\mathcal{F}$ has a (k_0,h)-collapse.

Proposition 3.2. *Let K be a field and R an $\mathbb{N}$-graded K-algebra. Let V be a finite-dimensional graded K-vector subspace of R with dimension sequence $(n_1, \ldots, n_d)$. Let $\kappa = (k_1, \ldots, k_d)$ and $\lambda = (h_1, \ldots, h_d)$ be elements of $\mathbb{N}^d$. Let k and h be positive integers. We write $\kappa + h$ for $(k_1 + h, \ldots, k_d + h)$.*

(a) If V is κ-strong and $h_i \le k_i$ for $1 \le i \le d$, then every graded K-vector subspace of V is λ-strong. Hence, if V is k-strong and $h \le k$, then every graded K-vector subspace of V is h-strong.

(b) If the sequence $F_1, \ldots, F_m$ is κ-strong, the sequence $G_1, \ldots, G_n$ is κ'-strong, and $\deg(F_i) \neq \deg(G_j)$ for all i, j, then $F_1, \ldots, F_m, G_1, \ldots, G_n$ is κ''-strong, where for each degree s that occurs, κ''_s is κ_s if s is the degree of some F_i and is κ'_s if s is the degree of some G_j.

(c) If $G_1, \ldots, G_h, F_1, \ldots, F_n$ is $(\kappa + h)$-strong and $\deg(G_i) \le \deg(F_j)$ for all i and j (which is automatic if the G_i are linear forms) then the image of the sequence $F_1, \ldots, F_n$ is κ-strong in $R/(G_1, \ldots, G_h)R$. If V is $(\kappa + h)$-strong and contains no linear forms, and W is a vector space generated by linear forms with $\dim_K W \le h$, then the image of V is κ-strong in $R/(W)$.

(d) If a family of forms of positive degree is k-safe, then so is every larger family.

(e) If a set of forms of positive degree is $(k+h)$-safe, then it is still k-safe if one omits h elements.

(f) Suppose that V is κ-strong, that $F \in V$ has degree e, and that F is part of a basis $\mathcal{B}$ for V consisting of forms. Let n' be the number of forms in $\mathcal{B}$ of degree $< e$, i.e. $n' = \sum_{j=1}^{e-1} n_j$, and let $G_1, \ldots, G_m$ be any forms in R having positive degrees $< e$. Suppose that $n' + m \le k_e$ (respectively, $< k_e$). Let I be an ideal of R generated by the $G_1, \ldots, G_m$, a subset of the elements of $\mathcal{B}$ different from F, and a set of forms of R of degree $> e$. Then F is nonzero (respectively, irreducible) modulo I.

(g) *If V is k-strong for $k \geq 1$, the elements of $V - \{0\}$ all have the same degree, and $F_1, \dots, F_h$ is any sequence of K-linearly independent elements of V, then each F_i is irreducible modulo the ideal generated by the F_j for $j \neq i$.*

Proof. (a) and (d) are immediate from the definitions. Part (b) follows from the fact that in checking the condition (including whether one has linear independence), one need only consider linear combinations of forms of the same degree.

For part (c), if one has a k_i-collapse of a homogeneous element F of degree i mod $(G_1, \dots, G_h)R$ or if F becomes 0, then there is a homogenous lifting, which means that $F = \sum_{s=1}^{k_i} P_s Q_s + \sum_{t=1}^{h} H_t G_t$ where $\deg(P_s) + \deg(Q_s) = \deg(F) = i$ for every s, and $\deg(H_t) + \deg(G_t) = i$ unless $H_t = 0$. The G_t of degree i have scalar coefficients: let G be the sum of all such terms on the right. The other G_t with nonzero coefficients have degree strictly smaller than i. Then $F - G$ is a nonzero form of degree i (since the original elements are linearly independent over K) that has a $(k_i + h)$-collapse in R, since the other nonzero terms have a factor of degree smaller than i. The second statement follows from the first, because $W + V = W \oplus V$ is also $(\kappa + h)$-strong (a vector space of linear forms is t-strong for all positive integers t).

For part (e), if the forms are contained in a k-generated homogeneous ideal I after forms $G_1, \dots, G_h$ are omitted, then the original forms are contained in the ideal $I + (G_1, \dots, G_h)R$.

In part (f), the parenthetical statement about reducibility follows from the first statement: if F reduces, we may include a representative of one of the factors among the G_j, which increases m by 1. If the conclusion of the main statement fails, then for a K-linear combination F^* of forms of degree e in $\mathcal{B}$ other than F, $F - F^* \in V$ would be 0 modulo the ideal J generated by $G_1, \dots, G_m$ and the elements of $\mathcal{B}$ of degree $< e$ (by a degree argument, we do not need to consider the generators of degree $> e$). Since J has at most $m + n'$ generators, this expresses $F - F^*$ as a sum of multiples of at most $m + n'$ forms of strictly lower degree. Since $F - F^*$ is a nonzero form of V, this contradicts the fact that $k_e \geq m + n'$. Part (g) is a very special case of (f) in which $n' = m = 0$. □

In the following result, note that 0 has a strict k-collapse for all $k \geq 0$, and so is an element of all the sets asserted to be Zariski closed in R_d.

Proposition 3.3. *Let K be an algebraically closed field, and let R be a polynomial ring in finitely many variables over K. Let $d \geq 1$ be an integer and let R_d denote the K-vector space of d-forms in R. Let k be a positive integer, and let $d_1, \dots, d_k$ be positive integers that are $\leq d$.*

(a) *The set of elements f of R_d that are contained in an ideal I with k generators $f_1, \dots, f_k$ such that $f_i \in R_{d_i}$ is Zariski closed in R_d. The set of points of $\mathbb{P}(R_d)$ represented by such a nonzero f is Zariski closed in $\mathbb{P}(R_d)$.*
(b) *Let $h \in \mathbb{N}$. The set of elements f of R_d that have a (k, h)-collapse is Zariski closed in R_d. The set of points of $\mathbb{P}(R_d)$ represented by such a nonzero f is Zariski closed in $\mathbb{P}(R_d)$.*
(c) *The set of elements f of R_d having a strict k-collapse is Zariski closed in R_d. The set of points of $\mathbb{P}(R_d)$ represented by such a nonzero f is Zariski closed in $\mathbb{P}(R_d)$.*

Proof. (a) The second statement implies the first. Let $e_i = d - d_i$ for $1 \le i \le k$. For the projective case, consider the subset Z of

$$Y = \mathbb{P}(R_{d_1} \times \cdots \times R_{d_k}) \times \mathbb{P}(R_{e_1} \times \cdots \times R_{e_k}) \times \mathbb{P}(R_d)$$

consisting of points with representatives $\bigl((f_1, \ldots, f_k), (g_1, \ldots, g_k), f)\bigr)$ such that $\sum_{i=1}^k f_i g_i = f$. It suffices to see that Z is closed in Y, since Y is projective and the set described in the theorem is its image under the third coordinate projection to $\mathbb{P}(R_d)$. Let a_λ be the coefficients of the f_i, let b_μ be the coefficients of the g_i, and let c_ν be the coefficients of f, where ν, for example, indexes the monomials of degree d in R. The coefficients of $\sum_{i=1}^k f_i g_i$ are polynomials of bidegree (1,1) in the a_λ and b_μ: suppose that the coefficient of the monomial indexed by ν is $h_\nu(a, b)$. If s denotes the dimension of R_d, Z is defined by the vanishing of the 2×2 minors of a $2 \times s$ matrix whose columns are indexed by the ν, and such that the column corresponding to ν is $\begin{pmatrix} h_\nu \\ c_\nu \end{pmatrix}$. These minors are $(1, 1, 1)$-homogeneous polynomials in the a_λ, b_μ and c_ν. Thus, Z is closed, and so is its projection.

(b) and (c). The set in (b) is the finite union of the sets corresponding to all choices of $d_1, \ldots, d_{k+h}$ such that for precisely h values of i, $d_i = d$, while for all other values of i, $d_i < d$. Each of these sets is closed by part (a), and, hence, so is the union. Part (c) is the special case of (b) where $h = 0$. □

Remark 3.4. See also Corollary 4.4 for an alternative proof of part (c) in the case of quadratic forms.

Proposition 3.5. *Let $K \subseteq L$ be algebraically closed fields. Let $R = K[x_1, \ldots, x_N]$ be a polynomial ring, and $S = L \otimes_K R \cong L[x_1, \ldots, x_N]$. Let $a \le k, h \in \mathbb{N}$.*

(a) Let F be a form of positive degree d in R, and let $d_1, \ldots, d_k$ and $e_1, \ldots, e_k$ be two sequences of positive integers such that $d_i + e_i = d$ for $1 \le i \le k$. Suppose that F can be written in the form $\sum_{i=1}^k G_i H_i$, where every $G_i \in S$ has degree d_i or is 0 and every $H_i \in S$ has degree e_i or is 0. Also suppose that $G_1, \ldots, G_a \in R$. Then F can be written in the same form over R without changing $G_1, \ldots, G_a$.

(b) Let F be a form in R. F has a k-collapse in R if and only if it has a k-collapse in S.

(c) A sequence of forms in R is k-strong if and only if it is k-strong in S.

(d) Let $\kappa = (k_1, \ldots, k_d) \in \mathbb{N}^d$. A graded vector space $V \subseteq \oplus_{i=1}^d R_i$ is κ-strong if and only if $L \otimes_K V$ is κ-strong in S.

(e) A finite set of forms in R (or a finite-dimensional graded K-vector subspace of R, or a finitely generated homogeneous ideal of R) is k-safe (respectively, (k, h)-safe) if and only if it is k-safe (respectively, (k, h)-safe) in S.

Proof. (a) Let $\widetilde{G}_i = G_i$ for $i \le a$, let $\widetilde{G}_{a+1}, \ldots \widetilde{G}_k \in R$ have degrees $d_{a+1}, \ldots, d_k$, respectively, in $x_1, \ldots, x_N$ with unknown coefficients, and let $\widetilde{H}_1, \ldots, \widetilde{H}_k$ be polynomials in $x_1, \ldots, x_N$ of degrees $e_1, \ldots, e_k$, respectively, with unknown coefficients. Equating coefficients of distinct monomials in the variables $x_1, \ldots, x_N$ in $F - \sum_{i=1}^k \widetilde{G}_i \widetilde{H}_i$ to 0 yields a finite system of polynomial equations over K in the finitely many unknown coefficients of $\widetilde{G}_{a+1}, \ldots, \widetilde{G}_k$ and $\widetilde{H}_1, \ldots, \widetilde{H}_k$. Since there is a solution in L, there is a solution in K by Hilbert's Nullstellensatz.

Note that the "only if" parts of (b) and the "if" parts of (c), (d), and (e) are obvious. We need to prove the other parts. (b) is immediate from (a), and (c) follows from (d).

To prove (d), suppose that a form in $(L \otimes_K V)_e \cong L \otimes_K V_e$ has a k_e-collapse. Choose a basis $G_0, \ldots, G_a$ for V_e. Then some nonzero L-linear combination of $G_0, \ldots, G_a$ has a k_e-collapse, and one of the G_i has a nonzero coefficient. We may assume without loss of generality that this coefficient is 1, and, by renumbering, that $i = 0$. Let $F = G_0$. Then have an expression $F = \sum_{i=1}^{a} F_i G_i + \sum_{j=1}^{k_e} H_j P_j$, where the F_i, H_j, and P_j are in S and satisfy certain degree constraints (in particular, F_i has degree 0 for $1 \leq i \leq a$, while the degrees of the polynomials H_j, P_j are strictly smaller than $\deg(F)$), and we may apply part (a).

The proof of (e) is similar to the proof of (a). The problem for a finite set of forms $F_1, \ldots, F_s$ is the same as for the K-vector space or the ideal they generate, and we work with the first case. If the F_i are all in an ideal generated by k forms $G_1, \ldots, G_k$ of positive degree over L, each F_i will be a linear combination of these over S: replace the coefficients of the G_j and of their multipliers in the expression for each F_i by unknowns. One is led to a system of equations over K which has a solution in L. Therefore it has a solution in K. In the case of (k, h)-safety there are additional constraints on the degrees, but the equational nature of the problem is unchanged. □

4. Quadratic forms

We briefly discuss quadratic forms in the polynomial ring $R = K[x_1, \ldots, x_N]$ in N variables over an algebraically closed field. There are differences between the case when the characteristic is not 2 and when it is 2. This will not have a large effect on our study of quadratic forms, but the problem turns out to be much greater when studying cubic forms.

Background and basic results.

Discussion 4.1. If the characteristic is not 2, a quadratic form F is determined by a symmetric matrix M of scalars: in fact, $M = \frac{1}{2}\big(\partial^2 F/\partial x_i \partial x_j\big)$. We refer to the rank of M as the *rank* of the quadratic form F, and note that the rows of $2M$ correspond to the partial derivatives $\partial F/\partial X_i$: the entries of the ith row give the coefficients of the linear form $\partial F/\partial X_i$. Hence, the rank of F is the same as the dimension of the vector space $\mathcal{D}F = \{DF : D \in \mathrm{Der}_K(R, R)\}$. If $X = \begin{pmatrix} x_1 & \ldots & x_N \end{pmatrix}^{\mathrm{tr}}$ then $\big(F\big) = X^{\mathrm{tr}} M X$. If $X = AY$, where Y is a (possibly different) basis for the linear forms of R, then $X^{\mathrm{tr}} M X = Y^{\mathrm{tr}}(A^{\mathrm{tr}} M A) Y$. By a change of basis, the matrix M can be brought to a very special form: the direct sum of an $r \times r$ identity matrix I_r and an $(N - r) \times (N - r)$ zero matrix. Thus, for a suitable choice of A, F is represented as the sum of r squares of mutually distinct variables. Since $x_1^2 + x_2^2$ can be written as $y_1 y_2$ after a change of basis, a quadratic form of rank r can be written as

$$x_1 x_2 + \cdots x_{2h-1} x_{2h}$$

if $r = 2h$ is even and as

$$x_1 x_2 + \cdots x_{2h-1} x_{2h} + x_{2h+1}^2$$

if $r = 2h + 1$ is odd.

If the characteristic of K is 2, quadratic forms are classified in [3]. There is a primarily expository version in [29]. In this case, we define the rank of the quadratic form F to be the least number of variables occurring in F after a linear change of coordinates. This can be taken as the definition of rank in all characteristics.

By the classification of quadratic forms over an algebraically closed field one has:

Proposition 4.2. *If K is an algebraically closed field of arbitrary characteristic and F is a quadratic form of rank r then, after a linear change of variables, F can be written either as*

$$x_1x_2 + \cdots + x_{2h-1}x_{2h} \quad (r = 2h)$$

or as

$$x_1x_2 + \cdots + x_{2h-1}x_{2h} + x_{2h+1}^2 \quad (r = 2h+1).$$

In both cases, the dimension of the K-vector space $\mathcal{D}F$ is r except in characteristic 2 when r is odd, in which case the dimension of $\mathcal{D}F$ is $r-1$.

F is in the ideal generated by $\mathcal{D}F$ except when the characteristic of K is 2 and r is odd, in which case it is in the ideal generated by $\mathcal{D}F$ and one additional linear form. □

From this we have at once:

Proposition 4.3. *If K is any algebraically closed field and F is a quadratic form of rank r, then F has a k-collapse if and only if $k \geq \lceil \frac{r}{2} \rceil$. Equivalently, F has a k-collapse if and only if the rank of F is at most $2k$.*

Also, if $\mathcal{D}F$ has dimension at most t, then F is not $\lceil \frac{t}{2} \rceil$-strong if the characteristic of K is not 2, and is not $\lceil \frac{t+1}{2} \rceil$-strong if the characteristic of K is 2. □

The following result was proved by a different method in Proposition 3.3(c). The proof below offers a perspective with some advantages for the case of quadratic forms in characteristic $\neq 2$.

Corollary 4.4. *Let K be any algebraically closed field of characteristic $\neq 2$ and let $R = K[x_1, \ldots, x_N]$ be a polynomial ring. Let $k \in \mathbb{N}_+$. Then the set of quadratic forms that have a strict k-collapse is closed in the vector space R_2 of 2-forms.*

Proof. This set is defined by the ideal of $2k+1$ size minors of the symmetric matrix M of Discussion 4.1. □

Discussion 4.5. Let K be an algebraically closed field, and R the polynomial ring $K[x_1, \ldots, x_N]$. It is well known that when the characteristic of K is not 2, the determinant of $(\partial^2 F/\partial x_i \partial x_j)$ vanishes if and only if the form has rank less than N, and this is also true in characteristic 2 when N is even. A similar criterion exists in characteristic 2 for the case where N is odd. If one lets the coefficients of the form be indeterminates over $\mathbb{Z}$, one can compute the determinant as a polynomial in these indeterminates. The coefficients turn out to be even integers, and so one can consider the polynomial over $\mathbb{Z}$ in the coefficients obtained by dividing by 2, sometimes called the *reduced discriminant* or the *half-discriminant*. It is then correct that the form has rank less than N over any field K, regardless of characteristic, if and only if the half-discriminant vanishes. Moreover, in the case where N is odd in characteristic 2, the half-discriminant agrees with the result of substituting for every x_i in F the Pfaffian of the matrix $(\partial^2 F/\partial x_i \partial x_j)$ corresponding to deletion of the ith row and column. See [11], [22], and [27]. Hence, the forms of rank less than N form a subvariety of codimension 1 in the vector space of all quadratic forms in all characteristics. Also, it follows that if F and G are any two linearly independent quadratic forms in R, then for some choice of $a, b \in K$, not both 0, the form $aF + bG$ has rank less than N: the vector space $KF + KG$ must have nontrivial intersection with the codimension one variety of quadratic forms of rank at most $N - 1$.

Proposition 4.6. *Let K be an algebraically closed field, and let F, G be quadratic forms in a polynomial ring $R = K[x_1, \dots, x_N]$ over K such that F has rank r.*

(a) For any G and in all characteristics, for all but at most r choices of $c \in K$, the rank of $cF + G$ is at least r.
(b) Suppose that the characteristic of K is $\neq 2$ or that r is even. If G is not in the ideal $(\mathcal{D}F)R$, then for all but at most r choices of $c \in K$, the rank of $cF + G$ is at least $r + 1$.
(c) If K has characteristic 2 and the rank of F is odd, then either G is the sum of a quadratic form in $(\mathcal{D}F)R$ and the square of a linear form, or for all but at most $r - 1$ choices of c, the rank of $cF + G$ is at least $r + 1$.

Proof. For any quadratic form H, we use M_H for the Hessian matrix of H.

We first prove (a) when K has characteristic not 2 or characteristic 2 and the rank r is even. After a change of basis, we may assume that the matrix M_F of F is the direct sum of an $r \times r$ matrix M_0 and a zero matrix, where M_0 is either an $r \times r$ identity matrix or the direct sum A of $r/2$ copies of $\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, depending on whether the characteristic is not 2 or 2. The determinant of the size r square submatrix in the upper left corner of $cM_F + M_G$ is a monic polynomial in c of degree r. This shows that for all but finitely many choices of c, the rank of $cF + G$ is at least r.

If G is not in $(\mathcal{D}F)R$ first suppose that M_G has a nonzero entry a_{ij} for $i, j > r$, which is always true if the characteristic is not 2. Consider the size $r + 1$ square submatrix of $cM_F + M_G$ that contains the $r \times r$ submatrix in the upper left corner and the i, j entry. The determinant of this submatrix is a polynomial in c whose highest degree term is $a_{ij}c^r$, and so this polynomial is nonzero except for at most r values of c, and the rank of $cF + G$ is at least $r + 1$ except for these values.

In the remaining cases for (a), (b), and (c) we may now assume that the field has characteristic 2 and that with $F_0 = x_1x_2 + \cdots + x_{2h-1}x_{2h}$, we have that $r = 2h$ and $F = F_0$ or that $r = 2h+1$ and $F = F_1 = F_0 + x_{2h+1}^2$. We may write $G = P + Q$ where P is in the ideal generated by $x_1, \dots, x_{2h} = (\mathcal{D}F)$ and $Q \in K[x_{2h+1}, \dots, x_N]$. It remains to prove part (a) when the rank of F is odd, part (b) when the rank of F is even and Q is a nonzero square (otherwise some a_{ij} for $i, j > r$ is nonzero, a case which has already been covered), and part (c). In considering part (c), we may assume that Q is not a square.

Part (a) for forms F of odd rank in characteristic 2 may be deduced as follows. We have assumed that F can be expressed in terms of $x_1, \dots, x_r$. It suffices to prove the statement after specializing the other variables to 0. But then $cF + c'G$ has maximum rank if and only if its half-discriminant is not 0, and this is a homogeneous polynomial of degree r in c and c' which is nonzero when $c = 1$, $c' = 0$. Hence, c^r has nonzero coefficient in K. It follows that when we substitute $c' = 1$, there are at r most values of c for which the half-discriminant vanishes. We have now proved part (a) in all cases.

We next prove the remaining case of part (b). We may assume that $r = 2h$ is even and that Q is a square. We may make a change of variables, and then we might as well assume that $Q = x^2$, where $x = x_{2h+1}$. Since the rank can only drop when we kill x_j for $j > 2h + 1$, we may assume that $x_1, \ldots, x_{2h+1}$ are all the variables. Then we may assume $F = F_0$ and $G = P + x^2$ where $P \in (x_1, \ldots, x_{2h})R$. Note that M_F is the direct sum of h copies of $\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and a 1×1 zero matrix. We want to prove that except possibly for $2h$ values of c, the rank of $cF + G$ is $2h + 1$.

By Discussion 4.5 a quadratic form H in $2h + 1$ variables has rank $2h + 1$ if and only if H is nonzero at $(p_1, \ldots, p_{2h+1})$, where p_i is the Pfaffian of the matrix obtained by omitting the ith row and column of the Hessian M_H. Consider the Pfaffians of $cM_F + M_G$. Then p_{2h+1} is monic in c of degree h (its square is monic of degree $2h$), while all of the other Pfaffians p_i, $1 \le i \le 2h$ are polynomials in c of degree at most $h - 1$ (their squares are polynomials of degree at most $2h - 2$) . When we substitute the p_i into $cF + G$, the largest power of c that can occur in cF is $c(c^{h-1})^2 = c^{2h-1}$. Likewise, the terms of G other than x^2 yield polynomials in c of degree at most $2h - 1$, because G is in the ideal generated by $x_1, \ldots, x_{2h}$. Thus, the x^2 term contributes a unique term of degree c^{2h} with coefficient 1, and so the value of $cF + G$ at $(p_1, \ldots, p_{2h+1})$ is a monic polynomial in c of degree $2h$. Hence, there are at most $r = 2h$ values of c such that $cF + G$ has rank strictly less than $2h + 1 = r + 1$.

Finally, to prove (c) we may assume that $F = F_1$ has rank $2h + 1$, and that $G = P + Q$, where $P \in (x_1, \ldots, x_{2h})R$ and $Q \in K[x_{2h+1}, \ldots, x_N] - \{0\}$ is *not* a square. Hence, Q has a term that is $ax_{2h+1}x_j$ for $j > 2h+1$ or, if not, a term that is ax_jx_k where $k > j > 2h + 1$, where in both cases, $a \in K - \{0\}$. In the latter case we may kill $x_k - x_{2h+1}$, and so reduce to the former case. Consider the size $2h+2$ square submatrix D of the matrix of $cF_1 + G$ corresponding to the rows and columns numbered $1, 2, \ldots, 2h + 1, j$. D has the block form $\begin{pmatrix} cA + B & C \\ C^{\mathrm{tr}} & a\Lambda \end{pmatrix}$, and where A is the direct sum of h copies of $\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The determinant of D is a polynomial in c in which the highest degree term is a^2c^{2h}. Hence, the rank of $cF + G$ is at least $2h + 2 = r + 1$ for all but at most $r - 1$ values of c. $\square$

Corollary 4.7. *Let K be an algebraically closed field and let V be a vector space of quadratic forms in a polynomial ring R over K. Let $F \in V$ have maximum rank among forms in V. If the characteristic is not 2, then then V is contained in the ideal $(\mathcal{D}F)$. If the characteristic of K is 2, then the image of V modulo the ideal $(\mathcal{D}F)$ is a vector space whose nonzero elements are squares of linear forms.*

Proof. If F has maximum rank and $G \in V$ is not in $(\mathcal{D}F)$, then by Proposition 4.6, when the characteristic is not 2 we can construct a form $cF + G$ of larger rank, while if the characteristic is 2 we can do that unless the image of G modulo $\mathcal{D}F$ is a square. $\square$

The following result, Theorem 4.8, plays an important role in the analysis of the case of cubics if the characteristic is not 2.

Theorem 4.8. *Let K be an algebraically closed field of characteristic $\neq 2$. Let V be a vector space of quadratic forms that is $(2k, 0)$-safe, i.e., not contained in an* *ideal generated by $2k$ or fewer linear forms. Then there exists a Zariski dense open subset of V consisting of quadratic forms with no k-collapse.*

Proof. By Corollary 4.4 the forms with a k-collapse are closed, and so it suffices to show that there is at least one element of V with no k-collapse. Assume that every element of V has a k-collapse. Let r denote the rank of a form $F \in V$, with r as large as possible. Then $k \geq \lceil \frac{r}{2} \rceil$ and $r \leq 2k$. By Corollary 4.7, V must be contained in $(\mathcal{D}F)R$. But then V is not $(r, 0)$-safe, and so is not $(2k, 0)$-safe, a contradiction. □

Note that Theorem 4.8 is false in characteristic 2. In fact, more generally, in any positive characteristic p, the forms $x_1^p, \ldots, x_N^p$ are $(N-1)$-safe, but any linear combination of them has a 1-collapse if K is perfect. However, one can still handle the case of cubics in characteristic 2 with a careful application of Corollary 4.7.

Existence of ${}^{\eta}\!A$ in degree 2. In this section we show the existence of the functions ${}^{\eta}\!A(n_1, n_2)$. We begin with an analysis of when a vector space of quadratic forms consists entirely of reducible elements, and the strength conditions needed to guarantee that the quotient of a polynomial ring by an ideal generated by one or two quadratic forms is reduced, or a domain, or a normal domain, or a UFD, and so forth. When the base field K is understood and S is a set of linear forms, we denote by $\langle S\rangle^2$ the degree 2 component of $K[S]$, i.e., the set of quadratic forms expressible as K-linear combinations of products of two elements of S. E.g., $\langle x, y\rangle^2 = Kx^2 + Kxy + Ky^2$.

Proposition 4.9. *Let V be a K-vector subspace of R_2. If every element of V is reducible, then either:*

(1) V is contained in a 1-linear prime of R, or
(2) V is contained in $\langle u, v\rangle^2$ for variables $u, v \in R_1$, or
(3) K has characteristic 2 and every element of V is the square of a linear form.

Proof. Let W be the span of all the squares in V. If W is all of V then we are done if the characteristic is 2, since (3) holds. If the characteristic is not 2 and there are three or more squares of independent linear forms we have a contradiction because $x^2 + y^2 + z^2$ is irreducible when x, y, z are variables. Thus, there are at most two, and (2) holds. Henceforth we assume that W is a proper subspace of V.

Choose an element, which must factor xy, of $V - W$. Let $T = \langle x, y\rangle^2$. We may assume that V is not contained in T, or (2) holds.

Now consider any other element of $V - (W \cup T)$ (V is spanned by these). Call this element uv. Then u, v, x, y cannot be independent, since $uv + xy$ would then be irreducible. Moreover, u, v are not both in $Kx + Ky$, or else $uv \in T$. Consider a K-linear relation on u, v, x, y. This relation cannot involve both u and v with nonzero coefficient. If it does, then we can solve for one of them: say $v = u + ax + by$ (the coefficient of u may be absorbed into u). Here, x, y, u must be variables. Then $u(u + ax + by) + cxy = u^2 + (ax + by)u + cxy$ must be reducible for all choices of c. This contradicts the following fact:

(†) If u, x, y are variables and $a, b, c \in K$, then for fixed a, b, $u^2 + (ax + by)u + cxy$ reduces only when $c = 0$ or when $c = ab$. In particular, there is a value of c for which it is irreducible.

(To see this, note that since $K[x, y]$ is normal, if there is a factorization it must be as $(u+L_1)(u+L_2)$ where $L_1, L_2 \in K[x, y]$. Then $L_1L_2 = cxy$ with $c \neq 0$ implies that $L_1 = a'x$ and $L_2 = b'y$, say, where $a'b' = c$. Since $a'x + b'y$ must be $ax + by$, we must have $a' = a$ and $b' = b$ as well. This establishes the conclusion of (†).)

Hence, one of u, v, say u, depends linearly on x, y, while x, y, v are variables. Thus, for every element uv of $V - (W \cup T)$, we have that one factor is in $Kx + Ky$ while the other is K-linearly independent of x, y. Suppose $u = ax + by$ where $a, b \neq 0$. Then $(ax + by)v + xy$ is irreducible, since it is linear in v and the coefficients of v are relatively prime. Thus, a or b must be in 0, and so every element of $V - (T \cup W)$ is in xR_1 or in yR_1. This yields that $V \subseteq T \cup W \cup xR_1 \cup yR_1$, and so V is contained in xR_1 or yR_1, and (1) holds. □

We next consider the case of a single quadratic form, where the behavior is well known.

Proposition 4.10. *Let R be a polynomial ring in finitely many variables over an algebraically closed field K and let F be a nonzero quadratic form of R. If the rank of F is $r \geq 1$, then the ideal generated by F and its partial derivatives in R has height r. Hence, if $r \geq 2$, the codimension of the singular locus in R/FR is $r - 1$, so that R/FR satisfies the Serre condition R_{r-2}. If F is k-strong for $k \geq 1$, then rank of F is $\geq 2k + 1$, and so R/FR satisfies R_{2k-1}. Hence, R/FR satisfies R_η if $k \geq \lceil (\eta + 1)/2 \rceil$.*

Proof. If the characteristic of K is not 2, or if the characteristic is 2 and F has even rank, the partial derivatives of F span a vector space of dimension r, and F is in the ideal they generate. Hence, the ideal generated by F and its partial derivatives has height r, and the height of the defining ideal of the singular locus in R/FR is $r - 1$. If the characteristic is 2 and F has odd rank, we may assume that F has the form $x_1x_{h+1} + \cdots + x_hx_{2h} + x_{2h+1}^2$. The ideal generated by F and its partial derivatives in R is $(x_1, \ldots, x_{2h}) + (x_{2h+1}^2)$, and the height is still $r = 2h + 1$. □

Remark 4.11. Note that by Corollary 1.5, if F has rank 5 or more (equivalently, if F is 2-strong), then R/FR is a UFD. This is a much more elementary result which follows from the classification of quadratic forms and Lemma 4.13 below.

We first note the following result of Nagata: see [33], Theorem 6.3 and its Corollary.

Lemma 4.12. *Let f be a nonzero prime element in a domain R. Then R is a UFD if and only if R_f is a UFD.* □

Lemma 4.13. *Let A be a UFD and $R = A[x_1, x_2]$ be a polynomial ring over A. Let $F = x_1x_2 + G$ where $G \in A[x_1]$ and $G(0) = a \in A - \{0\}$ is irreducible. Then $S = R/FR$ is a UFD.*

Proof. First note that F is irreducible in the UFD R, since it is linear in x_2 and the coefficients have greatest common divisor 1. Hence, S is a domain. The hypothesis also yields that x_1 is prime in S (the quotient is $(A/aA)[x_2]$, a domain). Hence, by Lemma 4.12, S is a UFD if and only if S_{x_1} is a UFD. But once we localize at the element x_1, the equation F is, up to a unit, $x_2 + x_1^{-1}G$, and killing FR yields $A[x_1]_{x_1}$, a UFD. □

The following result describes behavior for a vector space of quadratic forms of dimension n. We state the general result, then prove the case where $n = 2$, which plays a special role in the proof of the general result.

Theorem 4.14. *Let K be an algebraically closed field and let R be a polynomial ring over K. Let $F_1, \ldots, F_n$ be linearly independent quadratic forms in R, where $n \geq 2$, and let V be the K-vector space they span. Let $I = (F_1, \ldots, F_n)R$. If V is k-strong for $k \geq n-1$, then $F_1, \ldots, F_n$ is a regular sequence, and R/I satisfies the Serre condition $\mathrm{R}_{2(k+1-n)}$. In particular, if $k \geq n-1$, then R/I is reduced, if $k \geq n$ then R is normal, and if $k \geq n+1$ then R/I is a UFD.*

The authors would not be surprised if the result above is in the literature, but do not know a reference for it. It is easy to obtain slightly weaker results. The authors would like to thank Igor Dolgachev for suggesting the use of Steinerians in the proof. The arguments we give use modifications of this idea.

Before giving the proof, we note that this result is best possible, as shown by the following example.

Example 4.15. Assume that $k \geq n-1$. Let $Y = (y_{ij})$ be an $n \times (k+1)$ matrix of indeterminates and let $x_1, \ldots, x_{k+1}$ be $k+1$ additional indeterminates. For $1 \leq i \leq n$, let F_i denote $\sum_{j=1}^{k+1} x_j y_{ij}$. Any nonzero K-linear combination of the F_i also can be written as $\sum_{j=1}^{k+1} x_j y'_j$ where $x_1, \ldots, x_{k+1}, y'_1, \ldots y'_{k+1}$ are indeterminates, and so has rank $2k+2$. Thus, the vector space V spanned by the F_i is k-strong. These are n general linear combinations of the generators of an ideal of depth $k+1$, and therefore form a regular sequence: see, for example, the first paragraph of [23].

The Jacobian matrix of $F_1, \ldots, F_n$ has Y as a submatrix, while the rest of the matrix consists of n rows each of which contains $x_1, \ldots, x_{k+1}$ and entries that are 0: moreover, the occurrences of the x_i in a given row are in columns that are different from where they occur in any other row. Thus, the columns that contain x_i give a permutation of the columns of the matrix $x_i \mathbf{1}_n$. For a suitable numbering of the variables, the Jacobian matrix has the form

$$\begin{pmatrix} y_{1,1} & \ldots & y_{1,k+1} & x_1 & \ldots & x_{k+1} & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ y_{2,1} & \ldots & y_{2,k+1} & 0 & \ldots & 0 & x_1 & \ldots & x_{k+1} & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & & \vdots & & \vdots & & \vdots & \\ y_{n,1} & \ldots & y_{n,k+1} & 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & x_1 & \ldots & x_{k+1} \end{pmatrix}$$

It follows that the radical of the ideal of minors of the Jacobian contains the ideal $(x_1, \ldots, x_{k+1})$, and, hence, all of the F_i. Thus, the ideal of minors of the Jacobian matrix is $I_n(Y) + (x_1, \ldots, x_{k+1})$. Under the assumption that $k \geq n-1$, i.e., that $k+1 \geq n$, we have from the main result of [14] or [24] that this ideal has height $(k+1) - n + 1 + (k+1) = 2(k+1) - n + 1$ in the polynomial ring. Thus, the height of its image in $R/(F_1, \ldots, F_n)$ is $2(k+1-n)+1$. This means that the quotient satisfies $\mathrm{R}_{2(k+1-n)}$, but not $\mathrm{R}_{2(k+1-n)+1}$.

We first prove Theorem 4.14 when $n = 2$, and then give the proof in the general case. In the case where $n = 2$ we change notation slightly to avoid subscripts and write F, G for F_1, F_2. Before proving the result when $n = 2$ we note:

Lemma 4.16. *Let $R = K[x_1, \ldots, x_N]$ where K is an algebraically closed field. Assume that F and G do not lie in a subring of R generated $N-1$ or fewer linear forms, i.e., roughly speaking, that we cannot decrease N. Assume also that*

$V(F,G)$ has a singularity other than the origin. Then we can make a linear change of variables and choose new generators for $V = KF + KG$ one of which has the form $x_1x_2 + Q(x_3, \ldots, x_n)$, and the other of which only involves $x_2, \ldots, x_N$.

Proof. Since the singular locus is not just the origin, after a linear change of coordinates we may assume that $p_0 = (1, 0, \ldots, 0)$ is a singular point of $V(F,G)$. Then x_1^2 does not occur in either F or G (since they vanish at p_0), and we may write them as $F = Ax_1 + Q$, $G = A'x_1 + Q'$, where $A, A', Q, Q' \in K[x_2, \ldots, x_N]$. Then the Jacobian matrix at p_0 has the form

$$\begin{pmatrix} 0 & c_2 & \ldots c_N \\ 0 & c_2' & \ldots c_N' \end{pmatrix}$$

where $A = \sum_{j=2}^N c_jx_j$ and $A' = \sum_{j=2}^N c_j'x_j$ with $c_j, c_j' \in K$. Since p_0 is a singular point, the Jacobian matrix has rank at most 1 at p_0, so that A, A' span a K-vector space of dimension at most one. Therefore, we may replace F, G by independent K-linear combinations such that A' is 0, and so we may assume that $F = x_1A + Q$ and $G = Q'$, where A, Q, and Q' involve only $x_2, \ldots, x_N$. If $A = 0$ we can reduce the number of variables N, and so we may assume that $A \neq 0$. After a change of variables in $Kx_2 + \cdots + Kx_N$, we may assume that $A = x_2$. We may write $Q = x_2B + Q''$ where Q'' involves only $x_3, \ldots, x_N$. Thus, $F = (x_1 + B)x_2 + Q''$ where Q'' involves only $x_3, \ldots, x_N$, and G involves only $x_2, \ldots, x_N$. Finally, we may replace x_1 by $x_1 + B$. $\square$

Discussion 4.17. Let F be a quadratic form in $K[x_1, \ldots, x_N]$, where K is algebraically closed. If the characteristic of K is different from 2 or if K has characteristic 2 and F has even rank, then $\mathcal{D}F$ is the smallest space of degree one forms such that F is in the polynomial ring they generate. If the characteristic of K is 2 and F has odd rank $2h+1$, after a change of variables F has the form $x_1x_2 + \cdots x_{2h-1}x_{2h} + x_{2h+1}^2$. In this case, there is still a smallest space W of forms of degree 1, the K-span of $x_1, \ldots, x_{2h+1}$, such that F is in the polynomial ring generated by these linear forms: it also may be described as $\mathcal{D}F + Kx_{2h+1}$. Here, while x_{2h+1} is not uniquely determined, the vector space W is, and may be described as the intersection of $\mathrm{Rad}(\mathcal{D}F, F)$ with the forms of degree 1 in the polynomial ring R. We shall use the notation $\widetilde{\mathcal{D}}F$ to denote this space. It is the same as $\mathcal{D}F$ whenever the characteristic is not 2 or F has even rank. The image of the ideal generated by $\widetilde{\mathcal{D}}F$ in R/F is always the defining ideal of the singular locus of F.

We now give the proof of Theorem 4.14.

Proof. We first give the proof when $n = 2$, and write $F := F_1$ and $G := F_2$, as mentioned above. Since F is 1-strong, it is irreducible, so that FR is prime. We have that G is irreducible and, in particular, nonzero mod F (although not necessarily prime) by part (g) of Proposition 3.2. Thus, it is obvious that F, G is a regular sequence. Note that once we know this, reduced is equivalent to R_0, and normal is equivalent to R_1. In the normal case, the domain condition is automatic: see Discussion 1.3.

We know that $k \geq 1$, and we want to show that R is reduced. If any form in V has rank 5 or more, say F, the quotient by it is a UFD (see Remark 4.11). Then, since G is irreducible in R/FR, $R/(F,G)R$ is a domain.

Therefore, we may assume that every nonzero element of V has rank 3 or rank 4. We may assume without loss of generality that $F = xy - z^2$ or $xy - wz$. Let the remaining variables be $\underline{x}$. Then $K[x,y,z,\underline{x}]/(F)$ (respectively, $K[w,x,y,z,\underline{x}]/(F)$) is isomorphic with $C[\underline{x}]$ where

$$C = K[u^2, v^2, uv] \subseteq K[u,v] = D$$

or

$$C = K[su, sv, tu, tv] \subseteq K[s,t,u,v] = D,$$

where s,t,u,v are new indeterminates. Note $C \to D$ is split over C, and this remains true when we adjoin $\underline{x}$. Thus, every ideal of $C[\underline{x}]$ is contracted from $D[\underline{x}]$.

Let $g = G(u^2, v^2, uv, \underline{x})$ (respectively, $G(su, sv, tu, tv, \underline{x})$). Then $R/(F,G) \cong C[\underline{x}]/(g) \hookrightarrow D[\underline{x}]/(g)$. It will suffice to show that g is square-free in the polynomial ring $D[\underline{x}]$, which we grade so that u,v (and s,t) have degree one and the elements of $\underline{x}$ have degree 2. If g is not square-free then either (1) $g = h^2$, where h has degree 2, or (2) $g = h^2 r$ where h has degree 1 and r has degree 2. We obtain a contradiction by showing that g factors in $C[\underline{x}]$, i.e., that G factors in R/FR, which contradicts Proposition 3.2 (g).

We first study cases (1) and (2) when $C = K[u^2, v^2, uv]$. In case (1), $h = Q + M$ where Q is quadratic in u,v and M is linear in $\underline{x}$. Then $Q \in C$, obviously, and we have the required factorization.

In case (2) for $C = K[u^2, v^2, uv]$, h is a linear form in u,v and r must be the sum of a quadratic form Q in u, v and a linear form M in the K-span of $x_1, \ldots, x_n$. Thus, $g = h^2(Q+M)$. Then $h^2 \in C$, and because of the splitting $C[\underline{x}] \to D[\underline{x}]$, g is a multiple of h^2 in $C[\underline{x}]$. Since the multiplier needed is uniquely determined, $Q+M$ must be in $C[\underline{x}]$, and we have a factorization of g in $C[\underline{x}]$.

Now assume $C = K[su, sv, tu, tv]$. Let the multiplicative group $G = \mathrm{GL}(1,K)$ of K act on the polynomial ring $D[\underline{x}] = K[s,t,u,v,\underline{x}]$ so that $a \in G$ sends $s \mapsto as$, $t \mapsto at$, $u \mapsto a^{-1}u$, $v \mapsto a^{-1}v$ and fixes all of the variables $\underline{x}$. The fixed ring of this action is $C[\underline{x}]$.

In case (1), $h = Q + M$ where Q is quadratic in u,v,s,t and M is linear in $\underline{x}$. Working mod $(\underline{x})$, we see that $Q^2 \in C$. Since Q^2 is invariant under the action of G, so is Q (the group G is connected), i.e., $Q \in C$. But then $h \in C[\underline{x}]$, which yields a factorization of G mod F.

In case (2), since $g = h^2 r$ is fixed by the action of G, the ideals generated by h and r in $D[\underline{x}]$ must be fixed by this action (they cannot be permuted, since G is connected). This implies that the action of G maps h to multiples of itself by elements of K, i.e., h is semi-invariant for the action, and the same holds for r. Hence, h must be $\alpha s + \beta t$ or $\alpha u + \beta v$, where α, $\beta \in K$ are not both 0. Then h^2 is a semi-invariant that is multiplied by a^2 or a^{-2} as $a \in G$ acts. Now r must have the form $Q + M$, where Q is quadratic in u,v,s,t and M is linear in the variable in $\underline{x}$, and since $r = Q + M$ is a semi-invariant that is multiplied by a^{-2} or a^2 as $a \in G$ acts, we must have that $M = 0$, and that if h involves u,v (respectively, s,t), then Q involves s,t (respectively, u,v). Then Q factors LL' where L, L' are linear forms in the appropriate pair of variables, and we have $g = (hL)(hL')$ where the factors are in $C \subseteq C[\underline{x}] \cong R/(F)$. As before, the fact that G factors mod (F) contradicts Proposition 3.2 (g). This completes the proof of the R_0 condition for $k \geq 1$.

In the remainder of the proof we may assume that F and G are not contained in a polynomial K-subalgebra of R generated by fewer than N linear forms: if they are, we carry out the proof in this subring. The codimension of the singular locus and the properties we are considering do not change when one omits the superfluous variables. Assume that $k \geq 2$ and that we know the result for smaller k. We may assume that R is generated by $\widetilde{\mathcal{D}}F$ and $\widetilde{\mathcal{D}}G$: again, omitting unneeded variables does not change the codimension of the singular locus. Moreover, we may assume that $N \geq 2k+2$: otherwise, by Discussion 4.5, some non-trivial linear combination of F and G has rank at most $2k$ and so is not k-strong. We write $J(F,G)$ for the Jacobian matrix of F, G. To prove that the singular locus in $V(F,G)$ has codimension at least $2k-1$, we want to prove that the height of the defining ideal in $R/(F,G)$ is at least $2k-1$. This is equivalent to showing that the height of the ideal $I = (F,G) + I_2\big(J(F,G)\big)$ is at least $2k+1$ in R. We assume, to the contrary, that the height of some minimal prime P of I is at most $2k$.

If the singular locus consists only of the origin, then it has codimension $N-2 \geq (2k+2)-2 = 2k$, and we are done. Hence, we may apply Lemma 4.16 and assume $F = x_1x_2 + Q(x_3, \ldots, x_N)$ while G involves only $x_2, \ldots, x_N$. We may write $G = x_2L + M$ where $L = \sum_{j=2}^N c_jx_j$ with the $c_j \in K$ and $M \in K[x_3, \ldots, x_N]$. Then the Jacobian matrix $J(F,G)$ is

$$\begin{pmatrix} x_2 & x_1 & Q_3 & \ldots & Q_N \\ 0 & L + c_2x_2 & M_3 + c_3x_2 & \ldots & M_N + c_Nx_2 \end{pmatrix},$$

where Q_j, M_j denote partial derivatives with respect to x_j. It follows that $x_2 \in P$ or that all the partial derivatives of G are in P. Since $G \in P$ as well, we have that $\widetilde{\mathcal{D}}G \in P$. Since G has no k-collapse, its rank is at least $2k+1$, and this gives a contradiction. Thus, $x_2 \in P$.

We next observe that $KQ + KM$ is $(k-1)$-strong in $K[x_3, \ldots, x_N]$: if a, b are not both 0 and $aQ + bM$ has a $(k-1)$-collapse, then

$$aF + bG = x_2(ax_1 + bL) + (aQ + bM)$$

has a k-collapse, a contradiction. This implies that $(Q,M) + I_2\big(J(Q,M)\big)$ has height at least $2k-1$ in $K[x_3, \ldots, x_N]$. Since $x_2, F, G \in P$ we also have that Q, $M \in P$. It follows that the contraction P_1 of P to $K[x_3, \ldots, x_N]$ has height at least $2k-1$. Since P contains P_1 and x_2, it follows that the contraction P_0 of P to $K[x_2, \ldots, x_N]$ has height at least $2k$. Since we are assuming that P has height at most $2k$, we must have that $P = P_0R$ has height exactly $2k$. Thus, $x_1 \notin P$. Working mod P and omitting columns that are 0 we have, since $x_2 \in P$ that $J(F,G)$ has the form

$$\begin{pmatrix} x_1 & Q_3 & \ldots & Q_N \\ L & M_3 & \ldots & M_N \end{pmatrix}.$$

Since $x_1M_j - LQ_j \in P_0R$ for $j \geq 3$, and $M_j, L, Q_j \in k[x_2, \ldots, x_N]$, and $P = P_0R$ is the expansion of an ideal from $K[x_2, \ldots, x_N]$, we m we must have that all of the $M_j \in P_0 \subseteq P$, and that all of the $LQ_j \in P$. Since $M \in P$ as well, we have that $\widetilde{\mathcal{D}}M \subseteq P$. We must have that $L \notin P$, since it follows otherwise that the second row of the Jacobian matrix $J(F,G)$ is in P, and then we would have $\widetilde{\mathcal{D}}G \subseteq P$, which we have already noted gives a contradiction (G has rank at least $2k+1$.) Since all the LQ_j are in P for $j \geq 3$, we also have that all the $Q_j \in P$ for $j \geq 3$. Since Q is in P, we have that $\widetilde{\mathcal{D}}Q \subseteq P$. Hence, the linear space $Kx_2 + \widetilde{\mathcal{D}}M + \widetilde{\mathcal{D}}Q \subseteq P$, and since P has height $2k$ and $\widetilde{\mathcal{D}}M + \widetilde{\mathcal{D}}Q \subseteq Kx_3 + \ldots + Kx_N$ does not meet Kx_2, we see that $\widetilde{\mathcal{D}}M + \widetilde{\mathcal{D}}Q$ has K-vector space dimension at most $2k-1$. It follows from Discussion 4.5 that there are scalars $a, b \in K$, not both 0, such that $aM + bQ$ has rank at most $2k-2$. But then $aF+bG=(ax_1+bL)x_2+(aM+bQ)$ has rank at most $(2k-2)+2=2k$, a contradiction.

In the remainder of this argument we assume that $n>2$. We use induction on n. We know inductively that $R/(F_1, \ldots, F_{n-1})$ is a domain and so that $F_1, \ldots, F_n$ is regular sequence. Let $X=V(F_1, \ldots, F_n)$ and let Z denote the set of points of X where the rank of the Jacobian matrix is precisely $n-1$. The singular locus is the union of Z and the closed set Y where the rank of the Jacobian matrix is at most $n-2$. We must show that the dimension of $Z\cup Y$ is at most $N-(2k+3-n)$. (We must show that the codimension of the singular locus in X is at least $2(k+1-n)+1$, i.e., that its dimension is at most $N-n-\bigl(2(k+1-n)+1\bigr)$, since X has dimension $N-n$.) To see this for Y, note that $F_1, \ldots, F_{n-1}$ vanish on Y, and the rank of the Jacobian matrix of $F_1, \ldots, F_{n-1}$ on Y is at most $n-2$. Thus, Y is contained in the singular locus of $X_0=V(F_1, \ldots, F_{n-1})$, and $F_1, \ldots, F_{n-1}$ is k-strong. By the induction hypothesis, the dimension of the singular locus of X_0 is at most $N-(2k+3-n+1)$, which is one less than we require.

Hence, to complete the argument it will suffice to show that the dimension of each irreducible component C of the locally closed set Z is at most $N-(2k+3-n)$. Given such a component C, we have a map $\theta : C \to \mathbb{P}^{n-1}_K$ that sends each point $z\in C$ to the (unique up to multiplication by a nonzero scalar) relation on the rows of the Jacobian matrix evaluated at c: the $n\times N$ matrix has rank exactly $n-1$. Either the image of θ has dimension $\leq n-2$, or the image contains a nonempty Zariski open set U in $\mathbb{P}^{n-1}_K$. Consider a point $v=[a_1 : \cdots : a_n]$ in the image of C, and let F_v denote $a_1F_1+\cdots+a_nF_n$, which is well-defined up to multiplication by a nonzero scalar in K. The set of points in C in the fiber is the same as the set of points of C in $W_v=V(F_1, \ldots, F_n, J_v)$, where J_v is generated by the partial derivatives of F_v. Hence, the defining ideal of W_v contains $\widetilde{\mathcal{D}}F_v$. From the k-strength hypothesis, the dimension of W_v is at most $N-(2k+1)$. Hence, if the image of θ has dimension $\leq n-2$, the dimension of C is at most $N-(2k+1)+(n-2)=N-(2k+3-n)$, as required.

Therefore we may assume instead that the image of C under θ contains a dense open subset U of $\mathbb{P}^{n-1}_K$. After decreasing U, if necessary, we may assume that the fibers all have the same dimension, and that the dimension of C is the dimension of a fiber plus $n-1$. Again by decreasing U if necessary, we may assume that for all points $v\in U$, F_v has the same rank. If this rank is $2k+2$ or more, then since the fiber over v is contained in $V(\widetilde{\mathcal{D}}F_v)$, each fiber has dimension at most $N-(2k+2)$, and the dimension of C is at most $N-(2k+2)+n-1=N-(2k+3-n)$, as required. Therefore we may also assume that the rank of every F_v is $2k+1$ for $v\in U$.

Let F, G be K-linearly independent elements of $KF_1+\cdots+KF_n$ that correspond to distinct points $v_0, v_1\in U$. Then for all but finitely many choices of $[a:b]$, $aF+bG$ has the form F_v with $v\in U$. We claim that for $[a:b]$ in general position, F is not in $(\widetilde{\mathcal{D}}F_v)$. To see this, suppose otherwise. Then the point of $X_1=V(\widetilde{\mathcal{D}}F_v)$ is a a singular point of $V(F,F_v)$. Since this true for all but finitely many points of the line L through v_0 and v_1 in $\mathbb{P}^{n-1}_K$, the singular locus of X_1 has dimension at least $N-(2k+1)+1$. This contradicts the result for $n=2$, which asserts the dimension is at most $N-(2k+1)$.

Thus, for almost all points $v\in L$, the fiber over v in C has dimension at most $N-(2k+2)$, since the height of $(F,\widetilde{\mathcal{D}}F_v)$ is at least one more than the height of

$(\widetilde{\mathcal{D}}F_v)$. This implies that the dimension of C is at most $N-(2k+2)+(n-1)$, as required. $\square$

Examples 4.18. Let K be any algebraically closed field.
(a) Let $X = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$ be a 2×3 matrix of indeterminates over K, let Δ_i denote the 2×2 minor obtained by omitting the i th column, and let C_i denote the i th column, $1 \le i \le 3$. Let $V = K\Delta_2 + K\Delta_3$. The minors Δ_2 and Δ_3 overlap in C_1, and if $a, b \in K$, not both 0, $a\Delta_2 + b\Delta_3$ is the determinant of a 2×2 matrix whose columns are C_1 and $aC_2 + bC_3$: the four entries of these two columns are algebraically independent. Hence, every nonzero element of V has rank 4, and V is 1-strong. The quotient $K[\underline{u}, \underline{v}]/(V)$ is reduced but is not a domain: (V) has two minimal primes of height 2, one generated by all three of the size 2 minors of X, and the other by the two variables that are the entries of the first column. This shows that 1-strong does not imply that the quotient by (V) is R_1 nor even a domain.
(b) Now adjoin indeterminates x, y (respectively x, y, and z) to the ring in the example above and let $F = xy + \Delta_2, G = y^2 + \Delta_3$ (respectively, $G = yz + \Delta_3$). If $a, b \in K$ then $aF+bG = y(ax+by)+a\Delta_2+b\Delta_3$ (respectively, $x(ay+bz)+a\Delta_2+b\Delta_3$) which has rank 5 or 6 (respectively, rank 6) when a, b are not both 0: this follows because $a\Delta_2 + b\Delta_3$ has rank 4 and $y(ax+by)$ (respectively, $x(ay+bz)$), which involves disjoint variables from $a\Delta_2 + b\Delta_3$, has rank 1 or 2 (respectively, rank 2). Hence, V is 2-strong but the quotient is not a UFD: the image of y is irreducible but not prime. This also shows that 2-strong does not imply R_3.

Definition 4.19. For every integer $\eta \ge 0$ we define a function α_η from $\mathbb{N}_+$ to $\mathbb{N}_+$ as follows. First, $\alpha_\eta(1) := \lceil (\eta+1)/2 \rceil$. If $n \ge 2$, $\alpha_\eta(n) = n - 1 + \lceil \eta/2 \rceil$

We note that α (respectively, α_η) is the same as A_2 (respectively, ${}^\eta A_2$) in the notation of Remark 1.7. It will be convenient to use the shorter notation here.

Theorem 4.20. *Let K be an algebraically closed field and let $R = K[x_1, \ldots, x_N]$ be a polynomial ring over K. Let η, $n \ge 1$ be integers, and let V be a vector space of quadratic forms of dimension n over K. If V is $\alpha_\eta(n)$-strong, the quotient by the ideal generated by any subspace of V is a complete intersection with singular locus of codiismension at least $\eta + 1$, i.e., satisfies the Serre condition R_η, and so is a normal domain. These quotients are unique factorization domains for $n \ge 1$ if the strength of V is at least $n + 1$.*

If V is $\alpha_1(n)$-strong, the quotient of R by the ideal generated by any subspace of V is a normal domain. If $n \ge 1$ and V is $(n-1)$-strong, then every subset of V consisting of elements linearly independent over K is a regular sequence.

Proof. The cases where $n = 1$ and $n \ge 2$ follow from Proposition 4.10 and Theorem 4.14, respectively. Note that if $n = 1$ and $\eta = 2$, we have that $\alpha_2(1) = 2$, and a 2-strong form has quadratic rank at least 5, and so the quotient is a UFD (see Remark 4.11). If $n \ge 2$, note that the level of strength, namely, $n - 1$, needed to insure that $n - 2$ of the forms constitute a UFD sequence guarantees also that the n forms are a regular sequence by Discussion 1.8. $\square$

We can now make ${}^\eta A(n_1, n_2)$ explicit:

Corollary 4.21. *Let $A(n_1, n_2) := \alpha(n_2) + n_1$, where $\alpha(n_2) := n_2 - 1$. If $\eta \ge 1$ is an integer, define ${}^\eta A(n_1, n_2) := \alpha_\eta(n_2) + n_1$. Let K be an algebraically closed field* *of arbitrary characteristic, and let $R = K[x_1, \dots, x_N]$. Let V be a graded K-vector subspace of R with dimension sequence (n_1, n_2), and let I be the ideal generated by V. If V is $A(n_1, n_2)$-strong, then a homogeneous basis for V is a regular sequence, so that R/I is a complete intersection.*

Moreover, if V is ${}^{\eta}\!A(n_1, n_2)$-strong then not only does the previous conclusion hold, but also the singular locus of R/I is of codimension at least $\eta+1$ in R/I, i.e., R/I satisfies the Serre condition R_η, and so is a normal domain. The ring R/I is a unique factorization domain if the strength of V is at least $n_1 + n_2 + 1$.

Proof. By Proposition 3.2(c), modulo the linear forms in V, the images of the quadratic forms in V form a vector space of dimension at most n_2 in a polynomial ring and the level of strength has decreased by at most n_1. The result is now immediate from Theorem 4.20, which is the case $n_1 = 0$. □

Existence of ${}^{\eta}\!B$ in degree 2. In this section we treat the situation where $\delta = (n_1, n_2)$. As usual, we fix an algebraically closed field K and consider $R = K[x_1, \dots, x_N]$ over K. Let V have dimension sequence δ. We study ${}^{\eta}\!B(n_1, n_2)$ and also the smallest choice of $B = B(n_1, n_2)$ that enables us to find a regular sequence $G_1, \dots, G_B$ of length B such that V is contained in $K[G_1, \dots, G_B]$.

In the sequel we allow ${}^{\eta}\!B$ to represent B as well, i.e., we allow η to be empty, so that we can treat both cases simultaneously. We make the same convention with α and α_η, which, by Corollary 4.21, are the same as $A(0, n_2)$ and ${}^{\eta}\!A(0, n_2)$, respectively.

If $n_2 = 0$, we evidently have that ${}^{\eta}\!B(n_1) = n_1$ for all η: the quotients are all polynomial rings. Take $\alpha(n) := n - 1$, which will give the smallest value of B obtainable by our methods. For $n \geq 1$, if a K-vector space V of quadratic forms has dimension n and is $\alpha(n)$-strong, then every set of K-linearly independent forms in V is a regular sequence by Corollary 4.21. The function α_η is given in Definition 4.19. Note that for all n, $\alpha(n) \leq \alpha_1(n) \leq \cdots \leq \alpha_\eta(n) \leq \cdots$.

Let $x_1, \dots, x_{n_1}$ be a K-basis for the degree 1 component of V and let $F_1, \dots, F_{n_2}$ be a K-basis for the degree 2 component. We now make explicit the idea of the first two paragraphs of §2. We shall show that the function defined recursively by the formulas ${}^{\eta}\!B(n_1, 0) = n_1$, ${}^{\eta}\!B(n_1, n_2) = {}^{\eta}\!B(2n_1 + 2\alpha_\eta(n_2), n_2 - 1)$ if $n_2 \geq 1$ has the property we want. Note that $B(n_1, n_2) \leq {}^{\eta}\!B(n_1, n_2)$ for all $\eta \geq 1$ because $\alpha_\eta(n) \geq \alpha(n)$ for all $\eta \geq 1$ and $n \geq 1$.

It is straightforward to see that if we apply the recursion h times, where $1 \leq h \leq n_2$, we get that

$$ {}^{\eta}\!B(n_1, n_2) = {}^{\eta}\!B\big(2^h n_1 + \sum_{t=0}^{h-1} 2^{h-t}\alpha_\eta(n_2 - t), n_2 - h\big). $$

Hence, setting $h = n_2$, we have that $n_2 - h = 0$, and the value of the right term is simply the value of the first input. Hence, by letting $s = n_2 - t$ in the summation in the second term below, we have

$$ (*) \quad {}^{\eta}\!B(n_1, n_2) = 2^{n_2} n_1 + \sum_{t=0}^{n_2-1} 2^{n_2-t}\alpha_\eta(n_2 - t) = $$

$$ 2^{n_2} n_1 + \sum_{s=1}^{n_2} 2^s (s - 1 + \lceil \eta/2 \rceil) + 2(\lceil (\eta+1)/2 \rceil - \lceil \eta/2 \rceil) $$

where the final term displayed, whose value is 2 or 0, depending on whether η is even or odd, is included to account for the difference in the formula for $\alpha_\eta(s)$ when $s = 1$. That term agrees with $1 + (-1)^\eta$. The formula for $B(n_1, n_2)$ is obtained by omitting all terms involving η in the display above.

A routine calculation yields that $B(n_1, n_2) = 2^{n_2}(n_1 + 2n_2 - 4) + 4$ and that

$$^\eta B(n_1, n_2) = 2^{n_2}(n_1 + 2n_2 - 4) + 4 + \lceil \eta/2 \rceil (2^{n_2+1} - 1) + 1 + (-1)^\eta =$$

$$2^{n_2}(n_1 + 2n_2 + 2\lceil \eta/2 \rceil - 4) - \lceil \eta/2 \rceil + 5 + (-1)^\eta.$$

Note that $^\eta B(n_1, n_2) \geq B(n_1, n_2) \geq n_1 + n_2$ (one gets equality if $n_2 = 0$).

We now return to the issue left hanging and show by induction on n_2 that $^\eta B(n_1, n_2)$ has the required property. If the image of the space V is $\alpha_\eta(n_2)$-strong mod the ideal $(x_1, \ldots, x_{n_1})R$, then $x_1, \ldots, x_{n_1}, F_1, \ldots, F_{n_2}$ is a regular sequence of length $n_1 + n_2$ with the required property. Since $^\eta B(n_1, n_2) \geq n_1 + n_2$, we only need to consider the case where the image of V is not $\alpha(n_2)$-strong. Then some nonzero K-linear combination F of the F_j has a k-collapse in the quotient for some $k \leq \alpha(n_2)$, which leads to an equation

$$F = \sum_{s=1}^{n_1} x_s y_s + \sum_{t=1}^{k} L_t L'_t$$

where the y_s and the L_t, L'_t are linear forms or zero. Hence, F is in the K-algebra generated by at most $2n_1 + 2\alpha_\eta(n_2)$ linear forms which include $x_1, \ldots, x_{n_1}$ among them. Let W be the K-vector space spanned by these linear forms, and $n_2 - 1$ elements that together with F form a basis for the component of V of degree 2. Then W has a dimension sequence that is at worst $(2n_1 + 2\alpha_\eta(n_2), n_2 - 1)$, and $K[V] \subseteq K[W]$. By the induction hypothesis we know that $K[W]$ is contained in a K-algebra generated by a regular sequence consisting of at most $^\eta B(2n_1+2\alpha_\eta(n_2), n_2 - 1)$ linear and quadratic forms. This completes the proof that $^\eta B(n_1, n_2)$ defined recursively as above has the required property. Hence:

Theorem 4.22. *Let K be an algebraically closed field, and $R = K[x_1, \ldots, x_N]$. Then we may take $B(n_1, n_2) = 2^{n_2}(n_1 + 2n_2 - 4) + 4$, and we may take $^\eta B(n_1, n_2) = 2^{n_2}(n_1 + 2n_2 + 2\lceil \eta/2 \rceil - 4) - \lceil \eta/2 \rceil + 5 + (-1)^\eta$.*

Hence, if we have n_1 linear forms and n_2 quadratic forms, they are contained in an algebra generated over K by a regular sequence of linear and quadratic forms of length at most $2^{n_2}(n_1 + 2n_2 - 4) + 4$. If I is an ideal generated by n quadratic forms, the projective dimension of R/I over R is at most $2^{n+1}(n - 2) + 4$. □

5. Key functions and recursions for $^\eta \underline{A}(\delta)$

The key functions $\mathfrak{K}_i$, $i \geq 3$, were defined in Definition 1.13 of §1. Our main result on obtaining the functions $^\eta A$ from the functions $\mathfrak{K}_i$ is stated in Theorem 1.17 of §1, and provides obvious motivation for proving the existence of key functions in such a way that one has explicit formulas for them. While [2] proves that these functions exist in general, the methods are not constructive and do not yield explicit bounds. This section gives the proof of Theorem 1.17. The rest of this paper is devoted to giving explicit formulas for the key functions $\mathfrak{K}_i$ when $i = 3$ and when $i = 4$ and the characteristic is not 2 or 3.

In order to prove Theorem 1.17, we need several preliminary theorems. The first two are Theorems 2.1 and 2.4 of [2], whose statements are given just below.

Theorem 5.1. *Let K be an algebraically closed field, let $R = K[x_1, \dots, x_N]$ be a polynomial ring. Let $V = V_1 \oplus \cdots \oplus V_d$, where V_i is spanned by forms of degree i, and suppose that V has finite dimension n. Assume that a homogeneous basis for V is a regular sequence in R. Let X be defined by the vanishing of all the elements of V. Let S be the family of all subsets of V consisting of nonzero forms with mutually distinct degrees, so that the number of elements in any member of S is at most the number of nonzero V_i. For $\sigma \in S$, let C_σ be the codimension of the singular locus of $V(\sigma)$ in $\mathbb{A}^N_K$. Then the codimension in $\mathbb{A}^N_K$ of the singular locus of X is at least $(\min_{\sigma \in S} C_\sigma) - (n-1)$.*

Theorem 5.2. *Let K be a field, let R be a polynomial ring over K, and let M be an $h \times N$ matrix such that for $1 \le i \le h$, the i th row consists of forms of degree $d_i \ge 0$ and the d_i are mutually distinct integers. Suppose that for $1 \le i \le h$, the height of the ideal generated by the entries of the i th row is at least b. (If the row consists of scalars, this is to be interpreted as requiring that it be nonzero.) Then the ideal generated by the maximal minors of the matrix has height at least $b-h+1$.*

We will use the improved version given below:

Theorem 5.3. *Let K be a field, let R be a polynomial ring over K, and let M be an $h \times N$ matrix such that for $1 \le i \le h$, the i th row consists of forms of degree $d_i \ge 0$ and the d_i are mutually distinct integers. Let I be a proper homogeneous ideal of R. Suppose that for $1 \le i \le h$, the height of the ideal generated by the entries of the i th row together with I is at least b. (If the row consists of scalars, this is to be interpreted as requiring that it be nonzero.) Then the ideal generated by the maximal minors of the matrix together with I has height at least $b-h+1$.*

Proof. We may enlarge the field to be algebraically closed without loss of generality. We may assume without loss of generality that $d_1 \le \dots \le d_h$. We may assume that the height of I is less than b, or the result is obvious.

We use induction on h: the case where $h = 1$ is immediate. (If one has a single nonzero row of scalars, the height of the ideal generated by the maximal minors is $+\infty$.) We therefore assume $h \ge 2$ and that the result holds for smaller h. Next, we reduce to the case where the number of variables in R is b, and every non-scalar row, together with I, generates an ideal primary to the homogeneous maximal ideal . Suppose that the number of variables is greater than b. For each i, choose a subset of the span of the i th row generating an ideal J_i such that $J_i + I$ has height b. Choose a linear form that is not in any of the minimal primes of any of the $J_i + I$. We may kill this form, and the hypotheses are preserved: the height of the ideal generated by the maximal minors does not increase by Remark **??**. We may continue in this way until the number of variables is b.

Let P be a minimal prime ideal of the sum of the ideal generated by the maximal minors of M and I. To complete the proof, it will suffice to show that the dimension of the ring R/P is at most $h-1$.

Let $\overline{M}$ denote the image of the matrix M over R/P. It is possible that all of the maximal minors of the matrix formed by a proper subset consisting of $h_0 < h$ of the rows of $\overline{M}$ vanish in R/P. But then the height of the ideal generated by the maximal minors of these rows together with I is at least $b-h_0+1$ by the induction hypothesis, and this shows that the dimension of R/P is at most h_0-1. Hence, we may assume that there is no linear dependence relation on any proper subset of the rows of the image of $\overline{M}$, while the rank of the image $\overline{M}$ is $h-1$. This implies that there are unique elements of the fraction field of R/P, call them $u_1, \ldots, u_{h-1}$, such that $\rho_h = \sum_{i=1}^{h-1} u_i \rho_i$, where ρ_i is the image of the i th row of M. More specifically, since the first $h-1$ rows of $\overline{M}$ are linearly independent over $\mathrm{frac}(R/P)$, we may choose $h-1$ columns forming an $h \times (h-1)$ submatrix $\overline{M}_0$ of $\overline{M}$ such that the $h-1$ size minor Δ of the first $h-1$ rows is not 0. The nonzero relation, unique up to multiplication by a nonzero scalar in $\mathrm{frac}(R/P)$, on the rows of the submatrix $\overline{M}_0$ is given by the vector whose entries are its $h-1$ size minors, which are homogeneous elements of R/P. This must give the relation on the rows of $\overline{M}$. Thus, every u_j can be written as a fraction with denominator Δ whose numerator is one of the other minors of $\overline{M}_0$. Let S be the ring $(R/P)[u_1, \ldots, u_{h-1}]$. Note that u_i has degree $d_h - d_i > 0$, so that S is a finitely generated $\mathbb{N}$-graded algebra over K with $S_0 = K$ generated over K by the images of the x_i and by the u_i. The Krull dimension of S is the same as that of R, since the fraction field has not changed, and that is the same as the height of the maximal ideal of S. But $S/(u_1, \ldots, u_{h-1})S$ is zero-dimensional, since the vanishing of the u_i implies the vanishing of all entries of ρ_h, and these generate an ideal primary to the maximal ideal of $K[x_1, \ldots, x_b]$. It follows that the Krull dimension of S is at most $h-1$, and, hence, the same holds for R/P, as required. $\square$

Definition 5.4. Let F be a form of positive degree in a polynomial ring R over a field. By the *J-rank* of F we mean the height of the ideal generated by F and all of its partial derivatives.

We make the convention that the J-rank of a nonzero linear form is $+\infty$. If $R = K[x_1, \ldots, x_N]$, where K is an algebraically closed field, the J-rank of F is the codimension of the singular locus of $V(F)$ in $\mathbb{A}^N_K$. We note that the J-rank of a quadratic form is the same as its rank: see Proposition 4.2 and the preceding discussion and Proposition 4.10.

Discussion 5.5. Let e_d denote the dimension sequence $(0, ,0, \ldots, 0, 1)$, with the 1 in the d th spot. It will be convenient to have a special name for the functions ${}^{\eta}A(e_d)$. In the definition below, $\mathfrak{J}_i(k)$ for $k \geq 2$ has the same defining property as ${}^{\eta}A(e_i)$ with $\eta = k-2$. We shall see that the existence of the functions $\mathfrak{K}_i$ implies the existence of the functions $\mathfrak{J}_i$, and this in turn implies the existence of all of the functions ${}^{\eta}A$.

Definition 5.6. Let $\mathcal{C}$ be a set of possible characteristics. Let $i \geq 1$. We shall say that $\mathfrak{J}_i : N_+ \to N_+$ is a *J-rank function* for $\mathcal{C}$ and for i if for every algebraically closed field K of characteristic in $\mathcal{C}$, for every $N \in \mathbb{N}_+$ and polynomial ring R in N variables over K, and for every form $F \in R$ of degree i, if F is $\mathfrak{J}_i(k)$-strong then the J-rank of F is at least k.

We shall say that a function from $\mathbb{N}^d \to \mathbb{N}$ is *nondecreasing*, if it is nondecreasing when the entries in all but one coordinate are held fixed.

Theorem 5.7. *Let $d \geq 1$ and for $1 \leq i \leq d$, let $\mathfrak{J}_i$ be a nondecreasing J-rank function for the set of characteristics $\mathcal{C}$. Let K be an algebraically closed field of characteristic in $\mathcal{C}$ and let R be a polynomial ring in N variables over K. Let $\delta = (n_1, \ldots, n_d)$ be a dimension sequence, let $n = \sum_{i=1}^{d} n_i$, and let h be the number of nonzero elements among $n_2, \ldots, n_d$. Then we may take ${}^{\eta}A_1(\delta) = 0$ and*

${}^{\eta}A_i(\delta) = \mathfrak{J}_i(h-1+2(n-n_1)+\eta)+n_1$ for $2 \leq i \leq d$. That is, if $V \subseteq R$ is a graded K-vector space with dimension sequence δ such that V_i is $\big(\mathfrak{J}_i(h-1+2(n-n_1)+\eta)+n_1\big)$-strong for $2 \leq i \leq d$, then R_η holds for the algebraic set defined by the vanishing of V or any set of forms in V.

Proof. If there is only one form F of degree d the result is clear: no condition is needed if $d = 1$, and otherwise we have $n_1 = 0$ and $n - n_1 = h = 1$, so that the condition required is $\mathfrak{J}_d(\eta + 2)$. This gives codimension $\eta + 2$ for the singular locus of F in $\mathbb{A}^N$, and so codimension $\eta + 1$ in $V(F)$.

We next note that by passing to the polynomial ring $R/(V_1)$, i.e., by first killing the 1-forms in V, we may assume that $n_1 = 0$. The levels of strength imposed all drop by at most n_1, by Proposition 3.2(c).

By induction on the dimension of V we know that any set of linearly independent forms in V generating a subspace of smaller dimension defines an algebraic set that satisfies R_η. Hence, any basis for V consisting of forms is a regular sequence. To show that R_η holds for the algebraic set X defined by V with dimension sequence $n_1, \ldots, n_d$, we need the codimension of the singular locus to be $\eta + 1$ in X, and hence to be $n + \eta + 1$ in the ambient space. In the worst case, by Theorem 5.1 we need the height of the ideal of minors of the Jacobian obtained from at most one form of each degree to be bounded below by $n + \eta + 1 + (n - 1)$, which means, by Theorem 5.3, that it suffices if the height of the span of every row is at least $h - 1 + 2n + \eta$ (there will be a maximum of h rows). If $n_1 = 0$, this means that one can simply take the i th entry of ${}^{\eta}\underline{A}(n_1, \ldots, n_d)$ to be $\mathfrak{J}_i(h - 1 + 2n + \eta)$, where h is the number of nonzero elements in the dimension sequence. Note that $n = n - n_1$ in this case. If $n_1 \neq 0$, one kills the ideal generated by the n_1 linear forms: the height of the ideal generated by the entries of the i th row cannot drop lower than $\mathfrak{J}_i(h - 1 + 2(n - n_1) + \eta)$, where we are now in the case where $n_1 = 0$ and n has been replaced by $n - n_1$. □

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Note that for the case where $n_1 \neq 0$, we may proceed alternatively without killing the linear forms. When we reduce to studying a Jacobian matrix for one form of each degree, we have at most $h + 1$ rows, one of which is a nonzero row of scalars. By Theorem 5.1, if we guarantee that the height of the ideal of minors is $n+\eta+1+(n-1) = 2n+\eta$, this will yield the required codimension for the singular locus in X. However, the number of rows the matrix may now be $h+1$, and so the strength condition on the forms of degree i in V is $\mathfrak{J}_i(h + 2n + \eta)$. In practice, the functions $\mathfrak{J}_i$ grow quickly enough that the result in Theorem 5.7 is better.

We first recall that a subset Y of the closed points X in a scheme of finite type over an algebraically closed field K is called *constructible* if it is a finite union of locally closed subsets of X. The image of a constructible set under a K-regular map is constructible. Constructible sets include open sets and closed sets in X, and the family of constructible sets is closed under finite union, finite intersection, and complementation within X. Note that if a constructible set is dense in a variety, then one of the locally closed sets in the union must be dense, and so the constructible set contains a dense open subset of the variety.

Theorem 5.8. *Let K be an algebraically closed field. Let $R = K[x_1, \ldots, x_N]$ be a polynomial ring over K. Let $\mathcal{M}$ be an $1 \times M$ matrix whose entries are d-forms in R, $d \geq 1$. Let $b, k \geq 1$ be integers. Let $\mathcal{V}$ denote the K-span of the entries of*

$\mathcal{M}$. Suppose that for every dense open subset U of $\mathcal{V}^b$, the b-tuples of elements of $\mathcal{V}$, some K-linear combination of the entries of some $v \in U$ with at least one nonzero coefficient has a k-collapse. Then there exists a K-vector subspace Θ of $\mathcal{V}$ of codimension at most $b-1$ such that every nonzero element of Θ has a strict bk-collapse.

Hence, if there is no vector subspace Θ of $\mathcal{V}$ of codimension at most $b-1$ such that every nonzero element of Θ has a strict bk-collapse, then there is an open dense subset U of the b-tuples of elements of $\mathcal{V}$ such that the entries of each element of U are linearly independent and k-strong.

Proof. We may replace $\mathcal{M}$ by a matrix whose entries are a basis for $\mathcal{V}$, and so we assume that the entries of $\mathcal{M}$ are K-linearly independent and that the vector space dimension of $\mathcal{V}$ is M. We may therefore assume that the nonzero entries of $\mathcal{M}$ are linearly independent. Moreover, we may omit the entries that are zero. We write $\mathcal{V}^b$ for the K-vector space of b-tuples of elements of $\mathcal{V}$. It should not cause confusion if we also think of elements of $\mathcal{V}^b$ as $1 \times b$ matrices whose entries are in $\mathcal{V}$. We write $\mathbb{P}(\mathcal{V}^b)$ for the corresponding projective space over K: it consists of b-tuples of elements $\mathcal{V}$, not all 0, modulo multiplication by scalars in $K^\times = K - \{0\}$. We write $\mathbb{P}(W_b)$ for the projective space associated with the vector space W_b of $b \times 1$ matrices over K. Given $v \in \mathcal{V}^b$ and $w \in W_b$, we write $w * v$ for the unique element of wv, which is an element of $\mathcal{V}$. For a given choice of w, $w * v$ is a K-linear combination of the entries of v: all of the entries of w occur as coefficients. By varying w, we get all K-linear combinations of entries of v. Let μ denote the map $W_b \times \mathcal{V}^b \to \mathcal{V}$ such that $(w, v) \mapsto w * v$.

Next, consider the subset $\mathcal{Z} \subseteq W_b \times \mathcal{V}^b$ consisting of pairs (w, v) such that $w * v$ has a strict k-collapse or is 0. The set of forms of degree d with a strict k-collapse together with 0 form a closed set by Proposition 3.3(c). $\mathcal{Z}$ is the inverse image of this set under the regular morphism μ, and so $\mathcal{Z}$ is closed in $W_b \times \mathcal{V}^b$. Since $\mathcal{Z}$ is closed under multiplying either coordinate by a nonzero scalar, it determines a closed set $X \subseteq \mathbb{P}(W_b) \times \mathbb{P}(\mathcal{V}^b)$. If the projection map $\pi_2 : X \to \mathbb{P}(\mathcal{V}^b)$ does not have dense image, the set of matrices representing points of the complement of its image (its image is closed) is a dense open set U in $\mathcal{V}^b$ such that every nonzero linear combination of the entries of any element of U is k-strong. Therefore we may assume that the projection map is surjective.

Since the projection map is surjective, the dimension of X is at least $bM - 1$. It follows that there is a point of $\mathbb{P}(W_b)$ such that the fiber of $\pi_1 : X \to \mathbb{P}(W_b)$ has dimension at least $(bM-1) - (b-1) = bM - b$. Fix a nonzero $b \times 1$ matrix w representing an element where the fiber has dimension at least $bM - b$, and so has codimension at most $bM - 1 - (bM - b) = b - 1$ in $\mathbb{P}(\mathcal{V}^b)$. Choose an irreducible closed subset X_1 of the fiber over w of codimension at most $b-1$ in $\mathbb{P}(\mathcal{V}^b)$. Let $Y_1 = Y$ be the affine cone over X_1: Y is also irreducible, and has codimension at most $b-1$ in the affine space $\mathcal{V}^b$. Let $Y_s = Y + \cdots + Y$, by which we mean $\{v_1 + v_2 + \cdots + v_s : v_j \in Y, 1 \le j \le s\}$. Then Y_s is a family of $1 \times b$ matrices v with entries in $\mathcal{V}$ such that every $w * v = w * v_1 + w * v_2 + \cdots + w * v_s$ has a strict sk-collapse or is 0. Note that Y_s is a cone in the affine space $\mathcal{V}^b$, and it is an irreducible constructible set, since it is the image of Y^k under a regular morphism. Let Z_s be the closure of Y_s. It is an irreducible closed set in the affine space $\mathcal{V}^b$, and it is also a cone. The image X_s of $Z_s - \{0\}$ is a closed irreducible set in $\mathbb{P}(\mathcal{V}^b)$.

The chain of closed varieties $Z_1 \subseteq Z_2 \subseteq \cdots Z_j \subseteq \cdots$ must have the property that for some s, $Z_s = Z_{s+1}$, since whenever it increases strictly the codimension drops, and so the number of increases cannot exceed the codimension, which is a most $b-1$. Thus, $s \leq b$. We claim that if $Z_s = Z_{s+1}$ then $Z_j = Z_s$ for all $j \geq s$. By induction, it suffices to show that $Z_{s+1} = Z_{s+2}$. But since $Z_s = Z_{s+1}$, we have that Y_s is dense in Y_{s+1}, and so $Y_s \times Y$ is dense in $Y_{s+1} \times Y$. But then the image of $Y_s \times Y$ is dense in the image of $Y_{s+1} \times Y$ under the map $(v', v) \mapsto v' + v$, which means that $Y_{s+1} = Y_s + Y$ is dense in $Y_{s+1} + Y = Y_{s+2}$, and this implies that $Z_{s+1} = Z_{s+2}$. Hence, $Z_s = Z_{2s}$. Similarly, since $Y_s \times Y_s$ is dense in $Z_s \times Z_s$, $Y_{2s} = Y_s + Y_s$ is dense in $Z_s + Z_s$. Hence, $Z_s + Z_s$ is contained in the closure of Y_{2s}, which is $Z_{2s} = Z_s$. It follows that the cone Z_s is a vector space, which we denote $\{w\} \times \mathcal{W}$, where $\mathcal{W} \subseteq \mathcal{V}^b$, and it has codimension at most $b-1$ in $\mathcal{V}^b$. The points in $v \in \mathcal{W}$ such that $w * v$ has a strict bk-collapse are dense. But this set is also the inverse image of a closed set in R_d under a regular map, and is therefore closed. Hence, for every $v \in \mathcal{W}$, $w * v$ has a strict bk-collapse. The map $v \mapsto w * v$ is clearly a surjection of $\mathcal{V}^b$ onto $\mathcal{V}$. Let Θ be the image of this map. Since $\mathcal{W}$ has codimension at most $b-1$ in $\mathcal{V}^b$, we have Θ has codimension at most $b-1$ in $\mathcal{V}$. Since every element of Θ has a strict bk-collapse, the proof is complete. □

Given a finite-dimensional vector space V over a field K, we say that a condition holds for elements of the vector space in general position if it holds for the elements of a dense open subset of V. In the theorem below, this is applied to the vector space of b-tuples of elements of $\mathcal{D}F$ for a form F.

Theorem 5.9. *Let $i \geq 2$ be an integer, and let K be an algebraically closed field of characteristic not dividing i. Let $\mathfrak{K}_i$ be a key function from $\mathbb{N}_+ \to \mathbb{N}_+$ for degree i for polynomial rings over K. Then for all $b \in \mathbb{N}_+$, if F is a form of degree i that has strength at least $\mathfrak{K}_i(bk) + b - 1$, then any b-tuple of elements of $\mathcal{D}F$ in general position has entries that are linearly independent over K and spans a vector space of strength at least k.*

Proof. Assume that F is a form of degree i that does not have a strict $(\mathfrak{K}_i(bk)+b-1)$-collapse. If the conclusion of the theorem fails, then by Theorem 5.8, there is a K-vector subspace Θ of $\mathcal{D}F$ of codimension at most $b-1$ such that every element of that K-vector subspace has a strict bk-collapse. We may make a linear change of variables and so assume that if $j < N-(b-1)$ then $\frac{\partial F}{\partial x_j} \in \Theta$. Let Q be the ideal generated the final substring of at most $b-1$ consecutive variables $x_{s+1}, \dots, x_N$ such that all the partial derivatives with respect to variables $x_1, \dots, x_s$ not in Q are in Θ. Since the formation of partial derivatives with respect to x_j not in Q commutes with killing Q, if we let $\overline{F}$ denote the image of F modulo Q in $K[x_1, \dots, x_s]$, where $N - s \leq b-1$, then we have that every element in the span of the partial derivatives of $\overline{F}$ has a strict bk-collapse. By the defining property of $\mathfrak{K}_i$, $\overline{F}$ has a strict $\mathfrak{K}_i(bk)$-collapse, and is in an ideal J generated by at most $\mathfrak{K}_i(bk)$ forms of smaller degree. This means that $F \in J + Q$, where $Q = (x_{s+1}, \dots, x_N)$ is generated by at most $b-1$ elements of degree 1, and so has an $(\mathfrak{K}_i(bk) + b - 1)$-collapse, a contradiction. □

Corollary 5.10. *Let K be an algebraically closed field for which we have a key function $\mathfrak{K}_i$ for some $i \geq 2$, where the characteristic of K does not divide i. Suppose also A_{i-1} is function such that a K-vector space of dimension n whose nonzero* *elements are forms of degree $i-1$ of strength at least $A_{i-1}(n)$ is generated by a regular sequence. Then we may take $\mathfrak{J}_i(k) := \mathfrak{K}_i\big(kA_{i-1}(k)\big) + k - 1$.*

Proof. Let F be a form of degree i. We may apply Theorem 5.9 with using k and $A_{i-1}(k)$ for the values of b and k. It follows that a vector subspace of $\mathcal{D}F$ generated by k elements in general position has strength $A_{i-1}(k)$, and so is generated by a regular sequence. $\square$

We are now ready to give the proof of Theorem 1.17.

Proof of Theorem 1.17. This is simply the result of combining Theorem 5.7 and Corollary 5.10. The condition in the definition of ${}^{\eta}A_2$ simply guarantees, when $n_1 = 0$, that the height of $\mathcal{D}F$ is at least b for every form $F \in V$. $\square$

In the case where one simply has a vector space of dimension n whose nonzero elements are all forms of degree d, this takes a simpler form.

Theorem 1.17 shows that for a class $\mathcal{C}$ of characteristics and degree at most d, if one has explicit key functions $\mathfrak{K}_i$ for the characteristics in $\mathcal{C}$ and $i \leq d$, then one obtains specific ${}^{\eta}A$ for degree sequences bounded by d. The results of § 2 then yield the functions ${}^{\eta}B$ up to degree d (i.e., the existence of small subalgebras) and, hence, we obtain explicit bounds for projective dimension, proving Stillman's conjecture in degree d, but in a constructive way. Moreover, these bounds are obtained for a situation more general than that of homogeneous ideals, as shown in § 2: one gets them for modules such that one has bounds on the size of the presentation, and without homogeneity assumptions.

In §6, we obtain our main results on cubics using both this result and other ideas. The cases of characteristics 2 and 3 require larger choices of ${}^{\eta}\underline{A}$. The difficulties are handled by using choices of $\mathfrak{J}_3$ arising form variant methods and applying Theorem 5.7. In §§7-9 we construct the functions $\mathfrak{K}_4$ for characteristic not 2, 3, which enables us to give explicit functions ${}^{\eta}A$, ${}^{\eta}B$, and $C(r, s, d)$ for degree $d \leq 4$ if the characteristic is not 2 or 3. The argument needed for $d = 4$ is very intricate.

6. The cubic case: $\mathfrak{K}_3$, $\mathfrak{J}_3$, and ${}^{\eta}\underline{A}(n_1, n_2, n_3)$

The main result of this section is Theorem 6.4 below. We first note:

Theorem 6.1. *For any algebraically closed field of characteristic $\neq 2, 3$ we may take $\mathfrak{K}_3(k) = 2k$. Hence, if K is a field of characteristic $\neq 2, 3$, we may take $\mathfrak{J}_3(k) = (2k+1)(k-1)$.*

Proof. The first statement is immediate from Theorem 4.8, and the second follows at once from Corollary 5.10 and the fact that by the last statement in Theorem 4.20, we may choose $A_2(k) = k - 1$. $\square$

Hence, by Theorem 5.9:

Theorem 6.2. *For any algebraically closed field K of characteristic $\neq 2, 3$, if b is a positive integer and F is a form of degree 3 in a polynomial ring over K such that F has strength at least $2bk + b - 1$, then any b-tuple of elements of $\mathcal{D}F$ in general position has entries that are linearly independent over K and span a vector space of strength at least k.*

We next want to show that one can construct a function of $\mathfrak{J}_3$ for cubics when K is an algebraically closed field of characteristic 2 or of characteristic 3.

Theorem 6.3. *Let R be a polynomial ring over an algebraically closed field. If K has characteristic 2, then we may take $\mathfrak{J}_3(k) = 2(k-1)(2k+1)$. If K has characteristic 3, we may take $\mathfrak{J}_3(k) = 2k^2 - k$.*

Proof. Let k be given and let F be a cubic form that is $\mathfrak{J}_3(k)$-strong according to the appropriate formula above. We shall show that the J-rank of F is at least k. By the last statement in Theorem 4.20, it will suffice if $\mathcal{D}F$ contains a k-tuple of linearly independent quadrics over K spanning a K-vector space of strength at least $k-1$. If this is not true, let $b = k$ and note that by the last sentence of Theorem 5.8, there is a vector space Θ of codimension at most $b-1 = k-1$ in $\mathcal{D}F$ such that every element has a $k(k-1)$-collapse. Choose an element $G \in \Theta$ such that the rank r of G is maximum. Since G has a $k(k-1)$-collapse, $r \leq 2k(k-1)$.

In the characteristic 2 case, let $r = 2h + \epsilon$, where ϵ is 0 or 1, and $2h$ is the dimension of $\mathcal{D}G$. We then have that $h \leq k(k-1)$. The ideal $(\mathcal{D}G)R$ is generated by $2h$ variables. By Corollary 4.7, the image of Θ mod the ideal $(\mathcal{D}G)R$, is a vector space consisting of squares of linear forms and 0: otherwise, a linear combination of G and another element of Θ will have larger rank. Let q be the dimension of the space W spanned by these linear forms. Note that if $q - 2h \geq k$, then the J-rank of F must be at least k, since the image of $\mathcal{D}F$ modulo $(\mathcal{D}G)$ will contain the image of the squares of the linear forms in W. Therefore, we may assume that $q \leq 2h + k - 1 \leq 2k(k-1) + k - 1$. Then Θ is in the ideal spanned by the $2h$ variables in $\mathcal{D}G$ and the q linear forms spanning W. It follows that $\mathcal{D}F$ is in the ideal generated by these $2h + q$ elements, along with at most $k-1$ elements of $\mathcal{D}F$: as many as needed to span $\mathcal{D}F/\Theta$. Since we may use Euler's formula (the characteristic is 2, and the degree of F is 3), we have that F is in the ideal generated by $2h+q+k-1 \leq 2k(k-1)+2k(k-1)+k-1+k-1 = 2(k-1)(2k+1)$ elements of lower degree, a contradiction.

In the characteristic 3 case, every element of Θ is in the ideal J generated by $\mathcal{D}G$, by Corollary 4.7. Choose a K-basis $y_1, \ldots, y_N$ for R_1 such that $y_1, \ldots, y_r$ is a basis for $\mathcal{D}G$: since the characteristic of K is not 2, the rank of G is the same as the K-vector space dimension of $\mathcal{D}G$. Now consider the image $\overline{F}$ of F modulo J in the polynomial ring $R/J \cong K[y_{r+1}, \ldots, y_N]$. If $j > r$, then $\partial\overline{F}/\partial y_j$ is simply the image in R/J of $\partial F/\partial y_j$. Hence, $\mathcal{D}\overline{F}$ is contained in the image of $\mathcal{D}F$ mod J. This is spanned by the images of at most $k-1$ elements of $\mathcal{D}F$, since $\Theta \subseteq J$. Thus $\mathcal{D}\overline{F}$ has K-vector space dimension at most $k-1$ over K.

We may take a new basis $z_1, \ldots, z_{N-r}$ for the forms of degree 1 in R/J such that $\partial\overline{F}/\partial z_h$ are 0 for $h > k-1$. This implies that every term of $\overline{F}$ that is not a scalar times the cube of a variable is in $(z_1, \ldots, z_{k-1})R$: if $cz_iz_jz_t$ or $cz_iz_j^2$ occurs in $\overline{F}$ for i,j,t (respectively, i,j) distinct and with $c \in K - \{0\}$, then $\partial F/\partial z_i$ has a cz_jz_t term (respectively, a cz_j^2 term), and so z_j or z_t (respectively, z_j) must be in J. The sum of the terms of $\overline{F}$ involving cubes of variables may be written as the cube of a single linear form L, and this implies that $\overline{F}$ is in the ideal generated by the k elements $z_1, \ldots, z_{k-1}$ and L. It follows that F is in the ideal generated by $r + k$ linear forms, and $r \leq 2(k-1)k$, which yields that F has a $\big(2(k-1)k + k\big)$-collapse, a contradiction. $\square$

We can now carry through the calculation of ${}^{\eta}\underline{A}$ in case $d \leq 3$.

First, we assume that $n_1 = 0$. Thus, $\delta = (0, n_2, n_3)$, and the case where $n_3 = 0$ has already been handled by Corollary 4.21.

By the last statement in Theorem 4.20, we may take $A_2(0, n_2) = n_2 - 1$ and hence $A_2(n_1, n_2) = n_1 + n_2 - 1$.

By Theorem 5.7, we have at once:

Theorem 6.4. *Let K be an algebraically closed field. Let $b = 2(n_2 + n_3) + \eta + 1$ if $n_2 \neq 0$, and $2(n_2 + n_3) + \eta$ if $n_2 = 0$. Then we may take*

$$^{\eta}\underline{A}(0, n_2, n_3) = \Big(0, \lceil \frac{b}{2} \rceil, \mathfrak{J}_3(b)\Big)$$

where $\mathfrak{J}_3(b) = (2b+1)(b-1)$ if the $\mathrm{char}(K)$ is not 2 or 3, $\mathfrak{J}_3(b) = 2(2b+1)(b-1)$ if $\mathrm{char}(K)$ is 2, and $\mathfrak{J}_3(b) = 2b^2 - b$ if $\mathrm{char}(K)$ is 3. More generally,

$$^{\eta}\underline{A}(n_1, n_2, n_3) = \Big(0, \lceil \frac{b}{2} \rceil + n_1, \mathfrak{J}_3(b) + n_1\Big).$$

□

For application to the degree four case in §10 we need the following:

Corollary 6.5. *Let $A_3(n) := \mathfrak{J}_3(2n-1)$, which is $2(4n-1)(n-1)$ if the characteristic is different from 2, 3. Then a sequence of n linearly independent cubic forms in a polynomial ring R over an algebraically closed field such that the vector space V spanned by the forms is $A_3(n)$-strong is a regular sequence.*

Proof. As in Remark 1.8, it suffices if any $n-2$ or fewer of the forms generate an ideal whose quotient ring is a UFD, and so it suffices if the strength of V is at least $^{\eta}A(0, 0, n-2)$ with $\eta = 3$. In this case, $b = 2(n-2) + 3 = 2n - 1$. □

Note that in all cases, $\mathfrak{J}_3(b)$ is quadratic in b with leading coefficient 2 or 4.

By the results of §2 we obtain as well explicit choices of $^{\eta}B$ and C. However, the construction of $^{\eta}B$ is recursive, and the discussion below shows that one gets extraordinarily large values from these techniques. It remains of great interest to find better bounds for the functions $^{\eta}B$. So far as we know, it is even possible that one can use a polynomial in n of degree d to bound the projective dimension of an ideal generated by n forms of degree at most d. If $d = 2$, a specific bound that is quadratic in n is presented in Question 6.2 of [25], but it is open question whether this bound is correct for the case of n quadrics.

Discussion 6.6. **The size of $\boldsymbol{B(n_1, n_2, n_3)}$.** The behavior of the value of $B(0, 0, n)$ obtained from the method of §2 is complicated, and the values are very large. The roles played by the linear and quadratic forms that arise in the process of repeated application of the method have a heavy impact. Assume the simplest case, which is that the characteristic is not 2 or 3. What can happen is that, for a certain constant positive integer a, one cubic form may collapse, producing roughly an^2 linear forms and an^2 quadratic forms. What may happen next is that all of the quadratic forms collapse. This may produce roughly $\mathcal{N} = an^2 2^{an^2}$ linear forms. To get another cubic to collapse (mod the linear forms one already has) the number $\mathcal{N}$ of linear forms must be added into the level of strength needed to get one more cubic to collapse. That means that getting one more cubic form to collapse will result in at least $\mathcal{N}$ new linear forms and $\mathcal{N}$ new quadratic forms. If all the quadratic forms collapse before another cubic collapses, one obtains $\mathcal{N}2^{\mathcal{N}}$ new linear forms, and this number must be added to the level of strength one needs for the cubics to guarantee a regular sequence. It should be clear that continuing in this way, where at each stage all the quadratic forms introduced collapse before the next cubic does, will produce, so far as one can tell *a priori*, something worse than n-tuple exponential behavior, since n repetitions are needed before all the cubics have collapsed. Because we believe that these results are far, far larger than needed to bound projective dimension, we have not made detailed estimates.

7. Sum decompositions of symmetric matrices

Overview of the argument for quartics. The results of this and the next three sections will handle the proof of the existence of ${}^{\eta}\underline{A}(n_1, n_2, n_3, n_4)$. The problem we confront is the construction of $\mathfrak{K}_4$. Roughly speaking, we must show that if F is quartic and every element of $\mathcal{D}F$ has a strict k-collapse, then F itself has a strict k'-collapse, where k' is "small" in the sense that it may depend on k but not on N. To prove this we study the Hessian $\mathfrak{H}$ of F. We let $\mathrm{GL}(N, K)$ act on the variables to put them in general position.

Given a matrix $\mathcal{M}$ with entries in a vector space over a field K, we refer to a K-linear combination of the columns of $\mathcal{M}$ as an LC_K-column of $\mathcal{M}$. The assumption that every element of $\mathcal{D}F$ has a k-collapse implies that every LC_K-column of the Hessian $\big(\partial^2 F/\partial X_i \partial X_j\big)$ has a collective $(k,\, k)$-collapse (cf. see the last three paragraphs of Definition 3.1), i.e., all of the entries of the LC_K-column are in the sum of an auxiliary ideal generated by at most k homogeneous polynomials of degree 1 and the auxiliary vector space spanned by at most k quadrics. The reason is that every LC_K-column consists of the partial derivatives of an appropriate linear combination G of the $\partial F/\partial X_i$. Since $G \in \mathcal{D}F$, G has a k-collapse, and we can write $G = \sum_{t=1}^{k} L_t Q_t$, where the L_t are linear forms and the Q_t are quadrics. The product rule implies that every $\partial G/\partial X_j$ is in the K-vector space sum of the ideal generated by the L_t and the K-vector space spanned by the Q_t.

We want to show that under this (k, k)-collapse condition on the LC_K-columns of the Hessian $\mathfrak{H}$, we can write $\mathfrak{H} = \mathfrak{H}_1 + \mathfrak{H}_2$, where these are symmetric matrices of quadrics such that (1) all of the LC_K-columns of $\mathfrak{H}_1$ have entries with "small" rank (independent of the column) and (2) the entries of $\mathfrak{H}_2$ span a vector space W of "small" dimension. This means that we will show, in effect, that a single vector space W of "small" dimension can be used for the collective (k', h')-collapse of all of the columns of $\mathfrak{H}$: k', h' are typically larger than the original k and h (which were k and k) but are given by functions of them independent of the number of variables generating the polynomial ring and also of the base field. Achieving condition (2) is studied in the next section. After that, in Section 9, we study symmetric matrices $\mathfrak{H}_1$ of quadrics in which every entry of every LC_K-column has small rank, independent of the column. We prove that for such a matrix, $G = (G_{ij})$, $\sum_{ij} x_i x_j G_{ij}$ has a small collapse. We then use the fact that an integer multiple (a restriction on the characteristic is needed to make certain the integer is nonzero in the base field) of F is congruent to G modulo an ideal with a small number of generators (a basis for W) to conclude that F has a small strict collapse. All of this is made precise in the sequel.

Some of the needed results are valid in higher degree cases, and we prove them in that generality.

It will be convenient to use general position arguments in which a certain finite family of elements is taken to be algebraically independent over K. For this purpose we shall sometimes pass to a larger algebraically closed extension field obtained from K by adjoining finitely many indeterminates and taking an algebraic closure. In many cases this is essentially equivalent to working on a Zariski dense open set of a variety parametrizing the family of elements. The following result gives such an equivalence.

Theorem 7.1. *Let $\mathfrak{H}$ be an $N \times N$ matrix of d-forms in a polynomial ring R over an algebraically closed field K. Let B denote a varying element in $GL(N,K)$ and let v denote a varying $N \times 1$ column over K.*

(a) For a given v, $\mathfrak{H}v$ has a collective (k,h)-collapse if and only if $B\mathfrak{H}v$ has a collective (k,h)-collapse for some $B \in GL(N,K)$ if and only if $B\mathfrak{H}v$ has a collective (k,h)-collapse for all $B \in GL(N,K)$.

(b) The following conditions are equivalent.

(1) For a dense open set U of $N \times 1$ matrices over K, $\mathfrak{H}v$ has a collective (k,h)-collapse for all $v \in U$.

(2) For a dense open set U of $N \times 1$ matrices over K and all $B \in GL(N,K)$, $B\mathfrak{H}v$ has a collective (k,h)-collapse for all $v \in U$.

(3) For a dense open set $\mathcal{U}$ of $GL(N,K)$, every column of $\mathfrak{H}A$ has a collective (k,h)-collapse for all $A \in \mathcal{U}$.

(4) For a dense open set $\mathcal{U}$ of $N \times N$ matrices over K and all $B \in GL(N,K)$, every column of $B\mathfrak{H}A$ has a collective (k,h)-collapse for all $A \in \mathcal{U}$.

(5) For a dense open set $\mathcal{U}$ of $GL(N,K)$, every column of $A^{\mathrm{tr}}\mathfrak{H}A$ has collective (k,h)-collapse for all $A \in \mathcal{U}$.

(6) Let $t_1, \ldots, t_N$ be indeterminates over K and let L denote the algebraic closure of $K(t_1, \ldots, t_N)$. Let T be the $N \times 1$ column matrix $\bigl(t_1 \ \ldots \ t_N\bigr)^{\mathrm{tr}}$. Then $\mathfrak{H}T$ has a collective (k,h)-collapse over L. Moreover, the coefficients of all elements occurring in this collective collapse may be taken in D, where D is a module-finite extension of $K[t_1, \ldots, t_N]_g$ and $g \in K[t_1, \ldots, t_N] - \{0\}$.

Proof. (a) is immediate from the observation that the entries of an $N \times 1$ column w have the same K-span as those of Bw for all $B \in \mathrm{GL}(N,K)$. Part (a) makes it immediately clear that (1) $\iff$ (2).

(2) $\Rightarrow$ (3) because $U \times \cdots \times U$ (N copies) with elements thought of as $N \times N$ matrices over K (we may think of the latter as N-tuples of columns) is a Zariski dense open set of the $N \times N$ matrices over K, and its intersection with $\mathrm{GL}(N,K)$ is therefore a dense open subset of $\mathrm{GL}(N,K)$. For the converse, consider a dense open set $\mathcal{U}$ of $\mathrm{GL}(N,K)$, again thought of as N-tuples of columns. The projection map from $\mathcal{U}$ to, say, its first column must have dense image in the space V of $N \times 1$ columns: otherwise, if the closure has dimension $N' < N$, the dimension of $\mathrm{GL}(N,K)$ would be at most $N' + (N-1)N < N^2$. The image therefore contains a dense open subset U of V. Thus, (1) and (3) are equivalent. The equivalence of (2), (4), and (5) is then immediate from part (a).

Let $\mathfrak{H} = \bigl(m_{ij}\bigr)$. The statement in (1) is equivalent to an equational statement formulated below. In the sequel, quantification on i, μ and ν is as follows: $1 \le i \le N$, $1 \le \mu \le k$, and $1 \le \nu \le h$. Let $\underline{d} = d_1, \ldots, d_k$ be a sequence of k positive integers such that for every μ, $1 \le d_\mu < d$. For every μ, let H_μ be a form of degree d_μ in $x_1, \ldots, x_N$ with unknown coefficients and for all μ, i let $H'_{\mu,i}$ be a form of degree $d - d_\mu$ in $x_1, \ldots, x_N$ with unknown coefficients. For all ν, let P_ν be a form of degree d in $x_1, \ldots, x_N$ with unknown coefficients, and for all ν, i, let $Z_{\nu,i}$ be an unknown. Denote all the unknown coefficients including the $Z_{\nu,i}$ as $Y_1, \ldots, Y_s$. For all $\underline{d}$, let $\mathcal{E}_{\underline{d}}$ denote the set of coefficients of the monomials in the x_j that occur in the elements

$$\sum_{j=1}^{N} m_{ij}t_j - \Big(\sum_{\mu=1}^{k} H_\mu H'_{i,\mu} + \sum_{\nu=1}^{h} P_\nu Z_{\nu,i}\Big),$$

for $1 \leq i \leq N$. These coefficients will be in $K[t_1, \ldots, t_N, Y_1, \ldots, Y_s]$. Then (1) is equivalent to the following statement. There exists $g \in K[t_1, \ldots, t_N] - \{0\}$ such that for every choice of $\underline{d}$, if the t_i are specialized to elements $c_1, \ldots, c_N$ in K such that $g(c_1, \ldots, c_N) \neq 0$, then at least one of the systems of equations obtained by setting all $\mathcal{E}_{\underline{d}}$ equal to 0 and setting $t_i = c_i$ for all i has a solution in K. Here, the condition $g(c_1, \ldots, c_N) \neq 0$ defines the open subset of linear combinations of the columns that have a collective (k, h)-collapse.

If we do not have a collective (k, h)-collapse for $\mathfrak{H}T$ over L then, for every choice of $\underline{d}$, $\mathcal{E}_{\underline{d}}$ defines the empty set over L. Hence, for every choice of $\underline{d}$, the elements $\mathcal{E}_{\underline{d}}$ generate the unit ideal in $L[y_1, \ldots, y_s]$. Since L is the directed union of the rings D where D is a module-finite extension of a ring of the form $K[t_1, \ldots, t_N]_g$ for some $g \in K[t_1, \ldots, t_N] - \{0\}$, we may choose a subring D of L module-finite over $K[t_1, \ldots, t_N]_g$ for some $g \neq 0$ such that each of the ideals $(\mathcal{E}_{\underline{d}})D[Y_1, \ldots, Y_s]$ is the unit ideal. Since an ideal becomes the unit ideal after extension to a module-finite overring iff it was already the unit ideal, it follows that we can choose $g \in K[t_1, \ldots, t_N] - \{0\}$ such that each of the ideals $(\mathcal{E}_{\underline{d}})K[t_1, \ldots, t_N]_g[Y_1, \ldots, Y_s]$ is the unit ideal. This is equivalent to the statement that the product of these ideals is the unit ideal, and we can conclude thats there exists $g \in K[t_1, \ldots, t_N] - \{0\}$ with a power in the product of the ideals $(\mathcal{E}_{\underline{d}})K[t_1, \ldots, t_N][Y_1, \ldots, Y_s]$.

Consider the dense open set in K^N where g does not vanish. It meets the dense open set U given in the statement of (1). Specialize the t_i to values c_i in K such that $c_1, \ldots, c_N$ is in the intersection. For some $\underline{d}$ the polynomials $(\mathcal{E}_{\underline{d}})$ vanish for a choice of values for $Y_1, \ldots, Y_s$ in K: because the point is in U, there is a collective (k, h)-collapse over K. But this forces g to vanish, a contradiction.

Now suppose that one has the collective (k, h)-collapse over L and, hence, over some D_g as described. Then for any point v of K^N (thought of as $N \times 1$ matrices over K) where g does not vanish, there is a maximal ideal $\mathfrak{m}$ of D_g lying over the maximal ideal of $K[t_1, \ldots, t_N]$ corresponding to the point v (since D_g is a module-finite extension of $K[t_1, \ldots, t_N]_g$). The quotient of D_g by $\mathfrak{m}$ is $\cong K$, and the specialization $D_g \twoheadrightarrow D_g/m \cong K$ yields a collective (k, h)-collapse of $\mathfrak{H}v$. $\square$

Discussion 7.2. Let $\mathfrak{H} = (m_{ij})$ be an $N \times N$ matrix of forms of degree d in a polynomial ring R over an algebraically closed field K. Let $t_1, \ldots, t_N$ be indeterminates over K and T be the column matrix $\big(t_1, \ldots, t_N\big)^{\mathrm{tr}}$. Let L be an algebraic closure of $K(t_1, \ldots, t_N)$. We refer to a collective (k, h)-collapse for $\mathfrak{H}T$ over $L \otimes_K R$ as a *generic* collective (k, h)-collapse for the columns of $\mathfrak{H}$. Suppose that

$$(*) \quad \sum_{j=1}^{N} m_{ij}t_j = \sum_{\mu=1}^{k} H_\mu H'_{\mu,i} + \sum_{\nu=1}^{h} P_\nu z_{\nu,i},\ 1 \leq i \leq N$$

displays the collapse. Here, every H_μ has degree d_i, $1 \leq d_i < d$, every $H'_{\mu,i}$ has degree $d - d_i$, every P_ν, $1 \leq \nu \leq h$ is a d-form and the $z_{\nu,i}$ are scalars.

We may choose $D \subseteq L$ where D is module-finite over $K[t_1, \ldots, t_N]_g$ for $g \in K[t_1, \ldots, t_N] - \{0\}$, so that D contains all the coefficients occurring in $(*)$. If h cannot be decreased, which we typically will be able to assume, the h-rowed matrix consisting of all coefficients of the P_ν has rank h, and some $h \times h$ minor (the entries will be in D) will be a nonzero element $\beta \in D$. By enlarging D, if necessary, we may assume that some minor β is a unit in D. The P_ν form a basis for an h-dimensional vector space over L that we denote W_L. We shall write W_D for the free D-module spanned by the P_ν. Let $\theta : D \to \Omega$ be a K-linear homomorphism from D to an algebraically closed field Ω containing K. By applying θ we obtain a collective (k, h)-collapse for $\mathfrak{H}\theta(T)$ over $\Omega \otimes_K R = \Omega[x_1, \ldots, x_N]$. We use images of the k elements of $D[x_1, \ldots, x_N]$ of degree strictly smaller than d and the images of the P_ν: the latter span a vector space of d-forms over Ω that we denote W_θ. This vector space may be identified with $\Omega \otimes_D W_D$.

A major case will be that where $\theta : D \twoheadrightarrow K$ is simply obtained by killing a maximal ideal of D. Note that if we have $\theta_0 : K[t_1, \ldots, t_N] \to \Omega$ such that $\theta_0(g) \neq 0$, where D is module-finite over $K[t_1, \ldots, t_N]_g$, then θ_0 extends to $K[t_1, \ldots, t_N]_g$ and, hence, to D.

By killing a varying maximal ideal $\mathfrak{m}$ of D, we obtain collective (k, h)-collapses for various LC_K-columns $\mathfrak{H}v$ of $\mathfrak{H}$, where v is a point of K^N with $g(v) \neq 0$. Consistent with the conventions above, we denote the h-dimensional K-vector space of forms spanned by the images of P_ν as W_θ, where $\theta : D \to D/\mathfrak{m} \cong K$. The next result analyzes when one has an element with a strict s-collapse in W_L.

Theorem 7.3. *Let notation and terminology be as in Discussion 7.2, so that we have fixed a generic (k, h)-collapse for $\mathfrak{H}$ over L, and suppose that it is defined over $D \subseteq L$ where D is module-finite of $K[t_1, \ldots, t_N]_g$, with $g \in K[t_1, \ldots, t_N] - \{0\}$. Fix a positive integer s.*

(a) The following two conditions are equivalent
  - *(1) There is a nonzero form of W_L with a strict s-collapse in $L \otimes_K R$.*
  - *(2) For some larger choice of $D' \supseteq D$, and every specialization $\theta : D' \to K$, W_θ has a nonzero form with a strict s-collapse. The same holds for every larger choice D'' of D'.*

(b) The following two conditions are also equivalent.
  - *(1) There is no nonzero form of W_L with a strict s-collapse in $L \otimes_K R$: that is, W_L is s-strong.*
  - *(2) For some larger choice of $D' \supseteq D$ and every specialization $\theta : D' \to K$, W_θ has no nonzero form with a strict s-collapse, i.e., W_θ is s-strong. The same holds for every larger choice D'' of D'.*

Proof. If the first condition in part (a) holds we can choose D' large enough to contain elements that are a basis for W_L over L such that these elements remain linearly independent over K for any specialization $L' \twoheadrightarrow K$ (by enlarging D' to contain the inverse of a suitable minor), to contain the coefficients of a nonzero element that has a strict s-collapse, the inverse of at least one of its nonzero coefficients, as well as the coefficients of the polynomials needed to exhibit the collapse. Then for every specialization $\theta : D' \twoheadrightarrow K$, the image of the element with the collapse is nonzero and has a strict s-collapse. This shows (1) $\Rightarrow$ (2) in part (a). The statement about larger choices D'' is then obvious, since any specialization $D'' \twoheadrightarrow K$ restricts to a specialization $D' \twoheadrightarrow K$.

Now assume that no nonzero element of W_L has a strict s-collapse. Again choose D' sufficiently large to contain the coefficients of a basis for W_L over L such that these elements remain linearly independent over K for any specialization $D' \to K$. Call this basis $\beta_1, \ldots, \beta_h$. Let $z_1, \ldots, z_h$ be new indeterminates. Let $P_1, \ldots, P_s, P'_1, \ldots, P'_s \in L[x_1, \ldots, x_N]$ be polynomials in $x_1, \ldots, x_N$ with new unknown coefficients y_j such that P_i has degree d_i and P'_i has degree $d-i$. The fact that only the zero element in W_L has an s-collapse means that for every choice of $d_1, \ldots, d_s$ and every choice of specialization of $z_1, \ldots, z_h, y_1, \ldots, y_j$ to values in L, if the unique entry Δ of $\begin{pmatrix} z_1 & \cdots & z_h \end{pmatrix}\mathfrak{H}T$ is 0, then all of the z_ν have value 0. Let γ_λ denote the coefficients of Δ when they are thought of as polynomials in $x_1, \ldots, x_N$. The γ_λ are polynomials over L in the z_ν and y_j. (Note that T has entries in L.) By Hilbert's Nullstellensatz, we then have that for every choice of $d_1, \ldots, d_s$, every z_ν has a power in the ideal generated by the γ_λ in $L[z_\nu, y_j : \nu, j]$. The same holds over D' for a sufficiently large choice of D': D' can be chosen so large as to work for every choice of $d_1, \ldots, d_s$. For this or any larger choice of D', if one has a specialization $\theta : D' \twoheadrightarrow K$, there is no nonzero element with a strict s-collapse in W_θ. This shows (1) $\Rightarrow$ (2) in part (b).

Since (a) part (1) and (b) part (1) give a mutually exclusive exhaustion of the possibilities, we have that in every instance either the statement in (a) part (2) or the statement in (b) part (2) holds, these are obviously mutually exclusive. It follows that in each part, statements (1) and (2) are equivalent. $\square$

When one has a collective (k, h)-collapse for a set of forms of degree d, the set of k forms of degree strictly smaller than d generates an ideal $\mathfrak{A}$: recall that is called the *auxiliary* ideal of the collapse, while the set of h forms of degree d span a vector space of dimension at most h that we call the *auxiliary* vector space of the collapse. The next result will be useful in modifying a collective (k, h)-collapse so that the representation of an element as a sum of an element in the auxiliary ideal and another in the auxiliary vector space is unique.

Theorem 7.4. *Let $a, k, h \in \mathbb{N}$ with $a, k > 0$ and let S be a set of d-forms over the polynomial ring R with a collective $(k,\, h)$-collapse with auxiliary ideal $\mathfrak{A}$ and auxiliary vector space V. Define a sequence (k_i, b_i) by the rule $k_0 = k$, $b_0 = 0$, $b_{i+1} = ak_i$, $k_{i+1} = k_i + b_{i+1}$. By a straightforward induction, $b_i = (a+1)^{i-1}ak$ for $i \geq 1$ and $k_i = (a+1)^i k$, $i \geq 0$. Choose a maximal sequence of linearly independent elements $v_1, \ldots, v_m \in V$ such that v_i has a strict b_i-collapse. Necessarily, $m \leq h$. Write $V = V_1 \oplus W$, where V_1 is the span of $v_1, \ldots, v_m$. Because v_i has a b_i-collapse for $1 \leq i \leq m$, for each such i we can choose an ideal $\mathfrak{J}_i$ with at most b_i homogeneous generators of positive degree smaller than the degree of v_i such that v_i is in $\mathfrak{J}_i$. Then S has a collective $(k', h-m)$-collapse with $k' = k_m = (a+1)^m k$. The auxiliary ideal is $\mathfrak{A} + \sum_{i=1}^m \mathfrak{J}_i$ and the auxiliary vector space is W.*

Proof. If no element of $V - \{0\}$ has a strict ak-collapse, then the sequence of v_i is empty, $m = 0$, and $W = V$. If some $v_1 \neq 0$ has an ak-collapse, then S has a collective $(k + ak,\, m-1)$ collapse, where the auxiliary ideal is the sum of $\mathfrak{A} + \mathfrak{J}_1$, which has at most $k + ak$-generators, and the auxiliary vector space is the span of $v_2, \ldots, v_m$ and W. By a straightforward induction on i, S has a collective $(k_i, m-i)$ collapse in which the auxiliary ideal is $\mathfrak{A} + \sum_{\nu=1}^i \mathfrak{J}_\nu$ and the auxiliary vector space is the span of $v_{i+1}, \ldots, v_m$ and W: one writes each element of S has the sum of an element $F \in \mathfrak{A} + \sum_{\nu=1}^i \mathfrak{J}_\nu$ and a sum $c_{i+1}v_{i+1} + c_{i+2}v_{i+2} + \cdots + c_m v_m + w$, where the c_ν are scalars and $w \in W$, and then takes $F + c_{i+1}v_{i+1}$ as the new strictly k_{i+1}-collapsible part, and $c_{i+2}v_2 + \cdots + c_m v_m + w$ as the part in the auxiliary vector space. The number of generators needed for the auxiliary ideal increases from k_i by the maximum number of generators for $\mathfrak{J}_{i+1}$ which is at most b_{i+1}, and it is clear that the dimension of the auxiliary vector space is at most $h - i$. In particular, it follows for $i = m$, that one has a collective $(k', h - m)$-collapse as stated. By the maximality of the sequence $v_1, \ldots, v_m$, no element of $W - \{0\}$ has a strict $ak' = ak_m = b_{m+1}$-collapse, or we could take that element to be v_{m+1}. $\square$

Proposition 7.5. *If a family of forms of degree d has a collective (k, h)-collapse that uses an auxiliary vector space V in which no nonzero element has a strict $2k$-collapse (i.e., V is $2k$-strong), then every K-linear combination of elements of the family can be written uniquely as the sum of an element with a strict k-collapse and an element of V.*

Proof. If $G_1 + v_1 = G_2 + v_2$ where the G_i have a strict k-collapse and the $v_i \in V$, then $G_2 - G_1 = v_1 - v_2 \in V$ has a strict $2k$-collapse, a contradiction unless $v_1 = v_2$. $\square$

Discussion 7.6. **Uniqueness of representation.** By combining Theorem 7.4 and Proposition 7.5, we can start with a collective (k, h)-collapse of a family of d-forms and, taking $a = 2$ (or a larger value) in Theorem 7.4, modify it to a collective $(3^m k, h - m)$-collapse for some m, $0 \le m \le h$ in which the representation of every form in the family as the sum of an element with a strict $(3^m k)$-collapse and an element in the auxiliary vector space of dimension $h - m$ is unique. Let $k' = 3^m k$ and $h' = h - m$. Note that $k' \le 3^h k$ and $h' \le h$. In the situation of Theorem 7.3, we may modify a generic collective (k, h)-collapse in this way to a (k', h')-collapse in which the auxiliary vector space is $2k'$-strong. By Theorem 7.3, the same condition holds for the collapses that arise by specialization as in Theorem 7.3 part (b)(2).

8. Using a single auxiliary vector space

Again let all notation be as in Discussion 7.2, so that L is an algebraic closure of $K(t_1, \ldots, t_N)$, and that we have a collective (k, h)-collapse for $\mathfrak{H}T$: we may assume that there is no such collapse for a smaller value of h, so that auxiliary vector space has dimension h.

Discussion 8.1. Suppose that t_{ij} are any N^2 algebraically independent elements in an algebraically closed field Ω. Let $\mathcal{T} = (t_{ij})$ and let $T^{(j)}$ denote the j th column of $\mathcal{T}^{\mathrm{tr}}$, whose i th entry is t_{ji}. For every j, $1 \le j \le N$, let L_j be the algebraic closure of $K(t_{j1}, \ldots, t_{jN})$ within Ω and fix a K-isomorphism of L with L_j that extends the K-isomorphism of $K[t_1, \ldots, t_N]$ with $K[t_{j1}, \ldots, t_{jN}]$ that sends $t_i \mapsto t_{ji}$, $1 \le i \le N$. Let θ_j be the composite $L \to L_j \hookrightarrow \Omega$. Specializing T to $T^{(j)}$ yields a collective (k, h)-collapse for $\mathfrak{H}T^{(j)}$: let W_j denote the auxiliary vector space over Ω that is used. We write $\mathfrak{A}_j$ for the auxiliary ideal in $L_j[x_1, \ldots, x_N]$, which will be generated by k or fewer polynomials of degree strictly less than d. Thus, we may write

$$\mathfrak{H}T^{(j)} = \mathfrak{C}^{(j)} + \mathfrak{D}^{(j)}$$

where the entries of columns $\mathfrak{C}^{(j)}$ and $\mathfrak{D}^{(j)}$ have coefficients in L_j, the entries of $\mathfrak{C}^{(j)}$ are in the ideal $\mathfrak{A}_j$, and the entries of $\mathfrak{D}^{(j)}$ span the vector space W_j. By left multiplying by $\mathcal{T} = \big(t_{ij}\big)$ we get a collective collapse for the j th column of $\mathcal{T}\mathfrak{H}\mathcal{T}^{\rm tr}$ that uses the same vector space W_j, since

$$\mathcal{T}\mathfrak{H}T^{(j)} = \mathcal{T}\mathfrak{C}^{(j)} + \mathcal{T}\mathfrak{D}^{(j)}.$$

Because we have a collective (k, h)-collapse for each of the columns of $\mathcal{T}\mathfrak{H}\mathcal{T}^{\rm tr}$, we can write

$$\mathcal{T}\mathfrak{H}\mathcal{T}^{\rm tr} = \mathfrak{P} + \mathfrak{Q}$$

where each column of the $N \times N$ matrix $\mathfrak{P}$ has a strict k-collapse and the j th column of the matrix $\mathfrak{Q}$ is in the vector space W_j generated over Ω by d-forms with coefficients in L_j. Each W_j has dimension h.

The next result plays a critical role in the proof of our main result for quartics.

Theorem 8.2. *Let $\mathfrak{H}$ be a symmetric matrix with entries that are 0 or else forms of degree d in a polynomial ring R over an algebraically closed field Ω. Let j be any integer, $1 \le j \le N$. Suppose that for $i = j$ and for $N - h \le i \le N$, the i th column $\mathfrak{H}^{(i)}$ of $\mathfrak{H}$ can be written as $\mathfrak{P}^{(i)} + \mathfrak{Q}^{(i)}$, where the entries of $\mathfrak{P}^{(i)}$ are in an ideal $\mathfrak{A}_i$ generated by at most k forms of degree strictly less than d, and the $\mathfrak{Q}^{(i)}$ span a vector space V_i over Ω of dimension at most h whose nonzero elements are d-forms. This explicitly shows that for $i = j$ and for $N - h \le i \le N$, the i th column of $\mathfrak{H}$ has a $(k,\, h)$-collapse using the ideal $\mathfrak{A}_i$ and the auxiliary vector space V_i. Also assume that if $i < N - h$ (respectively, $i \ge N - h$), V_i is spanned by the bottommost h elements of $\mathfrak{Q}^{(i)}$ (respectively, the bottommost $h+1$ elements of $\mathfrak{Q}^{(i)}$). Let $W_\Omega = W$ be the vector space spanned over Ω by the entries of the bottommost $h + 1$ elements of the $\mathfrak{Q}^{(i)}$ for $N - h \le i \le N$. Let $\mathfrak{d} := dim_\Omega(W)$. Then the j th column of $\mathfrak{H}$ has a $\big(k(h + 1), \mathfrak{d}\big)$-collapse using the ideal $\mathfrak{A} = \mathfrak{A}_j + \sum_{i=N-h+1}^{N} \mathfrak{A}_i$ and the auxiliary vector space W.*

Proof. The statement is obvious for elements of the j th column if $j \ge N - h$: each is the sum of an element with a strict k-collapse using $\mathfrak{A}_j$ and an auxiliary vector space contained in W. Now consider any element F of the vector space spanned by the j th column of $\mathfrak{H}$ for $j < N - h$. It is the sum of an element F_0 with a strict k-collapse using the ideal $\mathfrak{A}_j$ for that column and an element q in the span of $\mathfrak{Q}^{(j)}$. Then $q = \sum_{i=N-h+1}^{N} c_i q_i$ where the $c_i \in \Omega$ and $q_{N-h+1}, \dots, q_N$ are the h bottommost elements in $\mathfrak{Q}^{(j)}$. Each of the q_i comes from writing an element H_i among the bottommost h entries of the j th column of $\mathfrak{H}$ as $G_i + q_i$, where G_i has a strict k-collapse using $\mathfrak{A}_j$. But by transposing the symmetric matrix $\mathfrak{H}$, we may also think of H_i as an element of one of the rightmost h columns of $\mathfrak{H}$, and, as such, we can write $H_i = P_i + Q_i$ where P_i has a strict k-collapse using the ideal $\mathfrak{A}_i$ for the i th column of $\mathfrak{H}$ and $Q_i \in W$. Thus, $q_i = H_i - G_i = P_i - G_i + Q_i$ and

$$F = F_0 + \sum_{i=N-h+1}^{N} c_i(P_i - G_i + Q_i),$$

where each term is either in $\mathfrak{A}_j$, or in $\mathfrak{A}_i$, $N - h + 1 \le i \le N$, or in W. Since $\mathfrak{A} = \mathfrak{A}_j + \sum_{i=N-h+1}^{N} \mathfrak{A}_i$ is the sum of $h + 1$ homogeneous ideals with at most k homogeneous generators of degree strictly less than d, $\mathfrak{A}$ is a homogeneous ideal with at most $k(h + 1)$ homogeneous generators of degree strictly less than d, and the j th column of $\mathfrak{H}$ therefore has a $\big(k(h+1),\, \mathfrak{d}\big)$-collapse using the ideal I and the auxiliary vector space W. $\square$

Theorem 8.3. *Let notation and hypothesis be as in Discussions 7.2 and 8.1.*

(a) *The vector space spanned over Ω by a column of $\mathfrak{Q}$ is spanned by any h off-diagonal entries of the column.*
(b) *The vector space $W_\Omega = W$ spanned by the $h+1$ rightmost columns of $\mathfrak{Q}$ is the same as the vector space spanned by the entries of the columns $\mathfrak{D}^{(j)}$ for $N-h \le j \le N$. It is also the same as the span of the entries of $\mathfrak{Q}$ with indices i, j satisfying $N-h \le i,\, j \le N$ with $i \ne j$, i.e., of the off-diagonal entries of the $(h+1)\times(h+1)$ submatrix of $\mathfrak{Q}$ in the lower right corner. The latter set of generators of W consists of elements that have coefficients in the field $\mathcal{L}$ that is the algebraic closure in Ω of the field $K(t_{ji} : N-h \le i \le N,\, 1 \le j \le \mathbb{N})$. Moreover, $\mathfrak{d} := dim_\Omega(W)$ is at most $h(h+1)$.*
(c) *Every column of $\mathcal{T}\mathfrak{H}\mathcal{T}^{\mathrm{tr}}$ has a collective $\big(k(h+1),\, \mathfrak{d}\big)$-collapse in which the $\mathfrak{d}$ elements of degree d can be taken to be any basis for W. In particular, these $\mathfrak{d}$ elements may be chosen from the off-diagonal elements of the square size $h+1$ submatrix of $\mathcal{T}\mathfrak{H}\mathcal{T}^{\mathrm{tr}}$ in the lower right corner, in which case they have coefficients in $\mathcal{L}$. The ideal of the collapse is generated by the ideals of the collapses of the given column and the rightmost h columns of $\mathcal{T}\mathfrak{H}\mathcal{T}^{\mathrm{tr}}$. Thus the auxiliary vector space for all these $\big(k(h+1),\, \mathfrak{d}\big)$-collapses, which is W, is independent of the choice of the column.*
(d) *Let Z_0 be any $N-(h+1)$ size square matrix over Ω and let Z be the direct sum of Z_0 with a size $h+1$ identity matrix I, so that, in block form, $Z = \begin{pmatrix} Z_0 & 0 \\ 0 & I \end{pmatrix}$. Suppose that ZT has entries that are algebraically independent over K. Then every column of $(Z\mathcal{T})\mathfrak{H}(Z\mathcal{T})^{\mathrm{tr}}$ has a $\big(k(h+1),\, \mathfrak{d}\big)$-collapse using W as the auxiliary vector space.*
(e) *We also have the $\big(k(h+1),\, \mathfrak{d}\big)$-collapses described in part (c) and (d) if Ω is replaced by any larger algebraically closed field Ω': W is replaced by $W'_\Omega := \Omega' \otimes_\Omega W$ but may be taken to have the same basis.*

Proof. (a) Consider the jth column $\mathfrak{D}^{(j)}$ of $\mathfrak{D}$, which consists of polynomials with coefficients in L_j. We can choose distinct integers $1 \le i_1 < \cdots < i_h \le N$ such that the entries of the column in the positions with indices $i_1, \ldots, i_h$ are a basis for the span of the entries of the column. Then h off-diagonal entries of the jth column of $\mathfrak{Q}$, which is $\mathcal{T}\mathfrak{D}^{(j)}$ are given by integers $1 \le a_1 < \cdots < a_h \le N$ such that the a_i are all different from j. Let $\mathcal{T}_0$ be the submatrix of $\mathcal{T}$ consisting of the rows of $\mathcal{T}$ indexed by $a_1, \ldots, a_h$. Then $\mathcal{T}_0$ is an $h \times N$ matrix of indeterminates $t_{a_i,b}$ with $a_i \ne j$. Thus, these indeterminates are algebraically independent over L_j. The fact that the entries of $\mathcal{T}_0\mathfrak{D}^{(j)}$ are linearly independent now follows from the fact that we can specialize the entries of $\mathcal{T}_0$ without affecting $\mathfrak{D}^{(j)}$. In particular we can specialize the entries of $\mathcal{T}_0$ so that its νth row has 1 as its i_ν entry and 0 for all of its other entries. Then $\mathcal{T}_0$ specializes to a matrix M such that the entries of $M\mathfrak{D}^{(j)}$ are the entries of $\mathfrak{D}^{(j)}$ indexed by $i_1, \ldots, i_h$, which are independent over L_j.

(b) It is now clear that the vector space over Ω spanned by the h bottommost off-diagonal entries of one of the rightmost $h+1$ columns over $\mathfrak{Q}$ is the same as the Ω-vector space spanned by all of the entries of that column. Since multiplying by the invertible matrix $\mathcal{T}$ does not affect the Ω-span of the entries of a column, this is also the same as the span of all entries of the $\mathcal{D}^{(j)}$ for $0 \le N-j \le h$.

(c) and (d). We may write $\mathcal{T}\mathfrak{H}\mathcal{T}^{\mathrm{tr}}$ in block form as $\begin{pmatrix} M_{11} & M_{12} \\ M_{12}^{\mathrm{tr}} & M_{22} \end{pmatrix}$ where M_{11} is square and symmetric of size $N-h-1$, M_{12} is $(N-h-1)\times(h+1)$, and M_{22} is square and symmetric of size $h+1$. Then $\mathcal{Z}\mathcal{T}\mathfrak{H}(\mathcal{Z}\mathcal{T})^{\mathrm{tr}} = \begin{pmatrix} Z_0 M_{11} Z_0^{\mathrm{tr}} & Z_0 M_{12} \\ M_{12}^{\mathrm{tr}} Z_0^{\mathrm{tr}} & M_{22} \end{pmatrix}$. In particular, the last $h+1$ columns are the columns of $Z\begin{pmatrix} M_{12} \\ M_{22} \end{pmatrix}$ and each of these consists of Ω-linear combinations of the entries of the last $h+1$ columns of $\mathcal{T}\mathfrak{H}\mathcal{T}^{\mathrm{tr}}$. Hence, each of the last $h+1$ columns of $\mathcal{Z}\mathcal{T}\mathfrak{H}(\mathcal{Z}\mathcal{T})^{\mathrm{tr}}$ has a collective (k,h)-collapse using the ideal for the corresponding column of $\mathcal{T}\mathfrak{H}\mathcal{T}^{\mathrm{tr}}$ and a subspace of W. We may now apply Theorem 8.2, taking $\mathfrak{H}$ to be $\mathcal{T}\mathfrak{H}\mathcal{T}^{\mathrm{tr}}$ for part (a) and $\mathcal{Z}\mathcal{T}\mathfrak{H}(\mathcal{Z}\mathcal{T})^{\mathrm{tr}}$ for part (d). (Note that (c) is actually the special case of (d) where Z_0 is an identity matrix.) The conditions in Theorem 8.2 that the auxiliary vector spaces be spanned by certain bottommost elements follow from part (a) and the fact that the entries of $Z\mathcal{T}$ (or $\mathcal{T}$) are algebraically independent over K.

(e) is obvious. □

Theorem 8.4. *Let notation and hypothesis as in Discussions 7.2 and 8.1. Let W be defined as in Theorem 8.3(b), and apply Theorem 7.4 with $a=3$ to find an integer $m\in\mathbb{N}$ and subspace W_0 of W of dimension $\mathfrak{d}_0=\mathfrak{d}-m$ such that every column of $\mathcal{T}\mathfrak{H}\mathcal{T}^{\mathrm{tr}}$ has a collective $(k',\mathfrak{d}_0)$-collapse with $k'=4^m k$ and no nonzero element of W_0 has a strict $3k'$-collapse, i.e., W_0 is $3k'$-strong. Then we can write $\mathcal{T}\mathfrak{H}\mathcal{T}^{\mathrm{tr}} = \mathfrak{P}+\mathfrak{Q}$ uniquely, where $\mathfrak{P}$ has the property that every column has a strict k'-collapse, and $\mathfrak{Q}$ has entries in W_0. The matrices $\mathfrak{P}$ and $\mathfrak{Q}$ are symmetric. Moreover, if Z_0 is an $N-h-1$ by $N-h-1$ matrix with algebraically independent entries over $K(t_{ij}:1\le i,j\le N)$ and Z is the direct sum of Z_0 with a size $h+1$ identity matrix, then every column of $Z\mathfrak{P}Z^{\mathrm{tr}}$ has a strict collective k'-collapse. In particular, if $\mathfrak{P}_0$ is the size $N-h-1$ square submatrix of $\mathfrak{P}$ in the upper left corner, then every column of $Z_0\mathfrak{P}_0Z_0^{\mathrm{tr}}$ has a strict collective k'-collapse.*

Proof. Let $M=N-h-1$ and let $Z_0=\big(z_{ij}\big)$ have entries that are algebraically independent over the algebraic closure $\mathcal{K}$ of $K(t_{ij}:1\le i,j\le N)$.

For $1\le i,j\le M$ with $i\ne j$ and $y\in\Omega$ let $E_{ij}(y)$ denote the elementary size M square matrix obtained by adding y times the jth row of the size M identity matrix to the ith row. Let $D_i(y)$ be the diagonal matrix with y in the ith spot on the diagonal and 1 in all other spots on the diagonal. We refer to all of these as *elementary matrices*. We may work over $\kappa(z_{ij}:1\le i,j\le M)$, where κ is the prime field of K, to write

$$(*)\quad Z_0=\prod_{j=1}^{M}\big(D_j(y_{jj})\big(\prod_{1\le i\le M,\, i\ne j}E_{ij}(y_{ij})\big)\big).$$

This simply corresponds to performing elementary operations on Z_0 until one gets an identity matrix, and then writing down the product of the inverses of the elementary matrices used. At the first step, $y_{11}=z_{11}$. The next $M-1$ steps subtract multiples of the first entry of the first row so that the other entries of the first row become 0. After sM steps the first s rows agree with those of the identity matrix. One takes $y_{s+1,s+1}$ to be the entry of the matrix obtained at the $(s+1,s+1)$ spot. One additional left multiplication by $D_{s+1}(y_{s+1,s+1})$ enables us to assume that the $(s+1,s+1)$ entry is 1. One then performs column operations to make all of the other of the entries of the $s+1$ row 0. The elements we need to invert in this process are never 0, because that does not happen even if we specialize the z_{ij} to the entries of the size M identity matrix. Since the z_{ij} are in the field (even the κ-algebra) generated by the y_{ij}, the M^2 elements y_{ij} are algebraically independent both over κ and over L: the z_{ij} and the y_{ij} generate the same field over κ or L.

Let $s = M^2$ and denote the product on the right hand side of $(*)$ as $E_s \cdots E_1$. Let Y_ν be the direct sum of E_ν and a size $h+1$ identity matrix, $1 \leq \nu \leq s$. Then $Z = Y_m \cdots Y_1$. By a straightforward induction on ν, each of the matrices

$$Y_1, Y_2Y_1, \ldots, Y_\nu \cdots Y_1, \ldots, Y_m \cdots Y_1 = Z$$

has entries that are algebraically independent over L, and it follows that each of the matrices

$$\mathcal{T}, Y_1\mathcal{T}, Y_2Y_1\mathcal{T}, \ldots, Y_\nu \cdots Y_1\mathcal{T}, \ldots, Y_m \cdots Y_1\mathcal{T} = Z\mathcal{T}$$

has algebraically independent entries over K. Let $\mathcal{T}_\nu = Y_\nu \cdots Y_1\mathcal{T}$, $1 \leq \nu \leq s$ and $\mathcal{T}_0 = \mathcal{T}$. Then for each of the matrices $\mathcal{T}_\nu$, $0 \leq \nu \leq s$, we have a decomposition: $\mathcal{T}_\nu\mathfrak{H}\mathcal{T}_\nu^{\mathrm{tr}}\mathfrak{P}_\nu + \mathfrak{Q}_\nu$, where $\mathfrak{P}$ and $\mathfrak{P}_\nu$ have the property that every column has a collective (k', h')-collapse, and the $\mathfrak{Q}_\nu$ have all entries in W_0.

The choice of $a = 3$ guarantees that these representations are unique. Since $\mathcal{T}\mathfrak{H}\mathcal{T}^{\mathrm{tr}}$ is symmetric, $\mathfrak{P}$ and $\mathfrak{Q}$ are symmetric as well: by Proposition 7.5, an element of $\mathcal{T}\mathfrak{H}\mathcal{T}^{\mathrm{tr}}$ can be written as the sum of an element with a strict k'-collapse and an element in W_0 in only one way (the element with the k'-collapse. need not be assumed to come from a specific auxiliary ideal — one only needs that every element of $W_0 - \{0\}$ is at least $2k'$-strong). We now show by induction on ν that for $\nu \geq 1$ we have

$$\mathfrak{P}_\nu = (Y_\nu \cdots Y_1)\mathfrak{P}(Y_\nu \cdots Y_1)^{\mathrm{tr}} \text{ and } \mathfrak{Q}_\nu = (Y_\nu \cdots Y_1)\mathfrak{Q}(Y_\nu \cdots Y_1)^{\mathrm{tr}}.$$

This yields the desired conclusion when $\nu = s$.

This comes down to showing that if the entries of $\mathcal{T}_\nu$ and the element y are algebraically independent and $Y = Y_{\nu+1}$ is the direct sum of an elementary matrix, either $E_{ij}(y)$ or $D_i(y)$, with a size $h+1$ identity matrix, then $Y\mathfrak{P}_\nu Y^{\mathrm{tr}} = \mathfrak{P}_{\nu+1}$ and $Y\mathfrak{Q}_\nu Y^{\mathrm{tr}} = \mathfrak{Q}_{\nu+1}$. Clearly, we have $\mathcal{T}_{\nu+1} = Y\mathfrak{P}_\nu Y^{\mathrm{tr}} + Y\mathfrak{Q}_\nu Y^{\mathrm{tr}}$. The last term is a matrix with entries in W_0. If Y is diagonal, every entry of $Y\mathfrak{P}_\nu Y^{\mathrm{tr}}$ is a multiple of an entry of $\mathfrak{P}$ and so has a strict k'-collapse. If $Y = E_{ij}(y)$, every entry of $Y\mathfrak{P}_\nu$ is either an entry of $\mathfrak{P}_\nu$ or a linear combination of two entries of $\mathfrak{P}_\nu$ in the same column. Hence every entry of $Y\mathfrak{P}_\nu$ has a strict k'-collapse. Every entry of $(Y\mathfrak{P}_\nu)Y^{\mathrm{tr}}$ is, likewise, a linear combination of at most two entries from a row of $Y\mathfrak{P}$, and so has, at worst, a strict $(2k')$-collapse. Thus, in all cases, every entry of $Y\mathfrak{P}_\nu Y^{\mathrm{tr}}$ has, at worst, a strict $(2k')$-collapse. Let F be an entry of $Y\mathfrak{P}_\nu Y^{\mathrm{tr}}$. Then we have found a representation $F = G_1 + H_1$ where G_1 has a strict $(2k')$-collapse and G_1 is in W_0. From the fact that $Y\mathcal{T}_\nu Y^{\mathrm{tr}} = Y_{\nu+1} = \mathfrak{P}_{\nu+1} + \mathfrak{Q}_{\nu+1}$ we also have $F = G_2 + H_2$, where G_2 has a strict k'-collapse and $H_2 \in W_0$. It follows that $G_2 - G_1 = H_1 - H_2 \in V_0$ is an element of V_0 with a strict $(3k')$-collapse. This is a contradiction, by our choice of a, unless $H_1 - H_2 = 0$. This implies that $H_1 = H_2$ and $G_1 = G_2$. This completes the proof. $\square$

9. Hessian-like matrices of quadrics with entries of stably bounded rank

Theorem 9.3 below plays a critical role in the construction of ${}^{\eta}\underline{A}(n_1, n_2, n_3, n_4)$ for algebraically closed fields of characteristic 0 or ≥ 5 in the next section. Note that in this result, we do not need to assume that $\mathfrak{P}_0$ is symmetric.

Lemma 9.1. *Let $\mathcal{K}$ be an algebraically closed field of characteristic different from 2. Let M, N be positive integers. Let $R = \mathcal{K}[x_1, \dots, x_N]$ be a polynomial ring over $\mathcal{K}$. Let $\mathcal{Z} = \bigl(z_{ij}\bigr)$ be an $M \times M$ matrix of indeterminates over $\mathcal{K}$. Let $\mathfrak{P}_0 = \bigl(f_{ij}\bigr)$ be an $M \times M$ matrix whose entries are quadratic forms in R and suppose that all the entries of $\mathcal{Z}\mathfrak{P}_0\mathcal{Z}^{\mathrm{tr}}$ have rank at most r_2 over the field $\mathcal{F} = \mathcal{K}(z_{ij} : 0 \leq i, j \leq M)$. Then all off-diagonal entries of $\mathcal{Z}\mathfrak{P}_0\mathcal{Z}^{\mathrm{tr}}$ have the same rank $r_1 \leq r_2$ over $\mathcal{F}$, and all diagonal entries have the same rank $r \leq r_1$. If $M = 1$ we make the convention that $r_1 := r$. For a Zariski dense open subset U of $GL(M, \mathcal{K})$, for all $\alpha \in U$ all off-diagonal (respectively, diagonal) entries of $\alpha\mathfrak{P}_0\alpha^{\mathrm{tr}}$ have rank r_1 (respectively, r). For an off-diagonal entry f in the (i, j) spot of $\alpha\mathfrak{P}_0\alpha$ with $\alpha \in U$, the i th row and j th column of $\alpha\mathfrak{P}_0\alpha^{\mathrm{tr}}$ are both contained in the ideal $(\mathcal{D}f)R$.*

Proof. Let $\mathcal{Z}_i$ be the i th row of $\mathcal{Z}$ and $\mathcal{Z}^j$ the j th column. It is clear that the maximum rank that can be achieved at the (i, j) spot in $\alpha\mathfrak{P}_0\alpha^{\mathrm{tr}}$ is the same as the rank at the (i, j) spot of $Z\mathfrak{P}_0Z^{\mathrm{tr}}$: rank ρ is achieved if and only if the size ρ minors of certain matrix of polynomials in the z_{ij} do not all vanish. (Each quadratic form in $\mathfrak{P}_0$ corresponds to a symmetric $N \times N$ matrix of constants in $\mathcal{K}$, and $Z_i\mathfrak{P}_0\mathcal{Z}^j$ will correspond to an $N \times N$ symmetric matrix whose entries are in $\mathcal{K}[z_{ij} : 0 \leq i, j \leq M]$). If $i \neq j$, by interchanging the i th and j th rows of α one sees that if a certain rank is achieved at the (i, i) spot in $\alpha\mathfrak{P}_0\alpha^{\mathrm{tr}}$, it is also achieved at the (j, j) spot. Moreover this shows as well that the rank achieved at the (i, h) spot is also achieved at the (j, h) spot if $h \neq i, j$, so that for any column, the highest rank that can be obtained at an off-diagonal spot is constant. Since the same is true for rows, the highest possible ranks r, r_1 are constant both for spots on and for spots off the main diagonal. We show that $r \leq r_1$ below.

Choose $\alpha \in U$ so that $\mathfrak{P}_1 = \alpha\mathfrak{P}_0\alpha^{\mathrm{tr}} = \bigl(g_{ij}\bigr)$ has maximum possible rank for all entries. If $i \neq h, j$ and $c \in \mathcal{K}$ by adding c times the h th row to the i th row and also c times the h th column to the i th column in $\mathfrak{P}_1$ we obtain $cg_{hj} + g_{ij}$ at the (i, j) spot. By Proposition 4.6, the rank of $cg_{hj} + g_{ij}$ will increase from r_1 for infinitely many values of c unless $g_{hj} \subseteq (\mathcal{D}g_{ij})R$, and it will be at least the rank of g_{hj} even when $h = j$. This shows that $r \leq r_1$ and that for h, j distinct from i, $g_{hj} \in (\mathcal{D}g_{ij})$, even if $h = j$. But $g_{ij} \in \mathcal{D}g_{ij}$ in general, so that the entire i th row of j th column of $\mathfrak{P}_1$ is in the ideal $\mathcal{D}g_{ij}$. The entire j th column is in $\mathcal{D}g_{ij}$ by a similar argument. □

Example 9.2. The ranks r_1 and r can be different. Let V be a $1 \times M$ row of linearly independent linear forms and let $\mathfrak{P}_0 = VV^{\mathrm{tr}}$. Then $A\mathfrak{P}_0A^{\mathrm{tr}} = (AV)(AV)^{\mathrm{tr}}$, and each diagonal entry is the square of a linear form, and so has rank 1. Each off-diagonal entry is the product of two linearly independent linear forms and has rank 2. If $\mathfrak{P}_0$ is skew-symmetric the diagonal entries are 0 while the off diagonal entries can have arbitrarily large rank.

We are now ready to prove the last of the critical results needed to establish the existence of $\underline{A}(n_1, n_2, n_3, n_4)$. We only need the case $a = 1$ to handle quartics, but there are potential applications to higher degree cases.

Theorem 9.3. *Let $\mathcal{K}$ be an algebraically closed field of characteristic different from 2. Let M, N, and a be positive integers. Let $R = \mathcal{K}[x_1, \dots, x_N]$ be a polynomial ring over $\mathcal{K}$. Let $\mathcal{Z} = \big(z_{ij}\big)$ be an $M \times M$ matrix of indeterminates over $\mathcal{K}$. Let $\mathfrak{P}_0 = \big(f_{ij}\big)$ be an $M \times M$ matrix whose entries are quadratic forms in R and suppose that all the entries of $\mathcal{Z}\mathfrak{P}_0\mathcal{Z}^{\mathrm{tr}}$ have rank at most r_2 over $\mathcal{F} = \mathcal{K}(z_{ij} : 0 \le i, j \le M)$. Let $V = \big(L_1 \ \dots \ L_M\big)$ be a $1 \times M$ row of forms of degree a in R. Then $f := \sum_{0 \le i,j \le M} L_iL_jf_{ij}$, the unique entry of $V\mathfrak{P}_0V^{\mathrm{tr}}$, has a strict $4r_2$-collapse in R (a strict $3r_2$-collapse if $\mathfrak{P}_0$ is symmetric). More precisely, if r_1 is the greatest possible rank of an off-diagonal entry of $\mathcal{Z}\mathfrak{P}_0\mathcal{Z}^{\mathrm{tr}}$ and r is the greatest possible rank of a diagonal entry, then f has a strict $(2r_1 + 2r)$-collapse (a strict $(r_1 + 2r)$-collapse if $\mathfrak{P}_0$ is symmetric).*

Proof. By Lemma 9.1 the ranks r and r_1 will be achieved for the specialization of $\mathcal{Z}$ to a matrix α in an open subset U of $\mathrm{GL}(M, \mathcal{K})$. Hence, we may replace $\mathfrak{P}_0$ by $\alpha\mathfrak{P}_0\alpha^{\mathrm{tr}}$ for such an α and assume that f_{11} has rank r. Because $Z\alpha^{-1}$ has algebraically independent entries over $\mathcal{K}$, our hypothesis is preserved. At the same time we replace V by $V\alpha^{-1}$.

Moreover, since the characteristic is different from 2, for each quadratic form f_{ij} we have a corresponding symmetric $N \times N$ symmetric matrix $\Lambda_{ij} = \big(\lambda_{st}^{ij}\big)$. After a $\mathcal{K}$-linear change of coordinates in R we may assume that Λ_{11} is diagonal, with the first r entries on the diagonal equal to 1 and the other diagonal entries 0. We note that in the discussion below we have $1 \le i, j \le M$ and $1 \le s, t \le N$. Sometimes there are additional restrictions on these indices.

Let Z be the first row of $\mathcal{Z}$. To simplify notation we let $z = z_{11}$ and $z_j = z_{1j}$ for $j > 1$. Then $G = \sum_{1 \le i \le M} z_i^2 f_{ii} + \sum_{1 \le i \ne j \le M} z_iz_jf_{ij}$ is the $(1, 1)$ entry of $\mathcal{Z}\mathfrak{P}\mathcal{Z}^{\mathrm{tr}}$ and has rank r. Call the corresponding $N \times N$ symmetric matrix Λ: it has coefficients in $\mathcal{K}[z_i : 1 \le i \le M]$. In Λ, the first r entries of the main diagonal have the form

$$(\#) \quad z^2 + \quad \text{lower degree terms in } z$$

Note that z occurs with degree at most one in other entries of Λ. Consider the $N - r$ size square submatrix C of Λ in the lower right corner. We want to study the constraints imposed on C and on Λ by the condition that Λ have rank at most r, i.e., that the $r + 1$ size minors of Λ vanish.

Fix $s, t > r$ corresponding to some entry of C. Consider the $r + 1$ size square submatrix $\Theta_{s,t}$ of Λ coming from the rows indexed $1, 2, \dots, r, s$ and the columns indexed $1, 2, \dots, r, t$. It contains the upper left $r \times r$ block whose main diagonal consists of terms as described in (#). When we calculate its determinant as a sum of $(r+1)!$ products of entries, the main diagonal provides a term of degree z^{2r+1} or z^{2r} depending on whether or not z occurs with nonzero coefficient in the (s, t) entry of Λ. We cannot obtain z^{2r+1} in any other way. Hence, if z occurs with nonzero coefficient in the (s, t) entry of Λ, this $r + 1$ size minor of Λ does not vanish, a contradiction. Hence, we may assume that for all $s > r$ and $t > r$, z does not occur in the (s, t) entry of Λ. This implies that the determinant of $\Theta_{s,t}$ has degree at most $2r$ as a polynomial in z. We may calculate the coefficient of z^{2r} in terms of the z_i for $2 \le i \le M$ and the scalars λ_{st}^{ij}.

As already mentioned, we obtain a term of degree $2r$ in z by using the first r terms on the main diagonal and the term in the (s,t) spot.

Since $\Lambda = \sum_i z_i^2\Lambda_{ii} + \sum_{i,j} z_iz_j\Lambda_{ij}$, the only other way to get z^{2r} when we compute the determinant of $\Theta_{s,t}$ is by taking $r-1$ terms from the main diagonal, omitting, say, the ρ th term, where $1 \le \rho \le r$, and two terms, one, $\Lambda_{s\rho}$, from the bottom row of $\Theta_{s,t}$ (which is part of the s th row of Λ) with column index ρ, and the other, $\Lambda_{\rho t}$, from the rightmost column of $\Theta_{s,t}$ (which is part of the t th column of Λ) with row index ρ. The contribution to the coefficient of z^{2r} from the term of this type in the determinant of $\Theta_{s,t}$ coming from one choice of ρ is

$$-(\text{the coefficient of } z \text{ in } \Lambda_{s\rho})(\text{the coefficient of } z \text{ in } \Lambda_{\rho t}).$$

The coefficient of $z = z_1$ in $\Lambda_{st} = \sum_{i,j}\lambda_s^{ij}tz_iz_j$ comes from the z_1z_j terms, $j \ge 2$, and the z_iz_1 terms, $i \ge 2$, so that the coefficient of z is

$$\sum_{j\ge 2}\lambda_{st}^{j1}z_j + \sum_{i\ge 2}\lambda_{st}^{1i}z_i$$

which we can rewrite as

$$\sum_{i\ge 2}(\lambda_{st}^{i1} + \lambda_{st}^{1i})z_i$$

or as

$$\sum_{j\ge 2}(\lambda_{st}^{j1} + \lambda_{st}^{1j})z_j.$$

Taking account also of the contribution to the coefficient of z^{2r} that is entirely from the main diagonal in $\Theta_{s,t}$, which is the product of z^{2r} with $\Lambda_{s,t}$ (we observed earlier that z does not occur in this entry), we have that the coefficient of z^{2r} in the determinant of $\Theta_{s,t}$ is

$$\sum_{i,j\ge 2}\lambda_{st}^{ij}z_iz_j - \sum_{\rho=1}^{r}\Bigg(\sum_{i\ge 2}(\lambda_{\rho s}^{i1} + \lambda_{\rho s}^{1i})z_i\sum_{j\ge 2}(\lambda_{t\rho}^{j1} + \lambda_{t\rho}^{1j})z_j\Bigg).$$

Since $\Theta_{s,t}$ has rank r, the displayed expression vanishes, i.e., for all $s,t > r$,

$$(\dagger)\qquad \sum_{i,j\ge 2}\lambda_{st}^{ij}z_iz_j = \sum_{\rho=1}^{r}\Bigg(\sum_{i\ge 2}(\lambda_{\rho s}^{i1} + \lambda_{\rho s}^{1i})z_i\sum_{j\ge 2}(\lambda_{t\rho}^{j1} + \lambda_{t\rho}^{1j})z_j\Bigg).$$

Since the z_j are indeterminates, we may substitute the L_j for the z_j in (†) to obtain that for all $s,t > r$,

$$(\dagger\dagger)\qquad \sum_{i,j\ge 2}\lambda_{st}^{ij}L_iL_j = \sum_{\rho=1}^{r}\Bigg(\sum_{i\ge 2}(\lambda_{\rho s}^{i1} + \lambda_{\rho s}^{1i})L_i\sum_{j\ge 2}(\lambda_{t\rho}^{j1} + \lambda_{t\rho}^{1j})L_j\Bigg).$$

Choose an off-diagonal entry g_1 in the first row of $\mathfrak{P}_0$ and an off-diagonal entry g_2 of the first column. Then by Lemma 9.1, the $2r_1$ linear forms needed to span $\mathcal{D}g_1$ and $\mathcal{D}g_2$ generate an ideal that contains all the f_{1j} and all of the f_{j1}. Note that if $\mathfrak{P}_0$ is symmetric, we may take $g_1 = g_2$ and we only need r_1 elements. (If $M = 1$, we use instead f_{11} as the generator of the ideal need.) Let $\mathfrak{A}$ be the ideal generated by these elements and $x_1, \ldots, x_r$. The congruences below are taken modulo $\mathfrak{A}$, which has at most $2r_1 + r$ generators ($r_1 + r$ if $\mathfrak{P}_0$ is symmetric).

$$f = \sum_{i,j}L_iL_jf_{ij} \equiv \sum_{i\ge 2, j\ge 2}L_iL_jf_{ij}$$

since the $f_{i,1},\ f_{1,j} \in \mathfrak{A}$. We can rewrite this as

$$\sum_{i\geq 2, j\geq 2} L_iL_j\Big(\sum_{s,t}\lambda^{ij}_{st}x_sx_t\Big)$$

which, since $x_1,\,\dots,\,x_r \in \mathfrak{A}$,

$$\equiv \sum_{i\geq 2, j\geq 2} L_iL_j\Big(\sum_{s>r,t>r}\lambda^{i,j}_{s,t}x_sx_t\Big)$$

since $x_1,\,\dots,\,x_r \in \mathfrak{A}$. This becomes

$$\sum_{i\geq 2, j\geq 2}\Big(\sum_{s>r,t>r}(\lambda^{ij}_{st}x_sx_tL_iL_j)\Big) =$$

$$\sum_{s>r,t>r}\Big(\sum_{i\geq 2, j\geq 2}(\lambda^{ij}_{st}L_iL_j)x_sx_t\Big) = \sum_{s>r,t>r}x_sx_t\Big(\sum_{i\geq 2, j\geq 2}(\lambda^{ij}_{st}L_iL_j)\Big)$$

Using (††) this becomes

$$\sum_{s>r,t>r}x_sx_t\sum_{\rho=1}^{r}\Big(\sum_{i\geq 2}(\lambda^{i1}_{\rho s}+\lambda^{1i}_{\rho s})L_i\sum_{j\geq 2}(\lambda^{j1}_{t\rho}+\lambda^{1j}_{t\rho})L_j\Big) =$$

$$\sum_{\rho=1}^{r}\Big(\sum_{s>r,t>r}x_sx_t\Big(\sum_{i\geq 2}(\lambda^{i1}_{\rho s}+\lambda^{1i}_{\rho s})L_i\Big)\Big(\sum_{j\geq 2}(\lambda^{j1}_{t\rho}+\lambda^{1j}_{t\rho})L_j\Big)\Big) =$$

$$\sum_{\rho=1}^{r}\Big(\sum_{s>r}\Big(x_s\sum_{i\geq 2}(\lambda^{i1}_{\rho s}+\lambda^{1i}_{\rho s})L_i\Big)\Big)\Big(\sum_{t>r}\Big(x_t\sum_{j\geq 2}(\lambda^{j1}_{t\rho}+\lambda^{1j}_{t\rho})L_j\Big)\Big),$$

which shows that f has a strict r-collapse modulo an ideal generated by at most $2r_1+r$ forms (r_1+r if $\mathfrak{P}_0$ is symmetric). This proves that f has a strict $2(r_1+r)$-collapse (a strict (r_1+2r)-collapse if $\mathfrak{P}_0$ is symmetric). □

10. The quartic case: $\mathfrak{K}_4$, and ${}^{\eta}\underline{A}(n_1,n_2,n_3,n_4)$

Before proving the existence of $\mathfrak{K}_4$ for characteristic not 2 or 3, we need two observations. One is that by iterating Euler's formula, we have that for a homogeneous polynomial F of degree d over a field K, $d(d-1)F = \sum_{i,j}x_ix_j\partial^2F/\partial x_i\partial x_j$.

Second, we need the following observation:

Lemma 10.1. *Let K be a field and let $K[x_1,\,\dots,\,x_N]$ be a polynomial ring. Let G be a form of degree $d > 1$. Assume that d is not zero in K. If G of degree at least 3 has a k-collapse such that $0 \leq h \leq k$ of the k terms in the collapse expression involve a linear factor, then $\mathcal{D}G$ has a collective $(2k-h,h)$-collapse. Hence, $\mathcal{D}G$ has a collective $2k$-collapse. In particular, if G has degree 3 and has a k-collapse, then $\mathcal{D}G$ has a collective (k,k)-collapse.*

Proof. Suppose that $G = \sum_{s=1}^{h}L_sM_s + \sum_{t=1}^{k-h}N_tP_t$ where the L_i are linear forms, the M_i have degree $d-1$, and the N_s, P_s have degrees between 2 and $d-2$. By the product rule, each partial derivative of G is in the ideal generated by the L_s, N_t, P_t ($k-h+k-h+h$ elements) and the M_s (h elements). The final two remarks follows at once. (For the final remark, note that every term in a collapse of degree 3 polynomial involves a linear factor.) □

We are now ready to prove the existence of $\mathfrak{K}_4$. If G is a form in $K[x_1, \ldots, x_N]$, ∇G denotes the column of partial derivatives $\partial G/\partial x_i$ of G.

Theorem 10.2. *Let K be an algebraically closed field of characteristic not 2 or 3. Then we may take $\mathfrak{K}_4(k) = 6k(k+1)4^{k(k+1)} + (k+1)^2$.*

Proof. Let $R = K[x_1, \ldots, x_N]$ and let F be a quartic form over K. Suppose that $\mathcal{D}F$ consists entirely of forms with a k-collapse. By Definition 1.13, what we need to show is that F has $\big(6k(k+1)4^{k(k+1)} + (k+1)^2\big)$-collapse. By Proposition 3.5(b), it suffices to show this after enlarging the field.

Let $\mathfrak{H}$ be the Hessian of F. A linear combination of columns of $\mathfrak{H}$ is the same as ∇G, where G is the corresponding linear combination of $\partial F/\partial x_1, \ldots, \partial F/\partial x_N$. It follows from Lemma 10.1 that all of the LC_K-columns of $\mathfrak{H}$ have a collective (k, k)-collapse: each has the form ∇G for some $G \in \mathcal{D}F$.

Let notation and hypotheses be as in Discussions 7.2 and 8.1. In particular, let $\mathcal{T}$ be as in Discussion 8.1, and let Z and Z_0 be as in the statement of Theorem 8.3, part (d). By Theorem 7.1, Theorem 8.2, and Theorem 8.3 with $h = k$, we first obtain a collective $(k(k+1), k(k+1))$-collapse for the columns of $\mathcal{T}\mathfrak{H}\mathcal{T}^{\mathrm{tr}}$ using a single auxiliary vector space. We also obtain from Theorem 8.4 that $\mathcal{T}\mathfrak{H}\mathcal{T}^{\mathrm{tr}}$ has a decomposition $\mathfrak{P}+\mathfrak{Q}$ such that, if (as indicated above) Z_0 is a size $N-k-1$ square matrix of new indeterminates and $\mathfrak{P}_0$ is the size $N-k-1$ square submatrix in the upper left corner of $\mathfrak{P}$ (this notation is the same as in the statement of Theorem 8.4) then the following hold.

(1) The elements of $Z_0\mathfrak{P}_0Z_0$ all have a strict $k(k+1)4^{k(k+1)}$-collapse.
(2) The elements of $\mathfrak{Q}$ are in a vector space W_0 of dimension at most $k(k+1)$ over the extended base field.
(3) Hence, the rank of each element of $Z_0\mathfrak{P}_0Z_0$ is at most $r = 2k(k+1)4^{k(k+1)}$.

The iterated Euler's formula discussed above for $d = 4$ yields that $\big(12F\big) = X\mathfrak{H}X^{\mathrm{tr}}$, where X is the row matrix with entries $x_1, \ldots, x_N$. Then,

$$(*) \quad \big(12F\big) = X(\mathfrak{P}+\mathfrak{Q})X^{\mathrm{tr}} = X\mathfrak{P}X^{\mathrm{tr}} + X\mathfrak{Q}X^{\mathrm{tr}}.$$

The entries of $\mathfrak{Q}$ are in W_0 and so they are in the ideal I_1 generated by at most $k(k+1)$ linear forms. Let I_2 be the ideal generated by $x_{N-k+1}, \ldots, x_N$. Modulo I_2, X becomes the concatenation of a $1\times(N-K)$ matrix X_1 and a $1\times k$ block of zeros. Modulo I_1, $\mathfrak{Q} \equiv 0$. So $X\mathfrak{H}X^{\mathrm{tr}} \equiv X(\mathfrak{P})X^{\mathrm{tr}}$, and modulo $I_1 + I_2$, $X\mathfrak{H}X^{\mathrm{tr}} \equiv X_1\mathfrak{P}_0X_1^{\mathrm{tr}}$.

Hence, from $(*)$, we have $12F \equiv X_1\mathfrak{P}_0X_1^{\mathrm{tr}}$. By Theorem 9.3 in the symmetric case, since r bounds the ranks of all elements, $X_1\mathfrak{P}_0X_1^{\mathrm{tr}}$ has a strict $3r$-collapse, i.e., a strict $6k(k+1)4^{k(k+1)}$-collapse. Since I_1+I_2 has $k(k+1)+(k+1)$ generators, this implies that $12F$ and, hence, F, has a strict $\big(6k(k+1)4^{k(k+1)} + k(k+1) + k+1\big)$-collapse, as required. □

Theorem 10.3. *For algebraically closed fields of characteristic $\neq 2, 3$, a choice of the function ${}^{\eta}\underline{A}(n_1, n_2, n_3, n_4)$ exists for all $\eta \geq 1$, and can be calculated from Theorem 1.17. Hence, choices of the functions ${}^{\eta}B(n_1, n_2, n_3, n_4)$ and $C(n_1, n_2, n_3, n_4)$ can also be made explicit.*

In fact, one obtains a closed form for the functions ${}^{\eta}\underline{A}$. The result is rather complicated, but Corollary 6.5, Corollary 1.18, and Theorem 10.2 imply at once:

Corollary 10.4. *For algebraically closed fields of characteristic not 2 or 3, if $\delta = (0, 0, 0, n)$, corresponding to an n-dimensional vector space whose nonzero forms all have degree 4, we may take*

$$^{\eta}A(\delta) = \mathfrak{K}_4\big((2n+\eta)A_3(2n+\eta)\big) + 2n + \eta - 1$$

with $\mathfrak{K}_4$ as in Theorem 10.2 and

$$A_3(2n+\eta) = 2(8n+4\eta-1)(2n+\eta-1).$$

11. Questions and conjectures

We believe that the values we have found for the the functions $^{\eta}A$ and $^{\eta}B$ are very far from best possible (except for $^{\eta}A$ in the case of quadrics). We strongly expect that the best possible Stillman bounds are far better than those we have found. The conjectures below express these expectations.

Conjecture 11.1. *For all characteristics the best possible bound for $\mathfrak{K}_d(k)$ with d fixed and k varying is at worst $c_d n^{\kappa(d)}$ for some positive constant c_d and function κ of d.*

Note that for $d = 3$, $c_3 = 2$ and $\kappa(3) = 1$. Of course, our result for $\mathfrak{K}_4$ is much worse than polynomial, but we believe that there is a polynomial bound.

Conjecture 11.2. *For all characteristics the best possible bound for $^{\eta}A$ for a given value of η for n polynomials of degree d, where d is fixed and n varies, is, at worst, $C_{d,\eta}n^{\lambda(d)}$ for some positive constant $C_{d,\eta}$ depending only on d and η and some function λ of d.*

This has been proved here for $d \le 3$, we may take $\lambda(d) = d - 1$. However, we do not feel that there is enough evidence to make a conjecture yet. Because of the formula in part (3) of Theorem 1.17, or by Corollary 1.18 we have that $^{\eta}A_d(0, \dots, 0, n) = \mathfrak{K}_d\big((2n+\eta)A_{d-1}(2n+\eta)\big) + 2n + \eta - 1$. This shows that Conjecture 11.1 implies Conjecture 11.2, and that one has, at worst, $\lambda(d) = \kappa(d)\big(\lambda(d-1)+1\big)$.

Conjecture 11.3. *For all characteristics, the bound for the projective dimension of an ideal generated by n forms of degree at most d, where d is fixed and n varies, is, at worst, $C_d n^d$ for some positive constant C_d depending only on d.*

Note that while the conjecture for $^{\eta}A$ has been verified here for $d \le 3$, but that the conjecture for Stillman bounds is not known even in the case $d = 2$.

The following is raised as Question 6.2 in [25]. We conjecture this explicitly: of course, it implies Conjecture 11.3 for the case $d = 2$.

Conjecture 11.4. *If R is a polynomial ring over a field and I an ideal of R generated by n quadrics and of height h, then $\mathrm{pd}_R(R/I) \le h(n-h+1)$.*

The ideal $I_{m,n,d}$ constructed in [30] has height m, is minimally generated by $m + n$ homogeneous polynomials of degree d, while its projective dimension is $m + n\binom{m+d-2}{d-1}$, which is a polynomial of total degree d in m, n. These examples for $d = 2$ show that one cannot improve the bound in Conjecture 11.4, and that in a bound that is polynomial in n for the projective dimension for fixed d must have degree at least d.

The problem of giving explicit bounds for $^{\eta}A$ remains in characteristic 2, 3 even when $d = 4$. The corresponding problem for $^{\eta}A$ and $^{\eta}B$ if $d > 4$ is untouched.

Moreover, so far as we know, there is almost nothing known about lower bounds for ${}^{\eta}B$, even for quadrics, except the obvious fact that it must exceed the Stillman bound on projective dimension. In particular, so far as we know, it is possible that a polynomial bound for ${}^{\eta}B$ exists in every degree. This problem is wide open, even for quadrics.

Question 11.5. We note that the results of [2] show that everything about the primary decomposition of an ideal generated by n forms of specified degrees is bounded in terms of n and the degrees of the forms: this includes the number of primes, and the numbers of minimal generators and their degrees for both the primes and primary ideals occurring. We want to point out that it is largely an open question what can happen even for a regular sequence of n quadrics. Since the multiplicity of the quotient is 2^n, the number of associated primes (which are the same as the minimal primes) is at most 2^n. This can happen, e.g., if the regular sequence is $x_1x_2, x_3x_4, \ldots, x_{2n-1}x_{2n}$. We do not know what might happen with the numbers of generators nor with their degrees.

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Department of Mathematics
University of Michigan
Ann Arbor, MI 48109–1043
USA

E-mail: hochster@umich.edu

Altair Engineering
1820 E. Big Beaver Rd.
Troy, MI 48083
USA

E-mail: antigran@gmail.com